ISSN 0005-1179, Automation and Remote Control, 2016, Vol. 77, No. 3, pp. 428–442. © Pleiades Publishing, Ltd., 2016.
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Approximate Controllability of Semilinear Non-Autonomous Evolutionary Systems with Nonlocal Conditions

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Received January 20, 2015

Abstract—We consider approximate controllability of semilinear non-autonomous evolutionary systems with nonlocal conditions. In this study, we use the theory of fractional powers and α-norms, so our results can be applied to systems where nonlinear terms include derivatives of spatial variables. We formulate and prove sufficient conditions for approximate controllability. We also give a sample application of our results.

DOI: 10.1134/S000511791603005X

1. INTRODUCTION

Controllability theory for abstract linear control systems in infinite-dimensional spaces is well developed; see, e.g., [1–3] and references therein. Most results deal with the so-called semilinear control system that consists of linear and nonlinear parts. The work [2] studied approximate controllability of an abstract semilinear system under certain conditions of inequality type that depend on the properties of system components. The work [1] considered the same system and showed that it is approximately controllable under a certain condition on the set of values of the controlling influence operator. In [4, 5], this result has been extended to systems with finite delay.

It has been shown in [6] that under a suitable condition on the resolvent operator, approximate controllability of a semilinear system follows from approximate controllability of its linear part. This resolvent condition has been used by many authors to study approximate controllability for nonlinear (functional) differential equations (see, e.g., [7–9]). In [7], this condition and Schauder fixed point theorem were used to study approximate and full controllability of an abstract system

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + Bu(t) + F(t, x_t, u), \quad t \in [0, T] \\ x_0 = \phi. \end{cases}$$

The works [9–13] also used the resolvent condition to study approximate controllability for semilinear impulse (stochastic) systems and differential systems of fractional order with delay.

It is known that there exist many practical models where coefficients of partial derivative operators depend on the time t, and some of them are often written as semilinear non-autonomous evolutionary equations. Previously, many authors have studied the existence and asymptotic behavior of solutions for such equations without a control [14–16]. The approximate controllability problem for such systems is certainly important, but it has not been studied too much so far due to its complexity [17, 18].

The purpose of this work is to study approximate controllability of non-autonomous evolutionary systems with nonlocal conditions of the form

$$\begin{cases} \dfrac{d}{dt}x(t) = -A(t)x(t) + Bu(t) + F(t, x(r(t))), \quad t \in [0, T] \\ x(0) + g(x) = x_0. \end{cases} \tag{1}$$

Here the state variable $x(\cdot)$ takes values in a Hilbert space X, and the control function $u(\cdot)$ is defined in the Banach space $L^2([0,T];U)$, where U is also a Hilbert space, and B is a bounded linear operator from U to X; $\{A(t) : 0 \leqslant t \leqslant T\}$ is a family of linear closed (not necessarily bounded) operators from X to X that generate an evolutionary system of linear operators, $F(\cdot,\cdot)$, and $r(\cdot)$ and $g(\cdot)$ are the corresponding functions that will be defined below.

This goal is motivated by the approximate controllability problem for system

$$\begin{cases} \dfrac{\partial}{\partial t}z(t,x) = a(t,x)\dfrac{\partial^2 z}{\partial x^2}(t,x) + Bu(t) + f\left(t, z(\cdot,x), \dfrac{\partial z}{\partial x}(\cdot,x)\right) \\ z(t,0) = z(t,\pi) = 0, \quad t \geqslant 0 \\ z(0,x) + \displaystyle\sum_{i=1}^{p} k_i(x)z(t_i,x) = z_0(x), \quad 0 \leqslant x \leqslant \pi, \end{cases} \tag{2}$$

which serves as a model for nonlinear heat flow in materials with memory, where $z(t,x)$ is the temperature at point x at time moment t and $f(\cdot,\cdot,\cdot)$ is the heat intake. In particular, the third equation represents nonlocal initial conditions, when initial data depend on the entire state, so we call (2) a nonlocal Cauchy problem. Nonlocal Cauchy conditions were first introduced and studied in [19]. Their practical value for various applications has been discussed in [20].

There have been plenty of works on the existence and regularity of solutions for evolutionary systems with nonlocal conditions [21, 22]. Approximate controllability for $A(t) = A$ has been studied in [23, 24].

The controllability problem for system (1) is studied under a resolvent assumption similar to the one shown in [7, 9, 23]. We should note that system (2) can be considered as an abstract Eq. (1), but the results established in [7] become invalid in this situation even when $A(t) \equiv A$, since function f in Eq. (2) includes spatial derivatives.

In Section 4 we show that if we let $X = L^2([0,\pi])$, then the third variable of function f is defined on $X_{\frac{1}{2}}(t_0)$ rather than X, so solutions cannot be considered on X as they are in [7]. Therefore, we use the theory of operators with fractional powers and α-norms [25, 26], i.e., Eq. (1) is bounded by the Banach space $X_\alpha(t_0)(\subset X)$, and we study the existence of soft solutions for it. We assume that function $g(x)$ is continuous, and its value is fully defined on the interval $[\tau^*, T]$ for a sufficiently small $\tau^* > 0$ rather than on $[0,T]$.

The paper is organized as follows. In Section 2 we give some preliminary information on linear evolutionary systems and approximate controllability. In Section 3 we study the approximate controllability of system (1). In Section 4 we give an sample application for our results.

2. PRELIMINARY INFORMATION

Let X be a Hilbert space with norm $\|\cdot\|$. Suppose that a family of linear operators $\{A(t) : 0 \leqslant t \leqslant T\}$ satisfies the following assumptions.

