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Uniform Tauberian Theorem in Differential Games

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Abstract—This paper establishes the uniform Tauberian theorem for differential zero-sum games. Under rather mild conditions imposed on the dynamics and running cost, two parameterized families of games are considered, i.e., the ones with the payoff functions defined as the Cesaro mean and Abel mean of the running cost. The asymptotic behavior of value in these games is investigated as the game horizon tends to infinity and the discounting parameter tends to zero, respectively. It is demonstrated that the uniform convergence of value on an invariant subset of the phase space in one family implies the uniform convergence of value in the other family and that the limit values in the both families coincide. The dynamic programming principle acts as the cornerstone of proof.

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1. INTRODUCTION

In [29] Hardy established that a bounded sequence satisfies the condition

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i=\lim_{\lambda\downarrow 0}\sum_{i=1}^{\infty}(1-\lambda)^{i-1}a_i$$

if at least one of these limits exists. Such theorems yield good approximations for the sums of series using faster summation methods. Owing to Hardy, these theorems were termed "Tauberian" in honor of A. Tauber who proved a similar result for convergent series in 1897. For any bounded continuous scalar function g, an analogous Tauberian theorem [28] states the following: the limit of its Cesaro mean

$$\frac{1}{T}\int_0^T g(t)dt$$

as $T\to\infty$ and the limit of its Abel mean

$$\lambda\int_0^\infty e^{-\lambda t}g(t)dt$$

as $\lambda\to+0$ coincide if at least one of the limits exists.

Instead of the limits of such means, the limits of their optimal values in a given dynamics can be investigated. Let us clarify the aforesaid. Under known dynamics and running cost, it is possible to introduce two parameterized families of control problems (in the general case, games). In one family, the payoff function is the Cesaro mean of the running cost with the game horizon (the number of steps in the discrete setting) as a parameter. In the other family, the payoff function is the Abel mean of the running cost with the discounting factor as a parameter. For a given parameter value, each family of games yields its own value; for brevity, such values are further called the optimal means.

Apparently, the asymptotic behavior of optimal means was pioneered in [19] within the stochastic framework. Mertens and Neyman [34] proved the Tauberian theorem for a stochastic two-player game with a finite number of states and actions: the optimal Cesaro means and Abel means have a same limit value. Such limits were considered in other stochastic problems; e.g., see [21, 41] for a modern bibliography on the subject.

The existence of limits for optimal means was repeatedly investigated in control theory; at least, note the papers [1, 15] and the book [23]. In contrast to [19, 34], the initial state is known for sure in the deterministic case. Particularly, an optimal mean is a certain function of this state. As a result, while studying the convergence of optimal means for a certain family of games, one should specify an appropriate topology.

In the ergodic case (or even in the nonexpansive-like case), optimal means converge on a compactly open topology to constants and the limit functions are independent of the initial state, see [13, 16] for details. This fails in the general case, e.g., in [27, 36]. Such limits arise for solutions of Hamilton–Jacobi equations [7, 8, 33].

In control problems, the paper [14] was the first to show the equality of uniform limits for optimal means when at least one of the limits is a constant. The general statement of the Tauberian theorem for controlled systems (actually, for a dynamic controlled system in a very general setting) was proved in [35]. Oliu-Barton and Vigeral demonstrated that the uniform convergence of one optimal mean on a strongly invariant set of initial positions implies the uniform convergence of the other optimal mean and that the both limits coincide. The cited work also provides many references to Tauberian theorems in control problems.

There exist a few publications on optimal mean limits in differential games. In the first place, take notice of [12, 17, 22]; a good survey in this field of research can be found in [20, Section 3.4]. According to a remark in [35], for differential antagonistic games the Tauberian theorem was obtained only in the ergodic case. The present paper fills this gap.

2. STATEMENT OF DIFFERENTIAL GAME

Consider a conflict-controlled system

$$\dot{x}(t) = f(x(t), a(t), b(t)), \quad x(t) \in \mathbb{X}, \quad a(t) \in A, \quad b(t) \in B, \quad t \geq 0, \tag{2.1}$$

operating in a certain finite-dimensional Euclidean space $\mathbb{X}$. Here A and B are finite-dimensional compact sets.

Denote by $\mathcal{A}$ and $\mathcal{B}$ the sets of all possible Borel measurable selectors $\mathbb{R}_{>0} \ni t \mapsto a(t) \in A$ and $\mathbb{R}_{>0} \ni t \mapsto b(t) \in B$, respectively.

Let the functions $f : \mathbb{X} \times A \times B \to \mathbb{X}$ and $g : \mathbb{X} \times A \times B \to [0, 1]$ meet the following assumptions:

(C) f and g are continuous;

(L) f and g satisfy the Lipschitz condition in the phase variable, i.e., for some $L > 0$ we have

$$||f(x', a, b) - f(x'', a, b)||_2 \leq L||x' - x''||_2 \quad \forall x', x'' \in \mathbb{X}, \quad a \in A, \quad b \in B.$$

Due to the condition (L), the function f enjoys the sublinear growth condition. Hence, for any pair $a \in \mathcal{A}$, $b \in \mathcal{B}$ and each initial condition $\omega = x(0) \in \mathbb{X}$, the system (2.1) has a unique local solution that can be uniquely extended to $\mathbb{R}_{\geq 0}$; designate this solution by $y[\omega, a, b] \in C(\mathbb{R}_{\geq 0}, \mathbb{X})$. Moreover, the identity $y[\omega, a, b](0) = \omega$ holds. For all $\omega \in \Omega$ and all $a \in \mathcal{A}$, $b \in \mathcal{B}$, collect solutions $y[\omega, a, b]$ into the set $Y[\omega]$.

Suppose that some (not necessarily closed) set $\Omega \subset \mathbb{X}$ has the following property:

(Ω) Ω is strongly invariant with respect to the system (2.1), i.e., $x(t) \in \Omega$ holds for all $\omega \in \Omega$, $x \in Y[\omega]$, $t \in \mathbb{R}_{\geq 0}$.

Denote by $Y[\Omega]$ the set of all possible solutions $y[\omega, a, b]$ for all $a \in \mathcal{A}, b \in \mathcal{B}, \omega \in \Omega$.

Further exposition also needs the Isaacs condition (the saddle point condition in a small game, see [5, p. 56]):

(S) $\forall x, s \in \mathbb{X}\colon \max_{a\in A} \min_{b\in B} [\langle s, f(x,a,b)\rangle + g(x,a,b)] = \min_{b\in B} \max_{a\in A} [\langle s, f(x,a,b)\rangle + g(x,a,b)].$

Note that this condition remains in force as the function g is multiplied by an arbitrary positive expression independent of (x, a, b), e.g., by a positive time-varying function.

It was demonstrated in classical theory of differential games [2, 5, 6, 32, 39] that this condition guarantees game value existence. Although there are many different ways for defining a game and the strategy sets of each player, the above condition guarantees the equivalence of a considerable number of such formalizations in the sense of game values. Particularly, this allows choice of several settings, constructive strategy design and correct handling of errors and random disturbances [3–5]. An excellent survey covering an appreciable number of formalizations can be found in [39, Ch. III]. The present paper uses a formalization based on the notion of nonanticipating strategies (single-valued quasi-strategies). The underlying reasons include the following:

1) this formalization has the well-proven existence of ε-optimal universal strategy;

2) this formalization does not need a constructive (particularly, numerical) design method for such a strategy;

3) this formalization suits both finite and infinite horizons.

Let us define the notion of a nonanticipating strategy. In the context of dynamic games, it was first suggested in the paper [38] and further developed in [25, 37, 40]. A modification of this notion adopted below was borrowed from [18].

A rule $\alpha : \mathcal{B} \mapsto \mathcal{A}$ is said to be a nonanticipating strategy of player 1 if, for all $b, b' \in \mathcal{B}$, $t > 0$, the identity $b|_{]0,t]} = b'|_{]0,t]}$ implies the identity $\alpha(b')|_{]0,t]} = \alpha(b)|_{]0,t]}$. Denote by $\mathfrak{A}$ the set of all possible nonanticipating strategies of player 1. Similarly, introduce the set $\mathfrak{B}$ of the nonanticipating strategies of player 2 as all possible mappings $\beta : \mathcal{A} \mapsto \mathcal{B}$ with the following property: for all $a, a' \in \mathcal{A}$, $t > 0$, the identity $a|_{]0,t]} = a'|_{]0,t]}$ implies the identity $\beta(a')|_{]0,t]} = \beta(a)|_{]0,t]}$.

Each nonanticipating strategy $\alpha \in \mathfrak{A}$ associates each initial value $x(0) = \omega$ with a funnel $Y[\omega, \alpha]$ of all possible paths $y[\omega, \alpha(b), b]$ induced by this strategy (with respect to all $b \in \mathcal{B}$); naturally enough, $Y[\omega, \alpha] \subset Y[\omega]$. Moreover, for each nonanticipating strategy $\alpha \in \mathfrak{A}$ and each initial value $x(0) = \omega$, determine a funnel $Z(\omega, \alpha) \subset C(\mathbb{R}_{\geq 0}, \Omega) \times \mathcal{A} \times \mathcal{B}$ of all possible processes induced from ω by the nonanticipating strategy α, i.e., triplets of the form $(y[\omega, \alpha(b), b], \alpha(b), b)$ (with respect to all $b \in \mathcal{B}$). Also introduce the set $Z(\Omega) \subset C(\mathbb{R}_{\geq 0}, \Omega) \times \mathcal{A} \times \mathcal{B}$ of all possible processes, i.e., triplets of the form $(y[\omega, a, b], a, b)$ (with respect to all $\omega \in \Omega, a \in \mathcal{A}, b \in \mathcal{B}$).

