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# State Estimation and Stabilization of Continuous Systems with Uncertain Nonlinearities and Disturbances

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**Abstract**—Nonautonomous control systems with uncertain nonlinearities subjected to bounded exogenous disturbances are considered. Based on the method of matrix comparison systems and the framework of differential linear matrix inequalities, we suggest a solution approach to the problems of state estimation, stability and boundedness with respect to given sets, as well as suppression of disturbances and initial deviations using linear state-feedback control.

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## 1. INTRODUCTION

As a matter of fact, the problems of state estimation and control design are major in control theory. State estimation for linear nonautonomous systems employs ellipsoid-based methods [1–3] yielding locally optimal or minimax guaranteed ellipsoidal estimates for the set of induced processes of a linear system taking into account uncertain initial deviations, exogenous disturbances, and parametric changes. There exist control design methods belonging to the suppression of arbitrary bounded exogenous disturbances and involving the framework of linear matrix inequalities (LMIs) [4, 5] and the method of invariant ellipsoids [1, 6–10]. Here optimal (or suboptimal) controller design is reduced to searching for the smallest invariant ellipsoid of a closed-loop system. These methods demonstrate highest efficiency for linear stationary control systems. In [11, 12] LMIs were used for the robust stabilization of systems with uncertain nonlinearities, but without consideration of exogenous disturbances. The paper [13] applied this approach to find invariant ellipsoids and construct state-feedback controllers for a class of nonlinear autonomous systems subjected to uncertain bounded exogenous disturbances.

In a series of works [14–20], the method of matrix comparison systems (MCSs) proposed in [21] for stability analysis of Lourier-type controlled systems was further developed for the dynamic analysis and state estimation of nonlinear controlled systems with uncertain disturbances, parametric and structural changes.

Stabilization of nonautonomous systems often uses control laws with time-varying coefficients. Scientific literature also describes stabilization based on control laws with time-dependent coefficients for linear autonomous systems, e.g., a finite-horizon linear-quadratic controller. To evaluate the controller's coefficients, it is necessary to solve numerically the differential matrix Riccati equation backward in time; this procedure causes definite difficulties, particularly, requires information on the model parameters in future time. Some research efforts were focused on suggesting control design methods where the controller's coefficients are defined via numerical solution of a differential matrix equation forward in time. On the other hand, for linear nonautonomous systems, the recent years yielded the methods of stability analysis, finite-horizon boundedness and control design using the framework of differential linear matrix equations and inequalities, see [22–26].

The present paper adopts the method of matrix comparison systems [14–20] and the framework of differential linear matrix inequalities (DLMIs) [26] for proposing the ways of state estimation, analysis of stability and boundedness with respect to given sets (both on finite and infinite horizons), evaluation of invariant ellipsoids and design of state-feedback control (with time-dependent coefficients) for a class of nonautonomous nonlinear systems. It is demonstrated that the partial solutions of a MCS or DLMI define time-evolving invariant ellipsoids for the solutions of an initial system with any nonlinearities satisfying a given constraint and with any exogenous disturbances having a bounded norm. The conditions of stability and boundedness with respect to given sets (actually predetermining the performance of the initial system) are expressed as constraints on the partial solutions of the MCS or DLMI on the whole horizon considered. The control design problem is reduced to the numerical solution of the DLMI with optimization for the trace of the matrix of the invariant or output-bounding ellipsoid at each step of time sampling.

In the autonomous case, the estimates of the limit state in the form of an invariant ellipsoid can be obtained via numerical solution of the DLMIs, numerical integration of the MCS or numerical solution of algebraic matrix equations representing the equilibrium equations for the corresponding MCS. It is possible to pass from the DLMIs to algebraic LMIs appearing similar LMIs obtained in the autonomous case by the method of invariant ellipsoids [10, 13]. By a simple example, we illustrate the capabilities and efficiency of the proposed solution approach to the above-mentioned problems. Numerical simulation is performed in Matlab using CVX and Ellipsoid Toolbox.

## 2. SYSTEMS WITH UNCERTAIN NONLINEARITIES AND BOUNDED DISTURBANCES

Consider a dynamic system described by the following state-space model in continuous time:

$$\begin{aligned} \dot{x} &= A(t)x + D(t)w + \Phi(t)\varphi(t,x), \quad x(t_0) = x_0, \\ z &= C(t)x, \end{aligned} \tag{1}$$

where $x \in R^n$ denotes the state vector, $w(t) \in W_r \subset R^r$ is the input disturbance, $z \in R^l$ means the controlled output vector, $A(t) \in R^{n\times n}$, $D(t) \in R^{n\times r}$, $\Phi(t) \in R^{n\times q}$, and $C(t) \in R^{q\times n}$ are known matrices with continuous bounded entries for all $t \in [t_0, T)$, $T > 0$ indicates a given constant (in the finite-horizon setting) or $T = \infty$ (in the infinite-horizon setting).

A nonlinear vector function $\varphi(t,x)$ satisfies the conical (generalized sectoral) condition of the form

$$\|\varphi(t,x)\|^2 \le \mu_0 + \mu_1 \|C_f(t)x(t)\|^2, \ \ \forall t \in [t_0, T), \quad x \in R^n, \tag{2}$$

where $C_f(t) \in R^{q\times n}$ is a known matrix with continuous bounded entries for all $t \in [t_0, T)$. Here and in the sequel, $\|\cdot\|$ stands for the Euclidean vector norm, $\mu_0, \mu_1 \ge 0$ are given parameters.

Note that, under $\mu_0 = 0$, the condition (2) covers sectoral nonlinearities and local Lipschitz nonlinearities defined so that they vanish at the origin.

Assume that the uncertain disturbances represent continuous functions bounded for each time, i.e.,

$$\|w(t)\|^2 \le 1, \ \ \forall t \ge t_0. \tag{3}$$

Introduce the following notation: $G_+$ as the set of symmetric ($S = S^{\mathrm{T}}$) nonnegative definite ($S \ge 0$) matrices $S \in R^{n\times n}$ and $G^+$ as the set of symmetric ($S = S^{\mathrm{T}}$) positive definite ($S > 0$) matrices $S \in R^{n\times n}$. As is generally known, $G_+$ is a bodily reproducing cone used to define a partial order on the space of symmetric matrices $S \in R^{n\times n}$.

## 3. STATE ESTIMATION PROBLEM

Suppose that at the initial time $t_0$ the system has an uncertain state belonging to an ellipsoid

$$E(Q_0) = \left\{x \in R^n : x^{\mathrm{T}} Q_0^{-1} x \leq 1\right\}, \tag{4}$$

where $Q_0 \in G^+$ is a given positive definite symmetric matrix.