Similarly, define $Z(\omega, \beta) \subset C(\mathbb{R}_{\geq 0}, \Omega) \times \mathcal{A} \times \mathcal{B}$ for $\omega \in \Omega, \beta \in \mathfrak{B}$.

Define a parameterized family of games with v_T, where T is a positive parameter. For each $T > 0$, choose the payoff function as the Cesaro mean of the function g on the horizon $[0, T]$:

$$v_T(x, a, b) \triangleq \frac{1}{T} \int_0^T g(x(t), a(t), b(t))\, dt \qquad \forall z \in (x, a, b) \in Z(\Omega).$$

Players 1 and 2 seek to maximize and minimize, respectively, the function v_T. Then, for any $\omega \in \Omega$, it is possible to define the numbers

$$\sup_{\alpha\in\mathfrak{A}} \inf_{(x,a,b)\in Z(\omega,\alpha)} v_T(x, a, b), \qquad \inf_{\beta\in\mathfrak{B}} \sup_{(x,a,b)\in Z(\omega,\beta)} v_T(x, a, b). \tag{2.2}$$

As it is demonstrated, e.g., in [6, 25, 31], the condition (S) guarantees that the value of the game exists for all $T > 0$. Particularly, the numbers from (2.2) coincide for each $\omega \in \Omega$. Therefore, for each $T > 0$, it is correct to define the value function (the optimal mean on the horizon $[0, T]$):

$$V_T(\omega) \stackrel{\triangle}{=} \sup_{\alpha\in\mathfrak{A}} \inf_{z\in Z(\omega,\alpha)} v_T(z) = \inf_{\beta\in\mathfrak{B}} \sup_{z\in Z(\omega,\beta)} v_T(z) \qquad \forall\omega\in\Omega.$$

Define a parameterized family of games with w_λ, where λ is a positive parameter. For each $\lambda > 0$, choose the payoff function as the Abel mean of the function g with the discounting factor λ:

$$w_\lambda(x, a, b) \stackrel{\triangle}{=} \lambda \int_0^\infty e^{-\lambda t} g(x(t), a(t), b(t))\, dt.$$

Players 1 and 2 seek to maximize and minimize, respectively, the function w_λ. Then, for any $\omega \in \Omega$, it is possible to define the numbers

$$\sup_{\alpha\in\mathfrak{A}} \inf_{(x,a,b)\in Z(\omega,\alpha)} w_\lambda(x, a, b), \qquad \inf_{\beta\in\mathfrak{B}} \sup_{(x,a,b)\in Z(\omega,\beta)} w_\lambda(x, a, b). \tag{2.3}$$

As it was demonstrated, e.g., in [18, Corollary VIII.2.2], the condition (S) guarantees that the value of the game exists for all $\lambda > 0$. Particularly, the numbers from (2.3) coincide for each $\omega \in \Omega$. Therefore, for each $\lambda > 0$, it is correct to define the value function (the optimal mean with the discounting factor λ):

$$W_\lambda(\omega) \stackrel{\triangle}{=} \sup_{\alpha\in\mathfrak{A}} \inf_{z\in Z(\omega,\alpha)} w_\lambda(z) = \inf_{\beta\in\mathfrak{B}} \sup_{z\in Z(\omega,\beta)} w_\lambda(z) \qquad \forall\omega\in\Omega.$$

Since g takes values from $[0, 1]$, the mappings $w_\lambda, W_\lambda, v_T, V_T$ also have the same property.

The following statement is the main result of the paper.

Theorem. *Assume that the conditions $(C), (L), (\Omega), (S)$ hold.*

The uniform convergence of $V_T(\omega)$ in $\omega \in \Omega$ to the limit

$$V_*(\omega) \stackrel{\triangle}{=} \lim_{T\to+\infty} V_T(\omega)$$

implies the uniform convergence of $W_\lambda(\omega)$ in $\omega \in \Omega$ to the limit

$$W_*(\omega) = \lim_{\lambda\to+0} W_\lambda(\omega);$$

moreover, these two limits coincide.

Conversely, the uniform convergence of $W_\lambda(\omega)$ in $\omega \in \Omega$ to the limit

$$W_*(\omega) = \lim_{\lambda\to+0} W_\lambda(\omega),$$

implies the uniform convergence of $V_T(\omega)$ in $\omega \in \Omega$ to the limit

$$V_*(\omega) \stackrel{\triangle}{=} \lim_{T\to+\infty} V_T(\omega);$$

moreover, these two limits coincide.

A preliminary background required for the proof of this theorem will be given in the next section. For the time being, note several corollaries.

Elimination of a player (e.g., by making either A or B a singleton) leads to the uniform Tauberian theorem for controlled systems. This theorem was established in [35] for a rather general dynamic system with one player. Oliu-Barton and Vigeral also demonstrated that the uniform convergence condition is essential even for controlled systems.

For theorem, the values V_T, W_λ have been defined via nonanticipating strategies (quasi-strategies). However and obviously, one can adopt any formalization equivalent in the sense of values; details were discussed in [39, Ch. III].

The conflict-controlled system (2.1) does not explicitly incorporate the parameter t. But the above theorem is naturally extended to systems of the form

$$\dot{x}(t) = f(t, x(t), a(t), b(t)), \quad x(t) \in \mathbb{X}, \quad a(t) \in A, \quad b(t) \in B, \quad t \geq 0$$

provided that the right-hand side enjoys Lipschitzness in t. In this case, it can be rewritten as

$$\dot{w}(t) = 1, \quad \dot{x}(t) = f(w(t), x(t), a(t), b(t)),$$
$$(w(t), x(t)) \in \mathbb{R} \times \mathbb{X}, \quad a(t) \in A, \quad b(t) \in B, \quad t \geq 0.$$

Moreover, the uniform convergence of V_T, W_λ is required on an invariant subset Ω of the set $\mathbb{R} \times \mathbb{X}$ instead of the set $\mathbb{X}$, as for the system (2.1).

Let us formulate another corollary of theorem. The value of the differential game can be described using Hamilton–Jacobi–Bellman equations. To this end, following [39, Section 11.1; 18, (VIII.1.16)], define the Hamiltonians

$$H(x, s) \stackrel{\triangle}{=} \max_{a \in A} \min_{v \in Q} \left[\langle s, f(x, a, b) \rangle + g(x, a, b) \right] \quad \forall x, s \in \mathbb{X};$$

$$\bar{H}(x, s) \stackrel{\triangle}{=} \min_{a \in A} \max_{b \in B} \left[-\langle s, f(x, a, b) \rangle - g(x, a, b) \right] \quad \forall x, s \in \mathbb{X}.$$

Now, the terminal problem

$$\frac{\partial u(t, x)}{\partial t} + H(x, D_x u(t, x)) = 0 \quad \forall t \leq 0, x \in \mathbb{X}, \tag{2.4}$$

$$u(0, x) \equiv 0 \quad \forall x \in \mathbb{X} \tag{2.5}$$

has a unique minimax (viscosity) solution $u \in C\left(\mathbb{R}_{\leq 0} \times \mathbb{X}\right)$, see [39]. Then, for all $(T, x) \in \mathbb{R}_{>0} \times \Omega$,

$$V_T(x) = \frac{u(-T, x)}{T}.$$

Similarly, as shown in [18], for all $\lambda > 0$ the Hamilton–Jacobi equation

$$\lambda \bar{u}(x) + \bar{H}(x, D_x \bar{u}(x)) = 0 \quad \forall x \in \mathbb{X} \tag{2.6}$$

has a unique viscosity solution $\bar{u}_\lambda$ in $BC(\mathbb{X})$ (in the class of bounded continuous functions); furthermore, for all $x \in \Omega$,

$$W_\lambda(x) = \lambda \bar{u}_\lambda(x).$$

Then theorem directly brings to the following.

Corollary. *Assume that the conditions* $(C), (L), (\Omega), (S)$ *hold.*

Let $u \in C\left(\mathbb{R}_{\leq 0} \times \mathbb{X}\right)$ *be the minimax (viscosity) solution of the problem* (2.4), (2.5). *For each* $\lambda > 0$, *let the function* $\bar{u}_\lambda \in BC(\mathbb{X})$ *be the viscosity solution of Eq.* (2.6).

Then the following conditions are equivalent:

(1) *there exists* $\lim_{\lambda \downarrow 0} \lambda \bar{u}_\lambda(x)$ *that is uniform in* $x \in \Omega$;

(2) *there exists* $\lim_{\lambda \downarrow 0} \lambda u(-1/\lambda, x)$ *that is uniform in* $x \in \Omega$;

(3) *each of* $\lim_{\lambda \downarrow 0} \lambda \bar{u}_\lambda(x)$, $\lim_{\lambda \downarrow 0} \lambda u(-1/\lambda, x)$ *exists for any* $x \in \Omega$ *and is uniform in* $x \in \Omega$; *moreover, these limits coincide.*

Generally speaking, under given $\lambda > 0$, it is easier to find the viscosity solution of Eq. (2.6) than the viscosity solution of the problem (2.4), (2.5) (applicable numerical methods can be found in [39, 18]). The presented corollary strengthens the corresponding results in [12, 17, 22].

3. PRELIMINARY BACKGROUND

First of all, a wider class of strategies is necessary due to some circumstances. In the games considered, for all $\varepsilon > 0$ and any initial position ω of each player, the above formalization guarantees the existence of strategies ($\alpha^\omega \in \mathfrak{A}$ and $\beta^\omega \in \mathfrak{B}$, respectively) that are ε-optimal for this position. However, we will need more than that, i.e., in this game, for each player, the existence of strategies that are ε-optimal for all initial positions $\omega \in \Omega$. This can be achieved by passing from nonanticipating strategies to nonanticipating operators. Such operators were introduced, e.g., in [6, p. 180; 10]; see the papers [11, 24] for examples of their usage.

A nonanticipating operator of player 1 (player 2) is an arbitrary mapping from Ω to $\mathfrak{A}$ (to $\mathfrak{B}$, respectively); denote them by $\mathbb{A}$ (by $\mathbb{B}$, respectively). By analogy with $Z(\omega,\alpha)$ and $Z(\omega,\beta)$ introduced earlier, for each nonanticipating operator ζ in $\mathbb{A}$ or $\mathbb{B}$, define

$$Z(Gr\,\zeta) \triangleq \bigcup_{\omega\in\Omega} Z(\omega,\zeta(\omega)).$$

Note that, for any bounded payoff function c (particularly, v_T or w_λ) and for all $\omega\in\Omega$, we have

$$\sup_{\alpha\in\mathfrak{A}} \inf_{z\in Z(\omega,\alpha)} c(z) = \sup_{\zeta\in\mathbb{A}} \inf_{z\in Z(\omega,\zeta(\omega))} c(z), \qquad (3.1)$$

$$\inf_{\beta\in\mathfrak{B}} \sup_{z\in Z(\omega,\beta)} c(z) = \inf_{\xi\in\mathbb{B}} \sup_{z\in Z(\omega,\xi(\omega))} c(z).$$

Then, for all $\lambda, T > 0$, it is possible to write

$$V_T(\omega) = \sup_{\zeta\in\mathbb{A}} \inf_{z\in Z(\omega,\zeta(\omega))} v_T(z) = \inf_{\xi\in\mathbb{B}} \sup_{z\in Z(\omega,\xi(\omega))} w_\lambda(z) \quad \forall\omega\in\Omega, \qquad (3.2)$$

$$W_\lambda(\omega) = \sup_{\zeta\in\mathbb{A}} \inf_{z\in Z(\omega,\zeta(\omega))} w_\lambda(z) = \inf_{\xi\in\mathbb{B}} \sup_{z\in Z(\omega,\xi(\omega))} w_\lambda(z) \quad \forall\omega\in\Omega. \qquad (3.3)$$

Furthermore, suppose that, in a certain game with a bounded payoff function (e.g., v_T or w_λ), for any $\varepsilon > 0$ and any initial position ω, each player has an ε-optimal strategy for this point ($\alpha^\omega\in\mathfrak{A}$ and $\beta^\omega\in\mathfrak{B}$, respectively). Determine nonanticipating operators $\zeta\in\mathbb{A}$ and $\xi\in\mathbb{B}$ according to the following rule: $\zeta(\omega)=\alpha^\omega$, $\xi(\omega)=\beta^\omega$. Now, they are ε-optimal nonanticipating operators for the players.

Note 3.1. The games with a bounded payoff function for each player have ε-optimal nonanticipating operators on the whole set Ω for any accuracy parameter $\varepsilon > 0$.

It is also needed to introduce operations for nonanticipating operators. First, for each time $\tau > 0$ and a pair of functions $a', a''\in\mathcal{A}$, define their concatenation, i.e., a function $a'\diamond_\tau a''\in\mathcal{A}$, by the rule

$$(a'\diamond_\tau a'')(t) \triangleq \begin{cases} a'(t), & 0\le t\le\tau \\ a''(t-\tau), & t>\tau. \end{cases}$$

Note that $a'\diamond_\tau a''\in\mathcal{A}$; moreover, for all $\tau>0$, each element $a\in\mathcal{A}$ can be expressed as $a = a'\diamond_\tau a''$ for some $a', a''\in\mathcal{A}$. The concatenation $b'\diamond_\tau b''\in\mathcal{B}$ for all $b', b''\in\mathcal{B}$ is defined by analogy.

Let us introduce the operation $\diamond_\tau$ for nonanticipating operators. Fix some $\tau>0$, $\zeta',\zeta''\in\mathbb{A}$. It suffices to define $\zeta'\diamond_\tau\zeta''$ at each point $\omega\in\Omega$. Fix $\omega\in\Omega$ and first define a mapping $\eta=\eta_{\zeta',\zeta'',\omega}: \mathcal{B}\times\mathcal{B}\to\mathcal{A}$ by the rule

$$\eta_{\zeta',\zeta'',\omega}(b',b'') \triangleq \zeta''\big(y\left[\omega,\zeta'(\omega)(b'),b'\right](\tau)\big)(b'') \quad \forall b',b''\in\mathcal{B}.$$

Actually, it is independent of $b'|_{]\tau,\infty[}$; particularly, for all $b''\in\mathcal{B}$, we have

$$\eta_{\zeta',\zeta'',\omega}(b'\diamond_\tau b'',b'') = \zeta''\big(y\left[\omega,\zeta'(\omega)(b'),b'\right](\tau)\big)(b''). \qquad (3.4)$$

Since the image of ζ'' belongs to $\mathfrak{A}$, for all $b'\diamond_\tau b'', \bar b'\diamond_\tau \bar b''\in\mathcal{B}, \delta>0$, we obtain

$$\begin{gathered}\left(b'|_{[0,\tau]} = \bar b'|_{[0,\tau]},\, b''|_{]0,\delta]} = \bar b''|_{]0,\delta]}\right) \\ \Rightarrow \left(\eta_{\zeta',\zeta'',\omega}(b'\diamond_\tau b'',b'')|_{[0,\delta]} = \eta_{\zeta',\zeta'',\omega}(\bar b'\diamond_\tau \bar b'',\bar b'')|_{[0,\delta]}\right).\end{gathered} \qquad (3.5)$$

For any $\omega\in\Omega$, determine the value of $(\zeta'\diamond_\tau\zeta'')(\omega)$; this is a mapping from $\mathcal{B}$ to $\mathcal{A}$ defined by the rule

$$(\zeta'\diamond_\tau\zeta'')(\omega)(b) \triangleq \zeta'(\omega)(b)\diamond_\tau \eta_{\zeta',\zeta'',\omega}(b,b'') \quad \forall b = b'\diamond_\tau b''\in\mathcal{B}. \qquad (3.6)$$

Due to (3.4) and the nonanticipativity of $\zeta'(\omega)$, it is correct to write

$$(\zeta' \diamond_\tau \zeta'')(\omega)(b) = \zeta'(\omega)(b') \diamond_\tau \eta_{\zeta',\zeta'',\omega}(b', b'') \quad \forall b = b' \diamond_\tau b'' \in \mathcal{B}.$$

Remember that the mapping $(\zeta' \diamond_\tau \zeta'')(\omega)$ becomes nonanticipating if, for any $t > \tau$ and any $b, \bar{b} \in \mathcal{B}$, the identity $b|_{[0,t]} = \bar{b}|_{[0,t]}$ implies the identity $\alpha(b)|_{]\tau,t]} = \alpha(\bar{b})|_{]\tau,t]}$. This is the case by virtue of (3.5). Therefore, $(\zeta' \diamond_\tau \zeta'')(\omega)$ forms a nonanticipating strategy. Owing to arbitrary choice of $\omega \in \Omega$, the operator $\zeta' \diamond_\tau \zeta''$ is nonanticipating, i.e., an element from $\mathbb{A}$. Consequently, the operation $\diamond_\tau$ is well-defined on the set $\mathbb{A}$.

Now, for all $\tau > 0$, $x', x'' \in C(\mathbb{R}_{\geq 0}, \Omega)$ such that $x'(\tau) = x''(0)$, introduce the function $x' \diamond_\tau x''$ by the rule

$$(x' \diamond_\tau x'')(t) \triangleq \begin{cases} x'(t), & t < \tau \\ x''(t - \tau), & t \geq \tau. \end{cases}$$

For all $z' \triangleq (x', a', b')$, $z'' \triangleq (x'', a'', b'')$ having the property $x'(\tau) = x''(\tau)$ (and only for such z', z''), define their concatenation

$$z' \diamond_\tau z'' \triangleq (x' \diamond_\tau x'', a' \diamond_\tau a'', b' \diamond_\tau b'').$$

Moreover, for all $\omega \in \Omega$, $a', a'' \in \mathcal{A}$, $b', b'' \in \mathcal{B}$, $x', x'' \in Y[\Omega]$, we have

$$\Big(x'|_{[0,\tau]} = y\left[\omega, a', b'\right]|_{[0,\tau]},\ x'' = y[x'(\tau), a'', b'']\Big)$$
$$\Leftrightarrow \Big(x' \diamond_\tau x'' = y[\omega, a' \diamond_\tau a'', b' \diamond_\tau b'']\Big).$$

Then, for all $\alpha', \alpha'' \in \mathfrak{A}$ and $a' = \alpha'(b')$, $a'' = \alpha'(b'')$, we obtain

$$\Big(x'|_{[0,\tau]} = y[\omega, \alpha'(b'), b']|_{[0,\tau]}, x'' = y\big[x'(\tau), \alpha''(b''), b''\big]\Big)$$
$$\Leftrightarrow \Big(x' \diamond_\tau x'' = y\big[\omega, \alpha'(b') \diamond_\tau \alpha''(b''), b' \diamond_\tau b''\big]\Big).$$