By assumption, the pair $(A(t), D(t))$ is controllable and the matrix $C(t)$ has full row rank.

The problem consists in obtaining an ellipsoidal estimate for the state set of the induced processes of the system (1) on a given time horizon that evolve from the ellipsoid $E(Q_0)$ under the nonlinearities satisfying the condition (2) and the uncertain disturbances of the form (3).

The state vector will be estimated as an evolutionary ellipsoid being invariant for the processes of the system (1).

Recall that an invariant ellipsoid for a dynamic system is an ellipsoid

$$E(Q(t)) = \left\{x \in R^n : x^{\mathrm{T}} Q^{-1}(t) x \leq 1\right\}, \quad Q(t) > 0, \tag{5}$$

with the property that any system trajectory $x(t, t_0, x_0)$ evolving from a point $x(t_0) = x_0 \in E(Q(t_0))$ belongs to this ellipsoid at any time $t$, i.e., $x(t, t_0, x_0) \in E(Q(t))$.

Invariant ellipsoids characterize an external estimate of the attainability domain at the current time $t$ under the impact of the uncertain exogenous disturbances $w(t)$ and the nonlinearities on the trajectories of the system (1). In this context, of definite interest are minimal (in some sense) invariant ellipsoids containing the trajectories or output $z(t)$ of the system.

**Theorem 1.** *An ellipsoid $E(Q(t))$, where $Q(t) = Q(t, t_0, Q_0)$ is the solution of the matrix system of differential equations*

$$dQ(t)/dt = AQ + QA^{\mathrm{T}} + \alpha Q + \frac{1}{\alpha - \frac{\mu_0}{\beta}} DD^{\mathrm{T}} + \beta \Phi \Phi^{\mathrm{T}} + \frac{\mu_1}{\beta} Q C_f^{\mathrm{T}} C_f Q, \tag{6}$$

*is invariant for the trajectories of the system* (2) *that evolve from the initial ellipsoid $E(Q_0)$.*

Note that, in the autonomous linear system, an invariant ellipsoid with a constant matrix $Q^*$ defined by solving the algebraic Lyapunov equation (or the corresponding LMIs) is an attractor; in other words, all solutions of the system with any initial conditions and exogenous disturbances satisfying (3) converge to this ellipsoid. In the nonlinear autonomous system, an invariant ellipsoid is an attractor only for the solutions with initial conditions belonging to the domain of attraction of a certain limiting set. The invariant ellipsoid gives the upper estimate of this limiting set.

We will use the MCS and DLMI to estimate the state set of the initial nonlinear nonautonomous system at current times for the trajectories evolving from the initial ellipsoid, as well as to estimate the domain of attraction in the autonomous case.

**Theorem 2.** *An ellipsoid $E(Q(t))$ is invariant for the solutions of the system* (1) *if its matrix $Q(t) = Q(t, t_0, Q_0) > 0$ satisfies the differential linear matrix inequalities*

$$\begin{bmatrix} -dQ(t)/dt + AQ + QA^{\mathrm{T}} + \alpha Q + \beta \Phi \Phi^{\mathrm{T}} & D & QC_f^{\mathrm{T}} & 0 \\ * & -\alpha I & 0 & I \\ * & * & -\frac{\beta}{\mu_1} I & 0 \\ * & * & * & -\frac{\beta}{\mu_0} I \end{bmatrix} \leq 0 \tag{7}$$

*for all $t \in [t_0, T)$ and some $\beta > 0$, $\alpha \geq \mu_0/\beta$.*

**Definition 1.** A matrix function $F(t, S)$ is said to be nondecreasing with respect to the cone $G_+$ if for any $S_1, S_2 \in R^{n\times n}$ such that $S_2 - S_1 \in G_+$ it follows that $F(t, S_2) - F(t, S_1) \in G_+$ (or, similarly, the condition $y^{\mathrm{T}}(S_2 - S_1)y \geq 0$ implies $y^{\mathrm{T}}[F(t, S_2) - F(t, S_1)]y \geq 0$ for all $y \in R^n$ and $t \in [t_0, T)$).

**Definition 2.** A function $F(t, S)$ is said to be quasimonotonically nondecreasing with respect to the cone $G_+$ if for any $S_1, S_2 \in R^{n\times n}$ such that $S_2 - S_1 \in G_+$ and for all $y \in R^n$ such that $y^{\mathrm{T}}(S_2 - S_1)y = 0$ it follows that $y^{\mathrm{T}}[F(t, S_2) - F(t, S_1)]y \geq 0$ for all $t \in [t_0, T)$.

**Lemma 1.** *The system of differential Eqs.* (6) *is the matrix comparison system for the initial system* (1).

And the question regarding the existence of positive definite solutions of the comparison system (6) arises immediately. The following lemma gives the answer.

**Lemma 2.** *Let the matrices $A(t)$, $D(t)$, $\Phi(t)$, and $C_f(t)$ be such that the comparison system* (6) *with $Q(t_0) = Q_0 \in G_+$ has a unique solution $Q(t) = Q(t, t_0, Q_0)$ on some interval $[t_0, T_1)$ $(T_1 \leq T)$. Then $Q(t_0) \geq 0$ (or $Q(t_0) > 0$) implies $Q(t) \geq 0$ ($Q(t) > 0$, respectively) for all $t \in [t_0, T_1)$.*

**Theorem 3.** *Suppose that $Q(t) = Q(t, t_0, Q_0)$, $Q(t_0) = Q_0 > 0$, is a solution of the matrix comparison system* (6) *(of the DLMI* (7)*) on the interval $[t_0, T_1)$. Then for all $t \in [t_0, T_1)$ the solutions of the initial system* (1) *with the initial conditions $x(t_0) = x_0 \in E(Q_0)$, the uncertain disturbances satisfying* (2) *and the nonlinearities of the form* (3) *obey the following estimates:*

1) $x(t, t_0, x_0) \in E(Q(t))$;
2) $x(t, t_0, x_0)x^{\mathrm{T}}(t, t_0, x_0) \leq Q(t, t_0, Q_0)$ *(an inequality with respect to the cone $G_+$)*;
3) $x^{\mathrm{T}}(t, t_0, x_0)x(t, t_0, x_0) \leq Trace(Q(t, t_0, Q_0))$;
4) $x_i^2(t, t_0, x_0) \leq Q_{ii}(t, t_0, Q_0), \quad i = 1, \ldots, n$;
5) $z^{\mathrm{T}}(t)z(t) \leq Trace(C^T Q(t, t_0, Q_0)C)$.

Based on the MCS properties established in [17], it is possible to show the invariance of some sets for the solutions of the MCS (6).