For all $\zeta', \zeta'' \in \mathbb{A}$, substitution $\alpha' = \zeta'(\omega)$, $\alpha'' = \zeta''(x'(\tau))$ yields

$$\Big(x'\big|_{[0,\tau]} = y\big[\omega, \zeta'(\omega)(b'), b'\big]\big|_{[0,\tau]},\ x'' = y\big[x'(\tau), \zeta''(x'(\tau))(b''), b''\big]\Big)$$
$$\Leftrightarrow \Big(x' \diamond_\tau x'' = y\big[\omega, \zeta'(\omega)(b') \diamond_\tau \zeta''(x'(\tau))(b''), b' \diamond_\tau b''\big]\Big).$$

It appears from (3.6) and the nonanticipativity of $\zeta'(\omega)$ that

$$(\zeta' \diamond_\tau \zeta'')(\omega)(b' \diamond_\tau b'') = \zeta'(\omega)(b') \diamond_\tau \eta_{\zeta',\zeta'',\omega}(b' \diamond_\tau b'', b'').$$

On the other hand, the expression (3.4) gives $\zeta''(x'(\tau))(b'') = \eta_{\zeta',\zeta'',\omega}(b' \diamond_\tau b'', b'')$. Hence,

$$\Big(x'|_{[0,\tau]} = y[\omega, \zeta'(\omega)(b'), b']|_{[0,\tau]},\ x'' = y[x'(\tau), \zeta''(x'(\tau))(b''), b'']\Big)$$
$$\Leftrightarrow \Big(x' \diamond_\tau x'' = y[\omega, (\zeta' \diamond_\tau \zeta'')(\omega), b' \diamond_\tau b'']\Big).$$

Actually, the following result has been established.

Note 3.2. For all $\omega \in \Omega$, $\zeta', \zeta'' \in \mathbb{A}, \tau > 0$, we have

$$Z\left(\omega, (\zeta' \diamond_\tau \zeta'')(\omega)\right) = \Big\{z' \diamond_\tau z'' \,|\, z' \triangleq (x', \zeta'(\omega)(b'), b') \in Z\big(\omega, \zeta'(\omega)\big),$$
$$z'' \triangleq (x'', \zeta''(x'(\tau))(b''), b'') \in Z\big(x'(\tau), \zeta'(x'(\tau))\big)\Big\},$$
$$Z(Gr\, \zeta' \diamond_\tau \zeta'') = \Big\{z' \diamond_\tau z'' \,|\, z' = (x', a', b') \in Z(Gr\, \zeta'),$$
$$z'' \in Z\big(x'(\tau), \zeta''(x'(\tau))\big)\Big\}.$$

Note that all considerations above can be repeated for player 2 (although, this is not required for further exposition).

In the sequel, for definiteness, for all $\tau', \tau'' > 0$, $\zeta, \zeta', \zeta'' \in \mathbb{A}$ set $\zeta \diamond_{\tau'} \zeta' \diamond_{\tau''} \zeta'' \stackrel{\triangle}{=} (\zeta \diamond_{\tau'} \zeta') \diamond_{\tau''} \zeta''$.

Consider an arbitrary process $z \in Z(\Omega)$ and an arbitrary number $T \geq 0$. Define a process z_T by the rule $z_T(t) \stackrel{\triangle}{=} z(t+T)$. Naturally, if $z = (x' = y[\omega, a, b], a, b)$, then $z_T = (x'_T, a_T, b_T)$ satisfies $x'_T = y[x'(\tau), a_T, b_T]$. And so, z_T is also a process.

Such a compact form seems convenient while arguing that, for all $T > 0$, $\lambda > 0$, $0 < T' < T$, $z \in Z(\Omega)$, we have

$$\begin{aligned} w_\lambda(z) - e^{-\lambda T} w_\lambda(z_T) &= \lambda \int_0^\infty e^{-\lambda t} g(z(t))\,dt - \lambda \int_0^\infty e^{-\lambda(t+T)} g(z(t+T))\,dt \\ &= \lambda \int_0^\infty e^{-\lambda t} g(z(t))\,dt - \lambda \int_T^\infty e^{-\lambda t} g(z(t))\,dt \\ &= \lambda \int_0^T e^{-\lambda t} g(z(t))\,dt; \end{aligned} \tag{3.7}$$

$$\begin{aligned} v_T(z) - \frac{T'}{T} v_{T'}(z_{T-T'}) &= \frac{1}{T}\int_0^T g(z(t))\,dt - \frac{T'}{TT'}\int_0^{T'} g(z_{T-T'}(t+T))\,dt \\ &= \frac{1}{T}\int_0^T g(z(t))\,dt - \frac{1}{T}\int_{T-T'}^{T} g(z(t))\,dt \\ &= \frac{1}{T}\int_0^{T-T'} g(z(t))\,dt. \end{aligned} \tag{3.8}$$

The constructed games obey the dynamic programming principle in a formulation given below.

Note 3.3. For any $\omega \in \Omega$, $\lambda > 0, T > 0$, we have

$$\begin{aligned} W_\lambda(\omega) &= \sup_{\zeta \in \mathbb{A}} \inf_{z \in Z(\omega, \zeta(\omega))} \Big[w_\lambda(z) - e^{-\lambda T} w_\lambda(z_T) + e^{-\lambda T} W_\lambda(z(T)) \Big] \\ &\stackrel{(3.7)}{=} \sup_{\zeta \in \mathbb{A}} \inf_{z \in Z(\omega, \zeta(\omega))} \left[\int_0^T \lambda e^{-\lambda t} g(z(t))\,dt + e^{-\lambda T} W_\lambda(z(T)) \right]. \end{aligned} \tag{3.9}$$

Note 3.4. For all $\omega \in \Omega, T' > 0, T > T'$, we have

$$\begin{aligned} V_T(\omega) &= \sup_{\zeta \in \mathbb{A}} \inf_{z \in Z(\omega, \zeta(\omega))} \left[v_T(z) - \frac{T'}{T} v_{T'}(z_{T-T'}) + \frac{T'}{T} V_{T'}(z(T-T')) \right] \\ &\stackrel{(3.8)}{=} \sup_{\zeta \in \mathbb{A}} \inf_{z \in Z(\omega, \zeta(\omega))} \left[\frac{1}{T} \int_0^{T-T'} g(z(t))\,dt + \frac{T'}{T} V_{T'}(z(T-T')) \right]. \end{aligned} \tag{3.10}$$

In terms of nonanticipating strategies, the dynamic programming principle for finite-horizon games (particularly, for the payoff function v_T) is well-known, see [26]. Such a principle for the payoff function w_λ follows from [18, Theorem VIII.1.9]. By (3.1), if the principle holds for nonanticipating strategies, it also remains in force for nonanticipating operators. Therefore, Notes 3.3 and 3.4 have been shown.

4. ESTIMATE $V_* \leq W$

This section demonstrates that, if the functions V_T uniformly converge to the limit V_*, then player 1 can guarantee the value of the payoff function w_λ not less than V_* for any given accuracy for sufficiently small λ. Prior to formulating a rigorous statement, let us describe the general idea of proof.

For given $\lambda > 0$, choose an interval $[0, \tau_k[$ and its partition into subintervals $[\tau_i, \tau_{i-1}[$. Using them, then construct a piecewise constant function h approximating the function $e^{-\lambda t}$ with a required accuracy. Subsequently, the payoff function w_λ will be replaced by a new functional c. This functional depends only on the integral of h along the path until the time τ_k and on the position at this time. It suffices to find a nonanticipating operator guaranteeing that this functional is not smaller than V_* with a required accuracy.

Such an operator is obtained by dynamic programming, i.e., the concatenation of almost-optimal operators in specially designed problems on smaller horizons. To this end, decompose the functional c into the sum (4.10). The last row in this sum has the form of the bracketed expression from (3.10). Now, Note 3.4 allows choosing on the interval $[\tau_{k-1}, \tau_k[$ an nonanticipating operator such that its processes in the last row of (4.10) can be estimated by a function of the position at τ_{k-1}. Now, for this interval, by the uniform convergence of V_T, this estimate (see (4.11)) obeys Note 3.4. Repetition of this procedure $k-1$ times yields the desired estimate.

Proposition 4.1. *Suppose the uniform convergence of v_T in $\omega \in \Omega$ to the limit*

$$V_*(\omega) \stackrel{\triangle}{=} \lim_{T\to+\infty} \sup_{\zeta\in\mathbb{A}} \inf_{z\in Z(\omega,\zeta(\omega))} v_T(z).$$

Then,

$$V_*(\omega) \leq \liminf_{\lambda\to+0} \sup_{\zeta\in\mathbb{A}} \inf_{z\in Z(\omega,\zeta(\omega))} w_\lambda(z).$$

Moreover, for all $\varepsilon > 0$ there exists $\bar\lambda > 0$ such that, if $0 < \lambda \leq \bar\lambda$, then, for some nonanticipating operator $\zeta \in \mathbb{A}$, all $z \in Z(Gr\,\zeta)$ satisfy $w_\lambda(z) > V_*(\omega) - \varepsilon$.