Denote by $F(t, Q)$ the matrix function in the right-hand side of the MCS (6), $H_t = \{Q \in G_+ : F(t, Q) \leq 0\, ;\ \forall t \in [t_0, t_0 + T)\}$.

**Lemma 3.** *For any matrix $P \in H_t$, the set $\bar{S}(P) = \{S \in G_+ : S \leq P \in H_t\}$ in the matrix space $G_+$ is positive invariant for the solutions $Q(t, t_0, S)$ of the MCS with $Q(t_0) = S$.*

In the autonomous case, designate by $F(Q)$ the right-hand side of the matrix comparison system (6). Note that $H = \{Q \in G_+ : F(Q) \leq 0\}$.

**Theorem 4.** *The set $H$ in the matrix space $G_+$ is positive invariant for the solutions $Q(t, t_0, Q_0)$ of the MCS with $Q(t_0) = Q_0 \in H$. Moreover, there exists a $Q^* \in H$ such that*

$$\lim_{t\to\infty} Q(t, t_0, Q_0) = Q^*.$$

Based on Theorem 4, it is possible to obtain invariant ellipsoid estimates for the initial domain from which the solutions converge to the limited or minimal invariant ellipsoid.

## 4. THE PROBLEM OF BOUNDEDNESS (STABILITY) WITH RESPECT TO GIVEN SETS

Assume that the set of initial states, the set of admissible exogenous disturbances, and the set of admissible states at times $t \in [t_0, T)$ are defined in the form of ellipsoids $E(R) = \{x \in R^n : x^{\mathrm{T}} R^{-1} x \leq 1\}$, $E_w(I) = \{w \in R^n : w^{\mathrm{T}} w \leq 1\}$, and $E(S(t)) = \{x \in R^n : x^{\mathrm{T}} S^{-1}(t) x \leq 1\}$, respectively, where $R$ and $S(t)$ are known symmetric positive definite matrices.

**Definition 3.** We say that the system (1) is bounded with respect to the given sets $[E(R),$ $E_w(I)$, $E(S(t))]$ (in the absence of uncertain exogenous disturbances, stable with respect to $[E(R),$ $E(S(t))]$) on the interval $[t_0, T)$ if for all $x_0 \in E(R)$ there exist solutions $x(t) = x(t, t_0, x_0)$ of the system (1) with the initial conditions $x(t_0) = x_0$ on $[t_0, T)$ that satisfy $x(t, t_0, x_0) \in E(S(t))$ for all $t \in [t_0, T)$, $w(t) \in E_w(I)$ and all nonlinearities of the form (2) ($x(t, t_0, x_0) \in E(S(t))$ for all $t \in [t_0, T)$ and all nonlinearities of the form (2) with $\mu_0 = 0$, respectively).

In contrast to the generally accepted definitions of absolute stability and boundedness, the above definitions involve concrete sets of initial conditions and concrete sets for the trajectories of the system with these initial conditions. In this regard, the definitions are analogous to the ones of stability and boundedness on finite horizon intensively studied in recent years, see [22–26]. In the case considered, time horizon can be either finite or infinite.

**Theorem 5.** *For any nonlinear functions of the form* (2)*, the system* (1) *is bounded with respect to the given sets* $[E(R), E_w(I), E(S(t))]$ (*in the absence of exogenous disturbances and* $\mu_0 = 0$*, stable with respect to* $[E(R), E(S(t))]$) *on the interval* $[t_0, T)$ *if there exists a solution* $Q(t) = Q(t, t_0, Q_0)$ *of the matrix comparison system* (6) (*or the DLMI* (7)) *with the initial conditions* $Q_0 \geq R$ *that satisfy the inequality* $Q(t) \leq S(t)$ *for all* $t \in [t_0, T)$.

In the absence of exogenous disturbances and $\mu_0 = 0$ in (2), the matrix comparison system acquires the form

$$dQ(t)/dt = AQ + QA^{\mathrm{T}} + \beta\Phi\Phi^{\mathrm{T}} + \frac{\mu_1}{\beta}QC_f^{\mathrm{T}}C_fQ, \quad \beta > 0. \tag{8}$$

If $Q(t) = Q(t, t_0, Q_0)$ is a solution of the matrix comparison system (8) (or the DLMI (7)) that satisfies the conditions of Theorem 5, then all conditions of Definition 3 hold and stability with respect to the given sets $[E(R), E(S(t))]$ takes place accordingly.

## 5. CONTROL DESIGN PROBLEM

Consider the continuous-time control system

$$\begin{aligned} &\dot{x} = A(t)x + B(t)u + D(t)w + \Phi(t)\varphi(t, x, u), \quad x(t_0) = x_0, \\ &z = C(t)x + B_z(t)u, \end{aligned} \tag{9}$$

where $x \in R^n$ denotes the state vector, $w(t) \in R^r$ is an exogenous disturbance satisfying the constraint (2), $u \in R^m$ means the control vector, and $z \in R^l$ indicates the controlled output. The nonlinear vector function $\varphi(t, x, u) \in R^q$ satisfies the inequality

$$\begin{gathered} \|\varphi(t, x, u)]\|^2 \leq \mu_0 + \mu_1 \|C_f(t)x(t) + B_f(t)u(t)\|^2, \\ \forall t \in [t_0 + T), \quad x \in R^n, \quad u \in R^m, \end{gathered} \tag{10}$$

where $A(t) \in R^{n\times n}$, $B(t) \in R^{n\times m}$, $D(t) \in R^{n\times r}$, $\Phi(t) \in R^{n\times q}$, $C(t) \in R^{q\times n}$, $B_z(t) \in R^{q\times m}$, $C_f(t) \in R^{q\times n}$, and $B_f(t) \in R^{q\times m}$ are known matrices with continuous bounded entries for all $t \in [t_0, T)$. By assumption, the pair $(A, B)$ is controllable and the pair $(A, C)$ is observable.

The problem is to find a state-feedback controller

$$u = K(t)x \tag{11}$$

that

1) stabilizes the closed-loop system and suppresses the initial deviations and the exogenous disturbances (in the sense of the minimal bounding ellipsoid for the output $z$), and

2) ensures the boundedness of the closed-loop system.

The design problem is reduced to optimization of a performance criterion subject to differential linear matrix inequalities. A common criterion used is the trace of the matrix defining the size of the invariant or output-bounding ellipsoid.

**Theorem 6.** *Consider the solution $Q^*(t) = Q^*(t, t_0, Q_0)$, $Y^*(t)$, $Z^*(t)$ of the problem*

$$\mathit{trace}\left[CQC^{\mathrm{T}} + CY^{\mathrm{T}}B_z^{\mathrm{T}} + B_zYC^{\mathrm{T}} + B_zZB_z^{\mathrm{T}}\right] \to \min$$

*subject to the constraints*

$$\begin{bmatrix} -dQ(t)/dt + AQ + QA^{\mathrm{T}} + BY + Y^{\mathrm{T}}B^{\mathrm{T}} + \alpha Q + \beta\Phi\Phi^{\mathrm{T}} & D & QC_f^{\mathrm{T}} + Y^{\mathrm{T}}B_f^{\mathrm{T}} & 0 \\ * & -\alpha I & 0 & I \\ * & * & -\dfrac{\beta}{\mu_1}I & 0 \\ * & * & * & -\dfrac{\beta}{\mu_0}I \end{bmatrix} \le 0, \qquad (12)$$

$$\begin{bmatrix} Z & Y \\ Y^{\mathrm{T}} & Q \end{bmatrix} \ge 0, \qquad (13)$$

*where minimization runs with respect to the matrix variables $Q(t) = Q^{\mathrm{T}}(t) \in R^{n\times n}$, $Y(t) \in R^{n\times m}$ and $Z(t) \in R^{l\times l}$, the scalar variable $\beta > 0$, and the scalar parameter $\alpha \ge \mu_0/\beta$. For each $t \in [t_0, T)$, this solution defines the matrix $Q^*(t)$ of the bounding ellipsoid for the state vector $x(t)$, the matrix $CQ^*C^{\mathrm{T}} + CY^{*\mathrm{T}}B_z^{\mathrm{T}} + B_zY^*C^{\mathrm{T}} + B_zZ^*B_z^{\mathrm{T}}$ of the bounding ellipsoid for the output $z(t)$ of the system* (9) *and the time-dependent gain matrix $K^*(t) = Y^*(t)Q^{*-1}(t)$ for the corresponding state-feedback controller. Moreover, if the matrix $Q^*(t)$ satisfies the constraints $Q_0 \ge R$ and $Q(t) \le S(t)$ for all $t \in [t_0, T)$ where $R$ and $S(t)$ are given positive definite symmetric matrices, then the above controller ensures the boundedness of the closed-loop system with respect to the sets $[E(R), E_w(I), E(S(t))]$ for all nonlinearities of the form* (10).

## 6. EXAMPLE

As an example, consider the disturbed oscillation model of a mathematical pendulum [13] described by

$$\ddot{x} + \omega_0^2 \sin x = u + w(t).$$

Here $u$ specifies control and $w$ is a bounded exogenous disturbance (the deviation angles from an equilibrium are arbitrary). Introducing the new variables

$$x_1 = x, \quad x_2 = \dot{x}_1$$

yields the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\omega_0^2 \sin x_1 + u + w(t). \end{aligned}$$

It will be treated as the system (1) with the nonlinearity $\varphi(x) = \sin x_1$ satisfying the constraints (2) with $\mu_0 = 0$ and $\mu_1 = 1$. To avoid large values of control, choose the vector $z = (x_1, x_2, u)^{\mathrm{T}}$ as the system output. In addition, take into account an uncertainty in the initial state of the system that is defined by the matrix $P_0$. For the given system, set the matrices in (1) by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 \\ -\omega_0^2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

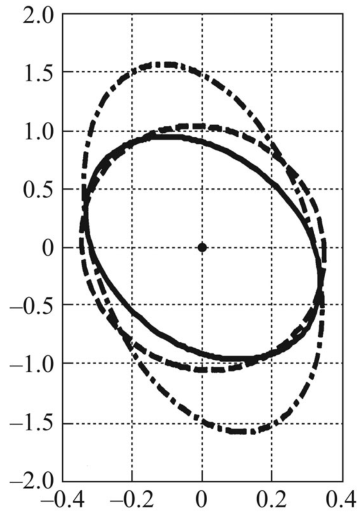


**Fig. 1.** Invariant ellipses for the pendulum.

Actually, this choice leads to Example 1 studied in the paper [13] if $C_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_f = [0]$.

For $\omega_0 = 3$ and $P_0 = 0.1I$, just like in [13], we have calculated the matrix

$$\widehat{Q} = \begin{bmatrix} 0.1121 & -0.0994 \\ -0.0994 & 0.9192 \end{bmatrix}$$

of the minimal invariant ellipse and the corresponding controller with the constant gains $\widehat{K} = [\ -12.7112 \quad -12.5433\ ]$ under $\beta = 0.1461$, $q = 0.4038$. For the same parameters $\beta$ and $q$, solving the LMI-based optimization problem has yielded the matrix

$$\widehat{Q}_P = \begin{bmatrix} 0.1160 & -0.1784 \\ -0.1784 & 2.4866 \end{bmatrix}$$

of the maximal invariant ellipse (in the sense of the trace criterion) lying in the domain of attraction of the initial nonlinear system.

Figure 1 demonstrates the calculated minimal invariant ellipse (firm line), the minimal invariant ellipse for the linearized system, i.e., with the controller designed for the linearized system (dashed line), and the invariant ellipse lying in the domain of attraction of the initial nonlinear system with the controller $u = \widehat{K}x$ (dashed-dot line).

Next, Fig. 2 shows the evolving invariant ellipses obtained by solving the optimization problems with the DLMI (Fig. 2a) and by solving the MCS (Fig. 2b) with the initial matrix

$$Q_0 = \begin{bmatrix} 0.1140 & -0.1389 \\ -0.1389 & 1.7029 \end{bmatrix}.$$

In the both cases, within $T \leq 10$ s the solutions converge to the minimal invariant ellipse with the matrix $\widehat{Q}$.

Using Theorem 5, we have constructed a state-feedback controller with time-dependent gain coefficients. To this end, on the interval [0.10 s] we have solved the DLMI with the initial matrix

$$Q_0 = \begin{bmatrix} 0.1260 & -0.1684 \\ -0.1684 & 2.4966 \end{bmatrix} \geq Q_P$$