Proof. *Step 1. Choice of constants and auxiliary estimates.* Fix a certain integer

$$k > 2. \tag{4.1}$$

On the one hand, by condition, there is $T_0 > k$ such that

$$\big|V_{T'}(\omega) - V_{T''}(\omega)\big| < \frac{1}{k^2}, \quad T', T'' > T_0/2 \quad \forall \omega \in \Omega. \tag{4.2}$$

On the other hand, it suffices to show that, under $\lambda < 2/T_0$, some nonanticipating operator $\zeta = \zeta^\lambda \in \mathbb{A}$ satisfies

$$w_\lambda(z) \geq V_*(z(0)) - \frac{2 + 3\ln k}{k} \quad \forall z \in Z(Gr\,\zeta^\lambda). \tag{4.3}$$

Fix arbitrary $\lambda < 2/T_0$ and set

$$p = \frac{1}{1 - \frac{\ln k}{k}}, \quad \delta = \frac{\ln p}{\lambda}, \quad T = \frac{\delta p}{p-1}, \quad \tau_i = i\delta \quad \forall i \in \overline{0,k}.$$

Then the inequalities $1 < \frac{p \ln p}{p-1} < p \stackrel{(4.1)}{<} 1 + \frac{2\ln k}{k} \stackrel{(4.1)}{<} 2$ give

$$1 < \lambda T = \frac{\lambda\delta p}{p-1} = \frac{p\ln p}{p-1} < p, \quad T^{-1} < \lambda < pT^{-1} < 2T^{-1}, \tag{4.4}$$

$$p^{-k} = \left(1 - \frac{\ln k}{k}\right)^k < \frac{1}{k}, \quad \frac{T-\delta}{T} = p^{-1}, \quad e^{\lambda\delta} = p < 1 + \frac{2\ln k}{k}. \tag{4.5}$$

By using (4.4) and $\lambda < 2/T_0$, it follows that $p^{-1}T > \lambda^{-1} > T_0/2$; therefore, the condition (4.2) holds for $T' = p^{-1}T$, $T'' = T$.

Step 2. Construction of a functional c approximating the payoff function w_λ. Now, define on $[0, \tau_k[$ a piecewise constant function h by the rule

$$h(t) = e^{-\lambda\tau_i} \stackrel{(4.5)}{=} p^{-i} \quad \forall t \in [\tau_i, \tau_{i+1}[, \quad i \in \overline{0, k-1}.$$

Then, $h(t)e^{\lambda t} \geq 1$; and, for all $t \in [0, \tau_k]$,

$$1 \leq h(t)e^{\lambda t} = h(\tau_i)e^{\lambda t} \leq h(\tau_i)e^{\lambda \tau_{i+1}} = p^{-i}p^{i+1} = p. \tag{4.6}$$

Consider the game with the payoff function

$$c(z) \triangleq \frac{1}{T}\int\limits_0^{\tau_k} h(t)g(z(t))dt + p^{-k}V_{p^{-1}T}(z(\tau_k)).$$

Note that, for all $x \in \mathbb{X}$, $a \in A$, $b \in \mathbb{B}$, we have $0 \leq g(x, a, b) \leq 1$, whence it appears that

$$0 \leq p^{-k}V_{p^{-1}T}(z(\tau_k)) \leq p^{-k} \overset{(4.5)}{\leq} \frac{1}{k} \overset{(4.1)}{\leq} \frac{\ln k}{k}.$$

Inequality (4.4) implies that $1 < \lambda T < p$ and, for any process z,

$$\begin{aligned} w_\lambda(z) &\overset{(4.6)}{\leq} \lambda \int\limits_0^{\tau_k} h(t)g(z(t))dt + e^{-\lambda\tau_k}w_\lambda(z_{\tau_k}) \\ &\leq \lambda T\, c(z) + \frac{\ln k}{k} \overset{(4.4)}{\leq} p\, c(z) + \frac{\ln k}{k}; \\ c(z) &\overset{(4.6)}{\leq} \frac{1}{T}\int\limits_0^{\tau_k} pe^{-\lambda t}g(z(t))dt + \frac{\ln k}{k} \\ &\leq \frac{p}{T\lambda}w_\lambda(z) + \frac{\ln k}{k} \overset{(4.4)}{\leq} p\, w_\lambda(z) + \frac{\ln k}{k}. \end{aligned}$$

Particularly, any process $z \in z(\Omega)$ satisfies

$$|w_\lambda(z) - c(z)| < \frac{\ln k}{k} + p - 1 \overset{(4.5)}{\leq} \frac{3\ln k}{k}. \tag{4.7}$$

Step 3. Design of nonanticipating operator ζ^*. According to Note 3.1, there exists a nonanticipating operator $\zeta \in \mathbb{A}$ that is $1/k^2$-optimal on the whole set Ω in the game with the payoff function

$$v_T(z) - p^{-1}v_{p^{-1}T}(z_\delta) + p^{-1}V_{p^{-1}T}(z(\delta)).$$

By Note 3.4, the value of this game coincides with V_T. Hence, for all $z \in Z(Gr\,\zeta)$, we have

$$v_T(z) - p^{-1}v_{p^{-1}T}(z_\delta) + p^{-1}V_{p^{-1}T}(z(\delta)) \geq V_T(z(0)) - \frac{1}{k^2}. \tag{4.8}$$

Moreover, the value of this game is independent of the process after δ and inequality (4.8) takes place for any concatenation of the form $z\diamond_\delta z'$ if $z \in Z(Gr\,\zeta)$. Then (4.8) holds for all $z \in Z(\omega, \zeta\diamond_\tau\zeta')$ with any operator $\zeta' \in \mathbb{A}$.

Let us demonstrate that the operator

$$\zeta^* \triangleq \zeta \diamond_{\tau_1} \zeta \diamond_{\tau_2} \cdots \diamond_{\tau_{i-1}} \zeta \diamond_{\tau_i} \cdots \diamond_{\tau_{k-1}} \zeta \in \mathbb{A}$$

guarantees that all its processes $z \in Z(Gr\,\zeta^*)$ satisfy

$$c(z) > V_T(z(0)) - \frac{2}{k}. \tag{4.9}$$

In combination with (4.7), this would give the required estimate (4.3).

Step 4. Iterative procedure. Recall that $\tau_{i+1} = \tau_i + \delta$, $h(\tau_i) = p^{-i}$, $p^{-1}T = T - \delta$. Now,

$$\frac{1}{T}\int\limits_{\tau_i}^{\tau_{i+1}} h(t)g(z(t))dt = \frac{1}{T}\int\limits_0^{\delta} h(\tau_i)g(z(t+\tau_i))dt$$
$$= \frac{h(\tau_i)}{T}\int\limits_0^{\delta} g(z(t+\tau_i))dt$$
$$\overset{(3.8)}{=} p^{-i}\Big[v_T(z_{\tau_i}) - p^{-1}v_{p^{-1}T}(z_{\tau_i+\delta})\Big]$$
$$= p^{-i}v_T(z_{\tau_i}) - p^{-i-1}v_{p^{-1}T}(z_{\tau_{i+1}}).$$

Then

$$\begin{aligned} c(z) &= v_T(z) - p^{-1}v_{p^{-1}T}(z_{\tau_1}) \\ &+ p^{-1}v_T(z_{\tau_1}) - p^{-2}v_{p^{-1}T}(z_{\tau_2}) + \ldots \\ &+ p^{-i}v_T(z_{\tau_i}) - p^{-i-1}v_{p^{-1}T}(z_{\tau_{i+1}}) + \ldots \\ &+ p^{-k+2}v_T(z_{\tau_{k-2}}) - p^{-k+1}v_{p^{-1}T}(z_{\tau_{k-1}}) \\ &+ p^{-k+1}v_T(z_{\tau_{k-1}}) - p^{-k}v_{p^{-1}T}(z_{\tau_k}) + p^{-k}V_{p^{-1}T}(z(\tau_k)). \end{aligned} \tag{4.10}$$

Note that $z_{\tau_{k-1}} \in Z(Gr\,\zeta)$ for any $z \in Z(Gr\,\zeta^*)$. Hence, $z_{\tau_{k-1}}$ satisfies inequality (4.8), i.e.,

$$v_T(z_{\tau_{k-1}}) - p^{-1}v_{p^{-1}T}(z_{\tau_k}) + p^{-1}V_{p^{-1}T}(z(\tau_k)) \geq V_T(z(\tau_{k-1})) - 1/k^2.$$

Due to (4.2), the value of the last row in (4.10) is not smaller than $p^{-k+1}V_{p^{-1}T}(z(\tau_{k-1})) - 2p^{-k+1}/k^2$. Thus and so, for any its process $z \in Z(Gr\,\zeta^*)$, the operator ζ^* guarantees that

$$\begin{aligned} c(z) &\geq v_T(z) - p^{-1}v_{p^{-1}T}(z_{\tau_1}) \\ &+ p^{-1}v_T(z_{\tau_1}) - p^{-2}v_{p^{-1}T}(z_{\tau_2}) + \ldots \\ &+ p^{-i}v_T(z_{\tau_i}) - p^{-i-1}v_{p^{-1}T}(z_{\tau_{i+1}}) + \ldots \\ &+ p^{-k+2}v_T(z_{\tau_{k-2}}) - p^{-k+1}v_{p^{-1}T}(z_{\tau_{k-1}}) + p^{-k+1}V_{p^{-1}T}(z(\tau_{k-1})) - \frac{2}{k^2}. \end{aligned} \tag{4.11}$$

Note that $z_{\tau_{k-2}} \in Z(Gr\,\zeta \diamond_\delta \zeta)$; hence, $z_{\tau_{k-2}}$ satisfies inequality (4.8), i.e.,

$$v_T(z_{\tau_{k-2}}) - p^{-1}v_{p^{-1}T}(z_{\tau_{k-1}}) + p^{-1}V_{p^{-1}T}(z(\tau_{k-1})) \geq V_T(z(\tau_{k-2})) - 1/k^2.$$

Taking into account (4.2), for $z \in Z(Gr\,\zeta^*)$, it follows that

$$\begin{aligned} c(z) &\geq v_T(z) - p^{-1}v_{p^{-1}T}(z_{\tau_1}) \\ &+ p^{-1}v_T(z_{\tau_1}) - p^{-2}v_{p^{-1}T}(z_{\tau_2}) + \ldots \\ &+ p^{-i}v_T(z_{\tau_i}) - p^{-i-1}v_{p^{-1}T}(z_{\tau_{i+1}}) + \ldots \\ &+ p^{-k+3}v_T(z_{\tau_{k-3}}) - p^{-k+2}v_{p^{-1}T}(z_{\tau_{k-2}}) + p^{-k-2}V_{p^{-1}T}(z(\tau_{k-2})) - \frac{4}{k^2}. \end{aligned}$$

Similarly, for all $l \in \overline{1, k-3}$, by $z_{\tau_l} \in Z(Gr\,\zeta \diamond_\delta \zeta')$, for some operator ζ', z_{τ_l} satisfies inequality (4.8). Hence, the condition (4.2) implies that

$$v_T(z_{\tau_l}) - p^{-1}v_{p^{-1}T}(z_{\tau_{l+1}}) + p^{-1}V_{p^{-1}T}(z(\tau_{l+1})) \geq V_T(z(\tau_l)) - 2/k^2.$$

Now, for all $z \in Z(Gr\,\zeta^*)$, for each $l \in \overline{1, k-3}$, we have

$$\begin{aligned} c(z) &\geq v_T(z) - p^{-1}v_{p^{-1}T}(z_{\tau_1}) \\ &+ p^{-1}v_T(z_{\tau_1}) - p^{-2}v_{p^{-1}T}(z_{\tau_2}) + \ldots \\ &+ p^{-l}v_T(z_{\tau_l}) - p^{-l-1}v_{p^{-1}T}(z_{\tau_{l+1}}) + p^{-l-1}V_{p^{-1}T}(z(\tau_{l+1})) - \frac{2(k-l)}{k^2}. \end{aligned}$$

Particularly, for τ_1, we obtain

$$c(z) \geq v_T(z) - p^{-1}v_{p^{-1}T}(z_{\tau_1}) + p^{-1}V_{p^{-1}T}(z(\tau_1)) - \frac{2(k-1)}{k^2}.$$

Now, the choice of the operator ζ^* guarantees that, for $z \in Z(Gr\,\zeta^*)$,

$$c(z) \geq V_T(z(0)) - \frac{2}{k},$$

i.e., the condition (4.9) holds. This completes the proof. $\square$

5. ESTIMATE $W_* \leq V$

Show that, if the functions W_λ uniformly converge to W_*, then player 1 can guarantee that the value of the payoff function v_T is not less than W_* for any given accuracy for sufficiently large T. Prior to formulating a rigorous statement, let us describe the general idea of proof.

For given $T > 0$, choose $\lambda > 0$ and partition the interval $[0, T[$ into some k subintervals $[\tau_i, \tau_{i+1}[$ whose lengths form a geometric progression. Using them, construct a close to 1 function h in the form of the product of the piecewise constant function and the function $e^{-\lambda t}$. Subsequently, the payoff function v_T is replaced by a new functional c. This functional depends only on the integral of h along the path until the time τ_k and on the position at this time. It suffices to find a nonanticipating operator guaranteeing that this functional is not smaller than W_* with a required accuracy.

Such an operator is obtained by dynamic programming, i.e., the concatenation of almost-optimal operators in specially designed problems on smaller horizons. To this end, decompose the functional c into the sum (5.10). The last row in this sum has the form of the bracketed expression from (3.9). Now, for the interval $[\tau_{k-1}, \tau_k[$, Note 3.3 allows choosing a nonanticipating operator such that its processes in the last row of (5.10) can be estimated by a function of the position at τ_{k-1}. Now, for this interval, by the uniform convergence of W_λ, this estimate (see (5.11)) obeys Note 3.3. Repetition of this procedure $k-1$ times yields the desired estimate.

Proposition 5.1. *Suppose the uniform convergence of w_λ in $\omega \in \Omega$ to the limit*

$$W_*(\omega) \overset{\triangle}{=} \lim_{\lambda \to +0} \sup_{\zeta \in \mathbb{A}} \inf_{z \in Z(\omega, \zeta(\omega))} w_\lambda(z).$$

Then,

$$W_*(\omega) \leq \liminf_{T \to +\infty} \sup_{\zeta \in \mathbb{A}} \inf_{z \in Z(\omega, \zeta(\omega))} v_T(z).$$

Moreover, for any $\varepsilon > 0$ there exists $\bar{T} > 0$ such that, if $T > \bar{T}$, then, for some nonanticipating operator $\zeta \in \mathbb{A}$, all $z \in Z(Gr\,\zeta)$ satisfy $v_T(z) > W_*(z(0)) - \varepsilon$.

Proof. *Step 1. Choice of constants and auxiliary estimates.* Consider an arbitrary integer $k > 1$. Then there exists a number $M > 1$ such that $k = M \ln M$. By condition, it is possible to choose such $T_k > 1$ that, for all $\lambda', \lambda'' < \frac{M}{T_k}$, we have

$$\left|W_{\lambda'}(\omega) - W_{\lambda''}(\omega)\right| < \frac{1}{k^2} \quad \forall \omega \in \Omega. \tag{5.1}$$

Now, it suffices to argue the following. For each $T > T_k$, any realization $z \in Z(Gr\,\zeta^T)$ of some nonanticipating operator $\zeta^T \in \mathbb{A}$ meets the condition

$$v_T(z) > W_*(z(0)) - \frac{2}{M} - \frac{2}{k}. \tag{5.2}$$

Fix arbitrary $T > T_k$ and set

$$p = e^{1/M}, \quad t_0 = \frac{T\left(1 - e^{-1/M}\right)}{1 - \frac{1}{M}}, \quad \lambda = \frac{1}{M t_0}, \quad \tau_0 = 0,$$
$$t_i = t_0 p^{-i}, \quad \tau_i = \tau_{i-1} + t_i \quad \forall i \in \overline{1, k}.$$

Note that $t_0 > 0$ due to $M > 1$. Observe that the sequence of t_i forms a monotonically decreasing geometric progression with partial sums defined by τ_i, moreover,

$$p^k = e^{\ln M} = M, \quad \tau_k = t_0 \frac{1 - p^{-k}}{1 - p} = \frac{T\left(1 - e^{-1/M}\right)}{1 - \frac{1}{M}} \times \frac{1 - \frac{1}{M}}{1 - e^{-1/M}} = T.$$

The inequality $1 - \frac{1}{M} < e^{-1/M} < 1 - \frac{1}{M} + \frac{1}{2M^2}$ and its corollary $1 - \frac{1}{2M} < M(1 - e^{-1/M}) < 1$ imply

$$1 - \frac{1}{M} < \lambda T = \frac{T}{Mt_0} = \frac{1 - \dfrac{1}{M}}{M\left(1 - e^{-1/M}\right)} < 1, \tag{5.3}$$

$$p^{-k} = e^{-\ln M} = \frac{1}{M}, \quad \lambda p^k = \lambda M = \frac{1}{t_0} \overset{(5.3)}{\leq} \frac{M}{T}. \tag{5.4}$$

By (5.4), all pairs of λp^i, instead of λ', λ'', satisfy the condition (5.1).

Step 2. Construction of a functional c close to the payoff function v_T. Now, define on $]0, \tau_k] =]0, T]$ a scalar function s by the rule

$$s(t) = e^{-\lambda p^i (t - \tau_{i-1})}$$

for all $t \in]\tau_{i-1}, \tau_i]$.

Note that, on each such interval, $s(t) \leq 1$,

$$s(t) \geq s(\tau_i) = e^{-\lambda p^i (\tau_{i+1} - \tau_i)} = e^{-\lambda p^i t_i} = e^{-\lambda t_0} = p^{-1} > 1 - \frac{1}{M}, \tag{5.5}$$

and, by virtue of (5.3), it appears that

$$1 \geq \lambda T s(t) \geq \left(1 - \frac{1}{M}\right)^2 \geq 1 - \frac{2}{M} \quad \forall t \in [0, T]. \tag{5.6}$$

Consider the game with the payoff function

$$c(z) \triangleq \lambda \int_0^T s(t) g(z(t)) dt + p^{-k} W_{\lambda p^{k-1}}(z(T)).$$

Note that any process $z \in Z(\Omega)$ satisfies

$$v_T(z) = \frac{1}{T} \int_0^T g(z(t)) dt \overset{(5.6)}{\geq} \int_0^T \lambda s(t) g(z(t)) dt \overset{(5.4)}{\geq} c(z) - \frac{1}{M};$$

$$v_T(z) - \frac{2}{M} \leq \left(1 - \frac{2}{M}\right) \frac{1}{T} \int_0^T g(z(t)) dt \overset{(5.6)}{\geq} \int_0^T \lambda s(t) g(z(t)) dt \leq c(z).$$

Particularly, for any process $z \in Z(\Omega)$, we obtain

$$|v_T(z) - c(z)| < \frac{2}{M}. \tag{5.7}$$

Step 3. Design of nonanticipating operator ζ^*. For each $i \in \overline{0, k-1}$, there exists a nonanticipating operator $\zeta_i \in \mathbb{A}$ that is $1/k^2$-optimal in the game with the payoff function

$$w_{\lambda p^i}(z) - p^{-1} w_{\lambda p^i}(z_{t_{i+1}}) + p^{-1} W_{\lambda p^i}(z(t_{i+1})).$$

Since $p = e^{\lambda t_0} = e^{\lambda p^i t_{i+1}}$ and the value of this game coincides with $W_{\lambda p^i}$, for all $z \in Z(Gr\,\zeta_i)$, we have

$$w_{\lambda p^i}(z) - p^{-1} w_{\lambda p^i}(z_{t_{i+1}}) + p^{-1} W_{\lambda p^i}(z(t_{i+1})) \geq W_{\lambda p^i}(z(0)) - \frac{1}{k^2}. \tag{5.8}$$

Moreover, the value of this game is independent of the process after the time t_{i+1}, and hence inequality (5.8) takes place for all processes of the form $z \diamond_{t_{i+1}} z'$ if $z \in Z(Gr\zeta)$. Then, for each operator $\zeta' \in \mathbb{A}$, (5.8) holds for all $z \in Z(Gr\zeta_i \diamond_{t_{i+1}} \zeta')$.