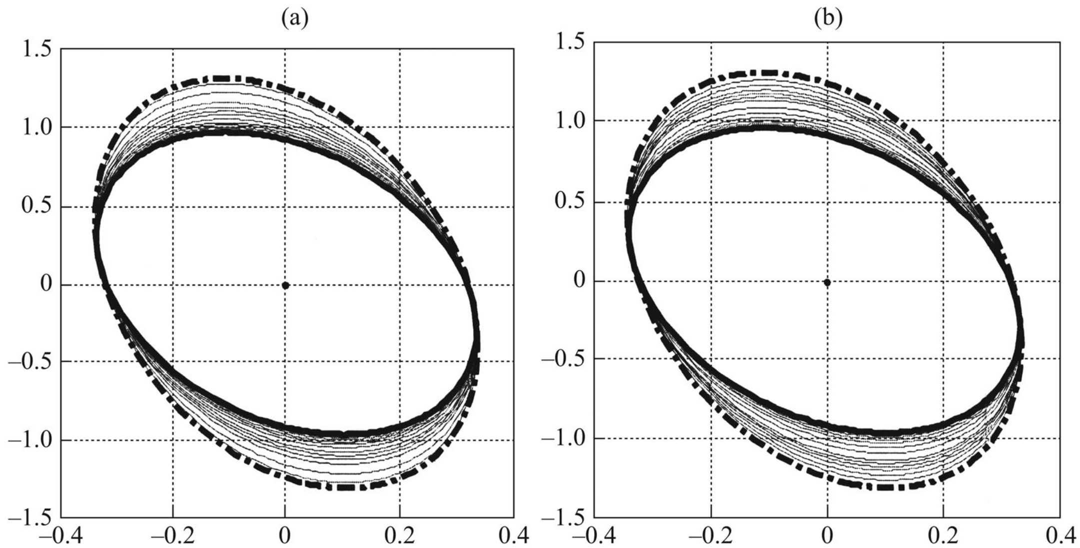


**Fig. 2.** Evolving invariant ellipses.

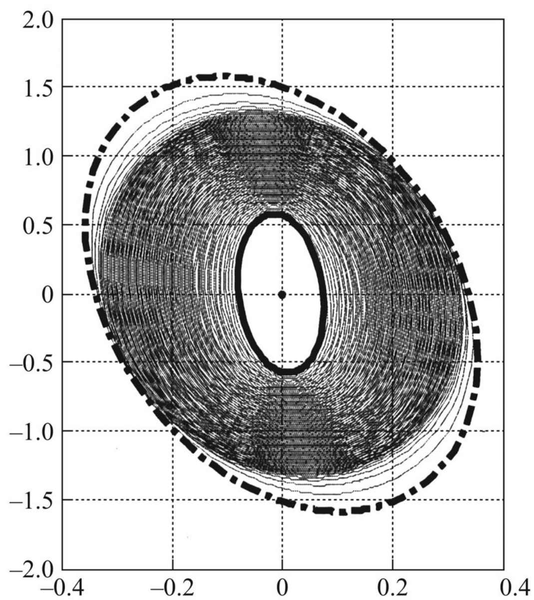


**Fig. 3.** Evolving invariant ellipse for system governed by controller with time-dependent gain coefficients.

and optimization of the output-bounding ellipsoid at each step. Figure 3 illustrates the ellipses representing the sections of the evolving invariant ellipse at the times $t_k$, $k = 1, \ldots, 100$, $t_1 = 0$, and $t_{k+1} = t_k + 0.1$. The initial ellipse ($k = 1$) is marked by dashed-dot line, whereas the terminal ellipse ($k = 100$) is indicated by firm line. It is associated with the matrix

$$Q_{100} = \begin{bmatrix} 0.0089 & -0.0076 \\ -0.0076 & 0.3276 \end{bmatrix} < \widehat{Q}.$$

Figure 4 demonstrates the graph of the parameter $\beta(t)$ (firm line), whose values decrease monotonically from 0.2298 to 0.0417, and the graph of the gain coefficients $K_1$ and $K_2$ of the controller (dashed and dashed-dot lines, respectively). If after 10 s we keep the controller with the constant

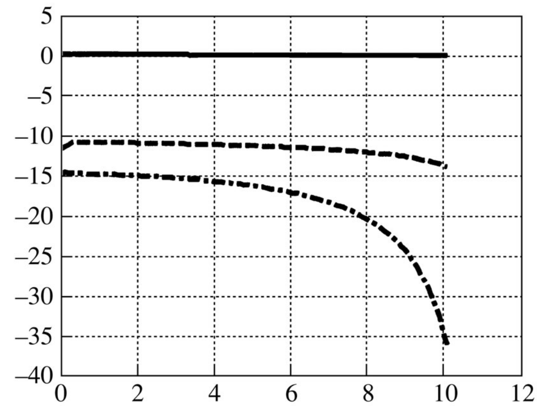


**Fig. 4.** Variation of parameter $\beta$ and gain coefficients in time.

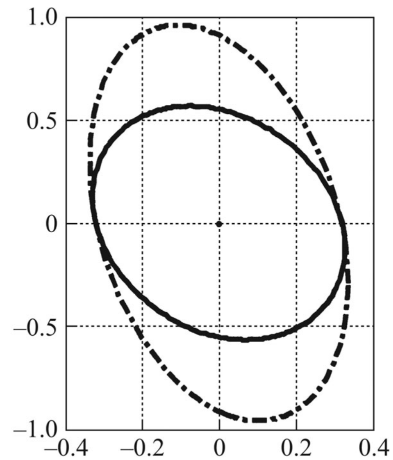


**Fig. 5.** Minimal invariant ellipses for pendulum.

gain coefficients $K = [-58.8643 \quad -14.6383]$ calculated at the last step, then the minimal invariant ellipse matrix for the nonlinear system with this controller is defined by

$$\widehat{Q}_1 = \begin{bmatrix} 0.0088 & -0.0028 \\ -0.0028 & 0.3029 \end{bmatrix} < \widehat{Q}.$$

Note that a small refinement via incorporating the matrix $C_f$ in formula (2) can be fruitful, both to improve the resulting estimates (e.g., in the form of the minimal bounding ellipsoids) and to reduce the dimensions of the constraints in the optimization problems. Particularly, for the nonlinear function $\varphi(x) = \sin x_1$ studied in our example, the constraint (3) is satisfied by choosing $C_f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $B_f = [0]$. By solving the optimization problem with the LMI, we have calculated the matrix

$$\widehat{Q}_2 = \begin{bmatrix} 0.1075 & -0.0407 \\ -0.0407 & 0.3220 \end{bmatrix} < \widehat{Q}$$

of the minimal invariant ellipse and the corresponding controller with the constant gain coefficients $\widehat{K} = [-5.4534 \quad -13.8586]$ under $\beta = 0.0464$ and $q = 0.4250$. Figure 5 shows the minimal invariant ellipses defined by the matrices $\widehat{Q}_2$ (firm line) and $\widehat{Q}$ (dashed-dot line). Therefore, by refining the nonlinearity estimate, we actually have improved the estimate of the limiting set for the initial system; this is important in high-accuracy stabilization problems. In addition, note that the dimension of the constraints matrix (12) has been decreased by 1.