Let us demonstrate that the operator

$$\zeta^* \stackrel{\triangle}{=} \zeta_0 \diamond_{\tau_1} \zeta_1 \diamond_{\tau_2} \cdots \diamond_{\tau_{i-1}} \zeta_{i-1} \diamond_{\tau_i} \cdots \diamond_{\tau_{k-1}} \zeta_{k-1} \in \mathbb{A}$$

guarantees for all its processes $z \in Z(Gr\,\zeta^*)$ that

$$c(z) > W_*(z(0)) - \frac{2}{k}. \tag{5.9}$$

In combination with (5.7), this would give the required estimate (5.2).

Step 4. Iterative procedure. Note that $e^{-\lambda p^i(\tau_{i+1}-\tau_i)} = p^{-1}$ for any $i \in \overline{0, k-1}$ due to (5.5). And so,

$$\lambda \int\limits_{\tau_i}^{\tau_{i+1}} s(t) g(z_t) dt = \lambda \int\limits_{\tau_i}^{\tau_{i+1}} e^{-\lambda p^i (t-\tau_i)} g(z(t))\, dt$$
$$\stackrel{(3.7)}{=} p^{-i} w_{\lambda p^i}(z_{\tau_i}) - p^{-i-1} w_{\lambda p^i}(z_{\tau_{i+1}}).$$

Summation over all intervals $[\tau_i, \tau_{i+1}]$ yields

$$\begin{aligned} c(z) &= w_\lambda(z) - p^{-1} w_\lambda(z_{\tau_1}) \\ &+ p^{-1} w_{\lambda p}(z_{\tau_1}) - p^{-2} w_{\lambda p}(z_{\tau_2}) + \ldots \\ &+ p^{-i} w_{\lambda p^i}(z_{\tau_i}) - p^{-i-1} w_{\lambda p^i}(z_{\tau_{i+1}}) + \ldots \\ &+ p^{-k+1} w_{\lambda p^{k-1}}(z_{T-t_{k-1}}) - p^{-k} w_{\lambda p^{k-1}}(z_T) + p^{-k} W_{\lambda p^{k-1}}(z(T)). \end{aligned} \tag{5.10}$$

Observe that $z_{\tau_k} \in Z(Gr\,\zeta_{k-1})$; hence, z_{τ_k} satisfies inequality (5.8), i.e.,

$$w_{\lambda p^{k-1}}(z_{\tau_{k-1}}) - p^{-1} w_{\lambda p^{k-1}}(z_{\tau_k}) + p^{-1} W_{\lambda p^{k-1}}(z(\tau_k)) \;\geq\; W_{\lambda p^{k-1}}(z(\tau_{k-1})) \;-\; \frac{1}{k^2}.$$

Then the row (5.10) is not smaller than $p^{-k+1} W_{\lambda p^{k-1}}(z(\tau_{k-1})) - p^{-k+1}/k^2$; by virtue of (5.1), it is not less than

$$p^{-k+1} W_{\lambda p^{k-2}}(z(\tau_{k-1})) - \frac{2}{k^2}.$$

Now, for any process $z \in Z(Gr\,\zeta^*)$, we have

$$\begin{aligned} c(z) &\geq w_\lambda(z) - p^{-1} w_\lambda(z_{\tau_1}) \\ &+ p^{-1} w_{\lambda p}(z_{\tau_1}) - p^{-2} w_{\lambda p}(z_{\tau_2}) + \ldots \\ &+ p^{-i} w_{\lambda p^i}(z_{\tau_i}) - p^{-i-1} w_{\lambda p^i}(z_{\tau_{i+1}}) + \ldots \\ &+ p^{-k+2} w_{\lambda p^{k-2}}(z_{t_{k-2}}) - p^{-k+1} w_{\lambda p^{k-2}}(z_{t_{k-1}}) + p^{-k+1} W_{\lambda p^{k-2}}(z(\tau_{k-1})) - \frac{2}{k^2}. \end{aligned} \tag{5.11}$$

Note that $z_{\tau_{k-2}} \in Z(Gr\,\zeta_{k-2} \diamond_{t_{k-1}} \zeta_{k-1})$, and it meets the condition (5.8), i.e.,

$$w_{\lambda p^{k-2}}(z_{\tau_{k-2}}) - p^{-1} w_{\lambda p^{k-2}}(z_{\tau_{k-1}}) + p^{-1} W_{\lambda p^{k-2}}(z(\tau_k)) \;\geq\; W_{\lambda p^{k-2}}(z(\tau_{k-2})) \;-\; \frac{1}{k^2}.$$

Next, the last row in (5.11) is not smaller than $p^{-k+2} W_{\lambda p^{k-2}}(\tau_{k-2}) - p^{-k+2}/k^2 - 2/k^2$, i.e., not less than $p^{-k+2} W_{\lambda p^{k-3}}(\tau_{k-2}) - 4/k^2$. Therefore, for any process $z \in Z(Gr\,\zeta^*)$, we obtain

$$\begin{aligned} c(z) &\geq w_\lambda(z) - p^{-1} w_\lambda(z_{\tau_1}) \\ &+ p^{-1} w_{\lambda p}(z_{\tau_1}) - p^{-2} w_{\lambda p}(z_{\tau_2}) + \ldots \\ &+ p^{-i} w_{\lambda p^i}(z_{\tau_i}) - p^{-i-1} w_{\lambda p^i}(z_{\tau_{i+1}}) + \ldots \\ &+ p^{-k+3} w_{\lambda p^{k-3}}(z_{t_{k-3}}) - p^{-k+2} w_{\lambda p^{k-3}}(z_{t_{k-2}}) + p^{-k-2} W_{\lambda p^{k-3}}(z(\tau_{k-2})) - \frac{4}{k^2}. \end{aligned}$$

Similarly, for all $l \in \overline{i, k-3}$ and some $\zeta' \in \mathbb{A}$, we have $z_{\tau_l} \in Z(Gr\,\zeta_l \diamond_{t_{l+1}} \zeta')$. And so, z_{τ_l} meets (5.8), and (5.1) yields

$$c(z) \geq w_\lambda(z) - p^{-1} w_\lambda(z_{\tau_1})$$
$$+ p^{-1} w_{\lambda p}(z_{\tau_1}) - p^{-2} w_{\lambda p}(z_{\tau_2}) + \dots$$
$$+ p^{-l} w_{\lambda p^l}(z_{\tau_l}) - p^{-l-1} w_{\lambda p^l}(z_{\tau_{l+1}}) + p^{-l-1} W_{\lambda p^{l-1}}(z(\tau_{l+1})) - \frac{2(k-l)}{k^2}.$$

Particularly, for the time τ_1 and any $z \in Z(Gr\,\zeta^*)$, we have

$$c(z) \geq w_\lambda(z) - p^{-1} w_\lambda(z_{\tau_1}) + p^{-1} W_*(z(\tau_1)) - \frac{2(k-1)}{k^2}.$$

It appears from (5.8) that

$$c(z) \geq W_*(z(0)) - \frac{2}{k},$$

i.e., inequality (5.9) holds for all $T > T_k$. □

6. PROOF OF THE MAIN THEOREM

Introduce the function $g^- = 1 - g$ and the corresponding functions v_T^-, w_λ^-. Also, define the sets $\mathbb{B}^- \overset{\triangle}{=} \mathbb{A}$, $\mathbb{A}^- \overset{\triangle}{=} \mathbb{B}$ and operations over them. Clearly, it follows from $g + g^- \equiv 1$ that, for all $T, \lambda > 0,\ \omega \in \Omega$, we have

$$v_T(z) + v_T^-(z) \equiv 1, \quad w_\lambda(z) + w_\lambda^-(z) \equiv 1,$$
$$\sup_{\zeta^- \in \mathbb{A}^-} \inf_{z \in Z(\omega, \zeta^-(\omega))} v_T^-(z) \equiv 1 - \inf_{\xi \in \mathbb{B}} \sup_{z \in Z(\omega, \xi(\omega))} v_T(z),$$
$$\sup_{\zeta^- \in \mathbb{A}^-} \inf_{z \in Z(\omega, \zeta^-(\omega))} w_\lambda^-(z) \equiv 1 - \inf_{\xi \in \mathbb{B}} \sup_{z \in Z(\omega, \xi(\omega))} w_\lambda(z).$$

Direct application of Propositions 4.1, 5.1 to the game with the running cost $1 - g^-$ brings to the following results.

Proposition 6.1. *Assume the uniform convergence of v_T in $\omega \in \Omega$ to the limit*

$$V_*(\omega) \overset{\triangle}{=} \lim_{T \to +\infty} \inf_{\xi \in \mathbb{B}} \sup_{z \in Z(\omega, \xi(\omega))} v_T(z).$$

Then $V_*(\omega) \geq \limsup_{\lambda \to +0} \inf_{\xi \in \mathbb{B}} \sup_{z \in Z(\omega, \xi(\omega))} w_\lambda(z)$.

Proposition 6.2. *Assume the uniform convergence of w_λ in $\omega \in \Omega$ to the limit*

$$W_*(\omega) \overset{\triangle}{=} \lim_{\lambda \to +0} \inf_{\xi \in \mathbb{B}} \sup_{z \in Z(\omega, \xi(\omega))} w_\lambda(z).$$

Then $W_*(\omega) \geq \limsup_{T \to +\infty} \inf_{\xi \in \mathbb{B}} \sup_{z \in Z(\omega, \xi(\omega))} v_T(z)$.

The one part of the proof is immediate from Propositions 4.1, 6.1, whereas the other follows from Propositions 5.1, 6.2.

Observe that the result of this theorem was announced in [9, 30].

REFERENCES

1. Gaitsgory, V.G., On the Use of the Averaging Method in Control Problems, *Differ. Equat.*, 1986, no. 11, pp. 1290–1298.

2. Krasovskii, N.N., *Upravlenie dinamicheskoi sistemoi: zadacha o minimume garantirovannogo rezul'tata* (Control of the Dynamic System: The Problem of Minimum of Guaranteed Result), Moscow: Nauka, 1985.

3. Krasovskii, N.N. and Kotel'nikova, A.N., Unification of Differential Games, Generalized Solutions of the Hamilton–Jacobi Equations, and a Stochastic Guide, *Differ. Equat.*, 2009, vol. 45, no. 11, pp. 1653–1668.

4. Krasovskii, N.N. and Kotel'nikova, A.N., Stochastic Guide for a Time-Delay Object in a Positional Differential Game, *Tr. Inst. Mat. Mekh. UrO RAN*, 2011, vol. 17, no. 2, pp. 97–104.

5. Krasovskii, N.N. and Subbotin, A.I., *Pozitsionnye differentsial'nye igry* (Positional Differential Games), Moscow: Nauka, 1974.

6. Subbotin, A.I. and Chentsov, A.G., *Optimizatsiya garantii v zadachakh upravleniya* (Optimization of the Guarantee in Control Problems), Moscow: Nauka, 1981.

7. Subbotina, N.N., Singular Approximations of Minimax and Viscosity Solutions to Hamilton–Jacobi Equations, *Tr. Inst. Mat. Mekh. UrO RAN*, 2000, vol. 6, no. 1, pp. 190–208.

8. Subbotina, N.N., The Method of Characteristics for Hamilton–Jacobi Equation and Its Applications in Dynamical Optimization, *Modern Math. Appl.*, 2004, vol. 20, pp. 2955–3091.

9. Khlopin, D.V., Tauberian Theorems for Conflict-Controlled Systems, in *Abstr. Int. Conf. "Systems Dynamics and Control Processes" Dedicated to the 90th Anniversary of N.N. Krasovskii*, Yekaterinburg, September 15–22, 2014, pp. 204–206.

10. Chentsov, A.G., On an Alternative in a Class of Nonanticipating Operators for a Differential Approach-Evasion Game, *Differ. Equat.*, 1981, vol. 16, no. 10, pp. 1167–1171.

11. Chentsov, A.G., Relationship between Different Variants of the Programmed Iterations Method: The Positional Variant, *Kibernetika*, 2002, no. 3, pp. 130–149.

12. Alvarez, O. and Bardi, M., Ergodic Problems in Differential Games, in *Advances in Dynamic Game Theory*, Boston: Birkhäuser, 2007, pp. 131–152.

13. Alvarez, O. and Bardi, M., Ergodicity, Stabilization, and Singular Perturbations for Bellman–Isaacs Equation, *Mem. Am. Math. Soc.*, 2010, vol. 960, pp. 1–90.

14. Arisawa, M., Ergodic Problem for the Hamilton–Jacobi–Bellman Equations, *Ann. Inst. Henri Poincare*, 1998, vol. 15, no. 1, pp. 1–24.

15. Artstein, Z. and Bright, I., Periodic Optimization Suffices for Infinite Horizon Planar Optimal Control, *SIAM J. Control Optim.*, 2010, vol. 48, no. 8, pp. 4963–4986.

16. Artstein, Z. and Gaitsgory, V., The Value Function of Singularly Perturbed Control Systems, *Appl. Math. Optim.*, 2000, vol. 41, no. 3, pp. 425–445.

17. Bardi, M., On Differential Games with Long-Time-Average Cost, in *Advances in Dynamic Games and Their Applications*, Boston: Birkhäuser, 2009, pp. 3–18.

18. Bardi, M. and Capuzzo-Dolcetta, I., *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*, Boston: Birkhäuser, 1997.

19. Blackwell, D., Discrete Dynamic Programming, *Ann. Math. Statist.*, 1962, vol. 33, no. 2, pp. 719–726.

20. Buckdahn, R., Cardaliaguet, P., and Quincampoix, M., Some Recent Aspects of Differential Game Theory, *Dyn. Games Appl.*, 2011, vol. 1, no. 1, pp. 74–114.

21. Buckdahn, R., Goreac, D., and Quincampoix, M., Existence of Asymptotic Values for Nonexpansive Stochastic Control Systems, *Appl. Math. Optim.*, 2014, vol. 70, no. 1, pp. 1–28.

22. Cardaliaguet, P., Ergodicity of Hamilton–Jacobi Equations with a non Coercive non Convex Hamiltonian in $\mathbb{R}^2/\mathbb{Z}^2$, *Ann. l'Inst. Henri Poincare, Non Linear Anal.*, 1998, vol. 27, no. 3, pp. 837–856.

23. Carlson, D.A., Haurie, A.B., and Leizarowitz, A., *Optimal Control on Infinite Time Horizon*, Berlin: Springer, 1991.

24. Chentsov, A.G. and Morina, S.I., Extensions and Relaxations, in *Mathematics and Its Applications*, Dordrecht: Kluwer, 2002, vol. 542.

25. Elliot, R.J. and Kalton, N., The Existence of Value for Differential Games, in *Memoir of the American Mathematical Society*, Providence: AMS, 1972, vol. 126.

26. Evans, L. and Souganidis, P.E., Differential Games and Representation Formulas for Solutions of Hamilton–Jacobi–Isaacs Equations, *Indiana Univ. Math. J.*, 1984, vol. 33, pp. 773–797.

27. Gaitsgory, V. and Quincampoix, M., On Sets of Occupational Measures Generated by a Deterministic Control System on an Infinite Time Horizon, *Nonlin. Anal. Theory, Meth. Appl.*, 2011, vol. 88, pp. 27–41.

28. Hardy, G.H., *Divergent Series*, Oxford: Oxford Univ. Press, 1949.

29. Hardy, G.H. and Littlewood, J.E., Tauberian Theorems Concerning Power Series and Dirichlet's Series Whose Coefficients are Positive, *Proc. London Math. Soc.*, 1914, vol. 13, pp. 174–191.

30. Khlopin, D.V., A Uniform Tauberian Theorem for Abstract Game, *Abstr. Fourth Int. School-Seminar "Nonlinear Analysis and Extremal Problems,"* June 22–28, 2014, p. 59.

31. Krasovskii, N.N. and Chentsov, A.G., On the Design of Differential Games. I, *Probl. Control Inform. Theory*, 1997, vol. 6, no. 5–6, pp. 381–395.

32. Krasovskii, N.N. and Subbotin, A.I., *Game-Theoretical Control Problems*, New York: Springer-Verlag, 1988.

33. Lions, P.-L., Papanicolaou, G., and Varadhan, S.R.S., Homogenization of Hamilton–Jacobi Equations, *Preprint*, 1986.

34. Mertens, J.F. and Neyman, A., Stochastic Games, *Int. J. Game Theory*, 1981, vol. 10, no. 2, pp. 53–66.

35. Oliu-Barton, M. and Vigeral, G., A Uniform Tauberian Theorem in Optimal Control, in *Advances in Dynamic Games*, Cardaliaguet, P. and Cressman, R., Eds., Boston: Birkhäuser, 2013, pp. 199–215.

36. Quincampoix, M. and Renault, J., On the Existence of a Limit Value in Some Non Expansive Optimal Control Problems, *SIAM J. Control Optim.*, 2011, vol. 49, no. 5, pp. 2118–2132.

37. Roxin, E., Axiomatic Approach in Differential Games, *J. Optim. Theory Appl.*, 1969, vol. 3, no. 3, pp. 153–163.

38. Ryll-Nardzewski, C., A Theory of Pursuit and Evasion, in *Advances in Game Theory*, Dresher, M. and Aumann, R.J., Eds., Princeton: Princeton Univ. Press, 1964, pp. 113–126.

39. Subbotin, A.I., *Generalized Solutions of First Order PDEs*, Boston: Birkhäuser, 1995.

40. Varaiya, P.P., On the Existence of Solutions to a Differential Game, *SIAM J. Control*, 1967, vol. 5, no. 1, pp. 153–162.

41. Vigeral, G., A Zero-Sum Stochastic Game with Compact Action Sets and No Asymptotic Value, *Dynamic Games Appl.*, 2013, vol. 3, no. 2, pp. 172–186.