## 7. CONCLUSIONS

This paper has further developed the method of matrix comparison systems and differential matrix inequalities with application to the problems of state estimation, analysis of stability and boundedness with respect to a given set of initial conditions and a given state- or output-bounding set, as well as to the problems of suppression of initial deviations and arbitrary bounded exogenous disturbances for a class of nonlinear nonautonomous systems. The problems of state estimation and boundedness with respect to given sets have been reduced to numerical solution of matrix comparison systems or differential matrix inequalities and verification of conditions expressed as linear matrix inequalities. Linear state-feedback controller design has been reduced to solution of differential linear matrix inequalities or semidefinite programming problems. The suggested approach extends to nonautonomous systems the method based on the ideology of invariant ellipsoids and the framework of linear matrix inequalities, with regard to nonlinear systems with bounded exogenous disturbances. Particularly, we have considered the suppression problem for initial deviations and exogenous disturbances using static (yet, time-dependent) linear state-feedback control minimizing the size of the invariant (or bounding) ellipsoids of a dynamic system. The presented example has indicated that, in a series of cases, application of the suggested approach improves the estimates of the state vector or output in comparison with the method of invariant ellipsoids [13].

The obtained results can be disseminated to discrete-time systems and can be used to solve other problems connected with state and performance estimation, as well as with the suppression of uncertain exogenous disturbances and parametric changes.

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*APPENDIX*

**Proof of Theorem 1.** Define the function $Q(t) = Q^{\mathrm{T}}(t) > 0$, $v(t,x) = x^{\mathrm{T}} Q^{-1}(t) x$, $Q(t_0) = Q_0$. For any $x_0 \in E(Q_0)$, we have $v(t_0, x_0) = x_0^{\mathrm{T}} Q_0^{-1} x_0 \le 1$. For the trajectories $x(t) = x(t, t_0, x_0)$ of the system (1) to stay in the ellipsoid $E(Q(t)) = \{x \in R^n : v(t,x) \le 1\}$, it suffices to require that

$$\dot{v}_{(1)}(t, x, w, \varphi) \le 0$$

holds for $v(t,x) = 1$ and for all admissible exogenous disturbances of the form (3) and all nonlinearities $\varphi(t,x)$ satisfying inequality (2). Here $\dot{v}_{(1)}$ denotes the derivative of the function $v(t,x)$ along the trajectories of the system (1). Calculate

$$\dot{v}_{(1)}(t,x) = x^{\mathrm{T}} \left\{ dQ^{-1}(t)/dt + A^{\mathrm{T}} Q^{-1} + Q^{-1} A \right\} x + w^{\mathrm{T}} D^{\mathrm{T}} Q^{-1} x$$
$$+ x^{\mathrm{T}} Q^{-1} D w + \varphi^{\mathrm{T}} \Phi^{\mathrm{T}} Q^{-1} x + x^{\mathrm{T}} Q^{-1} \Phi \varphi.$$

Next, using the obvious inequality

$$z^{\mathrm{T}} y + y^{\mathrm{T}} z \le \varepsilon z^{\mathrm{T}} z + \frac{1}{\varepsilon} y^{\mathrm{T}} y$$

that is true for any vectors $y, z \in R^n$ and any scalar or any continuous time-varying function $\varepsilon > 0$, obtain

$$\dot{v}(t,x) \le x^{\mathrm{T}} \left\{ dQ^{-1}(t)/dt + A^{\mathrm{T}}Q^{-1} + Q^{-1}A \right\} x + \frac{1}{\beta} x^{\mathrm{T}} Q^{-1} D D^{\mathrm{T}} Q^{-1} x + \beta w^{\mathrm{T}} w + \frac{1}{\gamma} x^{\mathrm{T}} Q^{-1} \Phi \Phi^{\mathrm{T}} Q^{-1} x + \gamma \varphi^{\mathrm{T}} \varphi,$$

where $\beta, \gamma > 0$ mean scalars or continuous time-varying functions. Taking into account the constraints imposed on the nonlinearities and the exogenous disturbances (see (2) and (3)) and the assumption $v(t,x) = 1$, we immediately have

$$\begin{aligned}
\dot{v}(t,x) \le x^{\mathrm{T}} &\left\{ dQ^{-1}(t)/dt + A^{\mathrm{T}}Q^{-1} + Q^{-1}A + \frac{1}{\beta} Q^{-1} D D^{\mathrm{T}} Q^{-1} + \frac{1}{\gamma} Q^{-1} \Phi \Phi^{\mathrm{T}} Q^{-1} \right\} x \\
&+ \gamma \mu_1 x^{\mathrm{T}} C_f^{\mathrm{T}} C_f x + \beta + \gamma \mu_0 + \alpha \left( x^{\mathrm{T}} Q^{-1} x - 1 \right) \\
&= x^{\mathrm{T}} \left\{ dQ^{-1}(t)/dt + A^{\mathrm{T}}Q^{-1} + Q^{-1}A + \frac{1}{\beta} Q^{-1} D D^{\mathrm{T}} Q^{-1} \right. \\
&\left. + \frac{1}{\gamma} Q^{-1} \Phi \Phi^{\mathrm{T}} Q^{-1} + \gamma \mu_1 C_f^{\mathrm{T}} C_f + \alpha Q^{-1} \right\} x + (\beta + \gamma \mu_0 - \alpha) \\
&\le x^{\mathrm{T}} \left\{ dQ^{-1}(t)/dt + A^{\mathrm{T}}Q^{-1} + Q^{-1}A + \frac{1}{\beta} Q^{-1} D D^{\mathrm{T}} Q^{-1} \right. \\
&\left. + \frac{1}{\gamma} Q^{-1} \Phi \Phi^{\mathrm{T}} Q^{-1} + \gamma \mu_1 C_f^{\mathrm{T}} C_f + \alpha Q^{-1} \right\} x \le 0,
\end{aligned}$$

provided that $\alpha > 0$ and $\beta + \gamma\mu_0 - \alpha \le 0$. The last inequality takes the form $0 < \beta \le \alpha - \gamma\mu_0$. Then it is possible to choose $0 < \beta = \beta_{\max} = \alpha - \gamma\mu_0$.

Find the matrix $Q(t)$ from the differential matrix equation

$$dQ^{-1}(t)/dt + A^{\mathrm{T}}Q^{-1} + Q^{-1}A + \frac{1}{\alpha - \gamma\mu_0} Q^{-1} D D^{\mathrm{T}} Q^{-1} + \frac{1}{\gamma} Q^{-1} \Phi \Phi^{\mathrm{T}} Q^{-1} + \gamma \mu_1 C_f^{\mathrm{T}} C_f + \alpha Q^{-1} = 0,$$

where $\gamma > 0$ and $\alpha > \gamma\mu_0$.

Perform pre- and post-multiplication of this equation by the matrix $Q(t) > 0$, introduce the notation $\beta = 1/\gamma$ and take advantage of the formula $Q(t) dQ^{-1}(t)/dt Q(t) = -dQ(t)/dt$ to obtain Eq. (6). This concludes the proof of Theorem 1.

Actually, the proof of Theorem 2 is similar to that of Theorem 1, except that the matrix $Q(t)$ is found not from Eq. (6) but from the differential matrix inequality

$$-dQ(t)/dt + AQ + QA^{\mathrm{T}} + \alpha Q + \frac{1}{\alpha - \frac{\mu_0}{\beta}} D D^{\mathrm{T}} + \beta \Phi \Phi^{\mathrm{T}} + \frac{\mu_1}{\beta} Q C_f^{\mathrm{T}} C_f Q \le 0, \tag{A.1}$$

where $\gamma > 0$, $\alpha > \gamma\mu_0$, and $t \in [t_0, T)$. By virtue of the Schur complement lemma, it follows from (A.1) that

$$\begin{bmatrix} -dQ(t)/dt + AQ + QA^{\mathrm{T}} + \alpha Q + \beta \Phi \Phi^{\mathrm{T}} + \frac{\mu_1}{\beta} Q C_f^{\mathrm{T}} C_f Q \le 0 & D \\ D^{\mathrm{T}} & \left( -\alpha + \frac{\mu_0}{\beta} \right) I \end{bmatrix} \le 0.$$

And finally, apply the Schur complement lemma twice more to get the DLMI (7).

**Proof of Lemma 1.** According to Lemma 10 from [16], the right-hand side of (6) is a quasi-monotonically nondecreasing matrix function with respect to the cone $G_+$. Therefore, (6) is the matrix comparison system of the initial system (1). Its partial solution $Q(t) = Q(t, t_0, Q_0) > 0$ (if any) defines a quadratic Lyapunov function $v(t,x) = x^{\mathrm{T}}Q^{-1}(t)x$ whose derivative along the trajectories of the system (1) is smaller or equal to 0, see the proof of Theorem 1. Hence, the ellipsoid $E(Q(t)) = \{x \in R^n : v(t,x) \leq 1\}$ evolving in time is invariant for the solutions of the system (1) with $x_0 \in E(Q_0)$ and gives an upper estimate for the attainability domain at time $t \in [t_0, T)$ for the solutions starting from the initial ellipsoid.

**Proof of Lemma 2.** Rewrite the comparison system (6) as

$$dQ(t)/dt = \left[A + \frac{\alpha}{2}I\right]Q + Q\left[A + \frac{\alpha}{2}I\right]^{\mathrm{T}} + \frac{1}{\alpha - \frac{\mu_0}{\beta}}DD^{\mathrm{T}} + \beta\Phi\Phi^{\mathrm{T}} + \frac{\mu_1}{\beta}QC_f^{\mathrm{T}}C_fQ \tag{A.2}$$

and consider the system

$$dS(t)/dt = \left[A + \frac{\alpha}{2}I\right]S + S\left[A + \frac{\alpha}{2}I\right]^{\mathrm{T}}. \tag{A.3}$$

Denote by $\Phi(t,\tau)$ the transition matrix of the system $dx/dt = [A(t) + \alpha/2I]x$. For $t$ and $\tau \in [t_0, T_1)$, the matrix $\Phi(t,\tau)$ satisfies the equations

$$\dot{\Phi}(t,\tau) = \frac{\partial}{\partial t}\Phi(t,\tau) = -\left(A(t) + \frac{\alpha}{2}I\right)^{\mathrm{T}}\Phi(t,\tau)$$

and

$$\dot{\Phi}^{\mathrm{T}}(t,\tau) = -\Phi^{\mathrm{T}}(t,\tau)\left(A(t) + \frac{\alpha}{2}I\right).$$

Let $P(t,\tau) = \Phi^{\mathrm{T}}(t,\tau)S(t)\Phi(t,\tau)$. It follows from $\det\Phi(t,\tau) \neq 0$ and $S(t_0) = Q_0 \geq 0$ (or $S(t_0) = Q_0 > 0$) that $P(t_0, \tau) \geq 0$ ($P(t_0, \tau) > 0$, respectively) for $\tau \in [t_0, T_1)$. Fix $\tau \in [t_0, T_1)$, then

$$\dot{P}(t,\tau) = \Phi^{\mathrm{T}}(t,\tau)\left\{\dot{S} - A(t)S - SA^{\mathrm{T}}(t) - \alpha S\right\}\Phi(t,\tau) = 0$$

if $S(t)$ is the solution of Eq. (A.3). This implies $0 \leq P(t_0,\tau) = P(t,\tau)$ ($0 < P(t_0,\tau) = P(t,\tau)$, respectively) for $\tau, t \in [t_0, T_1)$, and hence, given $\tau = t$, $0 \leq P(t,t) = S(t)$ ($0 < P(t,t) = S(t)$, respectively) for $t \in [t_0, T_1)$.

Now, let $Q(t)$ be the solution of Eq. (A.2) with $Q(t_0) = Q_0 \geq 0$ ($Q(t_0) = Q_0 > 0$, respectively). Since the matrix $\frac{1}{\alpha - \frac{\mu_0}{\beta}}DD^{\mathrm{T}} + \beta\Phi\Phi^{\mathrm{T}} + \frac{\mu_1}{\beta}QC_f^{\mathrm{T}}C_fQ$ is nonnegative definite for all $t \in [t_0, T_1)$, the above solution satisfies the differential inequality

$$dS/dt = A(t)S + SA^{\mathrm{T}}(t) + \alpha S \leq dQ/dt.$$

Then Theorem 4 (on differential matrix inequalities) from [16] dictates that $Q(t) \geq S(t) \geq 0$ ($Q(t) \geq S(t) > 0$, respectively) for all $t \in [t_0, T_1)$.

**Proof of Theorem 3.** Suppose that $Q_0 > 0$. By Lemma 2, on the interval $[t_0, T_1)$ there exists a unique positive definite solution $Q(t) = Q(t, t_0, Q_0)$ of the matrix comparison system (6) (or the DLMI (7)). According to Theorems 1 and 2, at each time this solution defines a nondegenerate ellipsoid $E(Q(t))$ being invariant for the solutions of the initial system (1) with the initial data $x(t_0) = x_0 \in E(Q_0)$. Therefore, estimate 1 holds.

Let estimate 1 be true. Then the solutions $x(t) = x(t, t_0, x_0)$ satisfy $0 \leq 1 - x^{\mathrm{T}}(t)Q^{-1}(t)x(t)$. By the Schur complement lemma, this inequality is equivalent to the following matrix inequality for all $t \in [t_0, T_1)$:

$$\begin{bmatrix} 1 & x^{\mathrm{T}}(t) \\ x(t) & Q(t) \end{bmatrix} \geq 0.$$

Hence, it appears that $Q(t) - x(t)x^{\mathrm{T}}(t) \geq 0$ for all $t \in [t_0, T_1)$. And so, estimate 2 holds.

Clearly, estimates 3–5 directly follow from estimate 2.

**Proof of Lemma 3.** Choose an arbitrary constant matrix $P \in H$. Then, given $V(t) \equiv P$ for all $t \in [t_0, T)$, we have $dV(t)/dt = 0 \geq F(t, V(t))$. According to the theorem on differential matrix inequalities, for any matrix $0 \leq S \leq P$ we obtain $P = V(t) \geq Q(t, t_0, S)$ for all $t \in [t_0, T)$.

**Proof of Theorem 4.** Due to the continuity and local Lipschitzness of the right-hand side of the MCS (6), for any matrix $P \in H_t$ there exists a unique solution $Y(t, P)$ on some interval $T_Y$. If $Y(t, P) \in H$ on a certain interval $T_1 \subset T_Y$, then $Y(t, P)$ is nonincreasing on $T_1$. Conversely, if $Y(t, P)$ is nonincreasing on $T_1$, then $F(t, Y(t, P)) = dY(t, P)/dt \leq 0$, i.e., $Y(t, P) \in H$ for all $t \in T_1$. By continuity of $F(Q)$ in $Q$, the set $H$ is closed. Therefore, $Y(t, P) \in H$ for all $t \in T_1$.

Now, suppose that there exists a time $\tau_1 > t_0$ such that $Y(\tau_1, P) \notin H$. Set $\tau_0 = \inf\{\tau \in (t_0, \tau_1] : Y(\tau, P) \notin H\}$. According to the aforesaid, on any interval $T_2 \subseteq [\tau_0, \tau_1]$ the solution $Y(t, P)$ is nonincreasing. Consequently, there exists $\tau' > \tau_0$ such that $Y(\tau') \nleq Y(\tau_0)$. This result contradicts the conclusion of Lemma 3 on the positive invariance of the set $\bar{S}(Y(\tau_0)) = \{S \in G_+ : S \leq Y(\tau_0) \in H\}$.

And finally, prove the second part of the theorem. Since $Y(t, P) \in H$ on the interval $T_Y$ and, by Lemma 2, is bounded below ($Y(t, P) \geq 0$), then the solution can be extended to any time interval. And so, the solution exists and is unique for all $t \in [t_0, +\infty)$. Thus, the solution $Y(t, P)$ represents a continuous matrix function that decreases for all $t \in [t_0, +\infty)$ and is bounded below. This guarantees the existence of $\lim_{t\to\infty} Q(t, t_0, Q_0) = Q^*$, where $Q^* \in H$ owing to the closedness of the set $H$.

**Proof of Theorem 5.** Let $Q(t) = Q(t, t_0, Q_0)$ be the solution of the matrix comparison system (6) (or the DLMI (7)) with the initial conditions $Q_0$. As $Q_0 \geq R$, we have $E(R) \subseteq E(Q_0)$, i.e., the ellipsoid $E(Q_0)$ contains the ellipsoid $E(R)$. Moreover, since $Q(t) \leq S(t)$, then $E(Q(t)) \subseteq E(S(t))$, i.e., the ellipsoid $E(Q(t))$ belongs to the ellipsoid $E(S(t))$. By Theorem 1 (by Theorem 2, accordingly), under $t \in [t_0, T)$ the ellipsoid $E(Q(t))$ contains all solutions of the initial system evolving from the given ellipsoid $E(Q_0)$. Hence, all conditions of Definition 3 are satisfied and boundedness with respect to the given sets $[E(R), E_w(I), E(S(t))]$ takes place.

**Proof of Theorem 6.** Substitute (11) into (9). Apply Theorem 2 to find an invariant ellipsoid $E(Q(t))$ with a matrix $Q(t)$ satisfying the DLMI (7). For the closed-loop system, this inequality takes the form

$$\begin{bmatrix} -dQ(t)/dt+(A+BK)Q+Q(A+BK)^{\mathrm{T}}+\alpha Q+\beta\Phi\Phi^{\mathrm{T}} & D & Q(C_f+B_fK)^{\mathrm{T}} & 0 \\ * & -\alpha I & 0 & I \\ * & * & -\dfrac{\beta}{\mu_1}I & 0 \\ * & * & * & -\dfrac{\beta}{\mu_0} \end{bmatrix} \leq 0 \qquad (A.4)$$

for all $t \in [t_0, T)$ and some $\beta > 0$, $\alpha \geq \mu_0/\beta$.

The variables $Q$ and $K$ enter the DLMI (A.4) nonlinearly. Introduce the matrix variable $Y = KQ$, $Y \in R^{n\times m}$. Then inequality (A.4) becomes linear with respect to $P$ and $Y$, see (12). In terms of the new variable $Y$, searching for the minimal ellipsoid (in the sense of the trace criterion) that bounds the output $z(t)$ is reduced to the problem

$$trace\left[CQC^{\mathrm{T}} + CY^{\mathrm{T}}B_z^{\mathrm{T}} + B_zYC^{\mathrm{T}} + B_zYQ^{-1}Y^{\mathrm{T}}B_z^{\mathrm{T}}\right] \to \min$$

subject to the constraint (12) for each $t \in [t_0, T)$.

According to Lemma 2 from [12], for each $t \in [t_0, T)$ the derived problem is equivalent to

$$trace\left[CQC^{\mathrm{T}} + CY^{\mathrm{T}}B_z^{\mathrm{T}} + B_zYC^{\mathrm{T}} + B_zZB_z^{\mathrm{T}}\right] \to \min$$

subject to the constraints (12) and (13), where $Z \in R^{m\times m}$ designates an auxiliary matrix variable.

Let the matrices $Q^*(t)$ and $Y^*(t)$ be the solution of the considered optimization problem with the DLMI (12) and the LMI (13) for $t \in [t_0, T)$. Then the matrix of the controller's gain coefficients is defined by $K^*(t) = Y^*(t)Q^{*-1}(t)$. If the matrix $Q^*(t)$ satisfies the additional constraints $Q_0 \geq R$ and $Q(t) \leq S(t)$ for all $t \in [t_0, T)$ where $R$ and $S(t)$ are given positive definite symmetric matrices, then by Theorem 4 the resulting controller ensures the boundedness of the closed-loop system with respect to the sets $[E(R), E_w(I), E(S(t))]$. This completes the proof of Theorem 6.

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