B_1. The domain $D(A)$ of the family $\{A(t) : 0 \leqslant t \leqslant T\}$ is dense in X and does not depend on t, $A(t)$ is a closed linear operator.

B_2. At every time moment $t\in[0,T]$, the resolvent $R(\lambda, A(t))=(\lambda-A(t))^{-1}$ of linear operator $A(t)$ exists for all λ such that Re $\lambda \leqslant 0$, and there also exists $K>0$ such that $\|R(\lambda, A(t))\| \leqslant K/(|\lambda|+1)$.

B_3. There exist $0<\delta \leqslant 1$ and $K>0$ such that $\|(A(t)-A(s))A^{-1}(\tau)\| \leqslant K|t-s|^{\delta}$ for all t, s and $\tau \in [0,T]$.

B_4. For each $t\in[0,T]$ and some $\lambda \in \rho(A(t))$, the resolvent set $R(\lambda, A(t))$ of linear operator $A(t)$ is a compact.

Under these assumptions, family $\{A(t): 0 \leqslant t \leqslant T\}$ generates a unique linear evolutionary system which is also called a linear evolutionary operator $\{W(t,s): 0 \leqslant s \leqslant t \leqslant T\}$, and there exists a family of bounded linear operators $\{\Phi(t,\tau)|0 \leqslant \tau \leqslant t \leqslant T\}$ with norm $\|\Phi(t,\tau)\| \leqslant C|t-\tau|^{\delta-1}$ such that $W(t,s)$ can be represented as

$$W(t,s) = e^{-(t-s)A(t)} + \int\limits_s^t e^{-(t-\tau)A(\tau)}\Phi(\tau,s)d\tau, \tag{3}$$

where $e^{-\tau A(t)}$ denotes the analytic semigroup with infinitesimal generator $(-A(t))$. The family of linear operators $\{W(t,s)\colon 0 \leqslant s \leqslant t \leqslant T\}$ satisfies the following conditions:

(a) $W(t,s)$ belongs to $\mathscr{L}(X)$, the set of bounded linear transformations on X for every $0 \leqslant s \leqslant t \leqslant T$, and for every $x \in X$ the mapping $(t,s) \to W(t,s)x$ is continuous;

(b) $W(t,s)W(s,\tau) = W(t,\tau)$ for $0 \leqslant \tau \leqslant s \leqslant t \leqslant T$;

(c) $W(t,t) = I$, where I is the unit linear operator;

(d) $W(t,s)$ is a compact operator for $t-s>0$, and there exists a constant $M \geqslant 1$ such that $\|W(t,s)\| \leqslant M$ for each $(t,s) \in [0,T]\times[0,T]$;

(e) $\frac{\partial}{\partial t}W(t,s) = -A(t)W(t,s)$ for $s<t$.

Condition B_4 guarantees that the generated evolutionary operator $W(t,s)$ satisfies property (d) (see [14, Proposition 2.1]). Assumptions B_1–B_3 mean that for every $t\in[0,T]$ the integral

$$A^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^{+\infty} s^{\alpha-1}e^{-sA(t)}ds$$

exists for every $\alpha \in (0,1]$, $A^{-\alpha}(t)$ is a bounded linear operator, and $A^{-\alpha}(t)A^{-\beta}(t) = A^{-(\alpha+\beta)}(t)$. Thus, we can define the fractional power as a closed linear operator $A^{\alpha}(t) = [A^{-\alpha}(t)]^{-1}$ such that $D(A^{\alpha}(t))$ is dense in X and $D(A^{\alpha}(t)) \subset D(A^{\beta}(t))$ for $\alpha \geqslant \beta$. $D(A^{\alpha}(t))$ becomes a Banach space with norm $\|x\|_{\alpha,t} = \|A^{\alpha}(t)x\|$, denoted by $X_{\alpha}(t)$. The following estimates hold [27]:

$$\|A^{\alpha}(t)A^{-\beta}(s)\| \leqslant C_{\alpha,\beta}, \tag{4}$$

where $C_{\alpha,\beta}$ is a constant related to T, δ, t and $s\in[0,T]$, $0 \leqslant \alpha < \beta \leqslant 1$;

$$\|A^{\beta}(t)e^{-sA(t)}\| \leqslant \frac{C_\beta}{s^\beta}e^{-\omega s}, \quad t>0, \ \ \beta \leqslant 0; \ \ \omega>0; \tag{5}$$

$$\|A^{\beta}(t)W(t,s)\| \leqslant \frac{C_\beta}{|t-s|^\beta}, \quad 0<\beta<\delta+1; \tag{6}$$

$$\|A^{\beta}(t)W(t,s)A^{-\beta}(s)\| \leqslant C'_\beta, \quad 0<\beta<\delta+1, \tag{7}$$

for $t>0$, where C_β and C'_β mean that these constants depend on the constant β. For convenience we require further that operators $A^{\alpha}(t)$ and $W(t,s)$ commute for every $0<\alpha \leqslant 1$, i.e., $A^{\alpha}(t)W(t,s) = W(t,s)A^{\alpha}(t)$. Note that this property does hold in many cases (see an example in Section 4). More detailed information can be found in [27–29].

Further, we will consider the problem on the subspace $X_\alpha(t_0)$ with norm $\|\cdot\|_\alpha$, $\alpha \in (0,1)$, $t_0 \in [0,T]$.

Definition 1. Function $x(\,\cdot\,; x_0, u) \in C([0,T], X_\alpha(t_0))$ is called a soft solution of Eq. (1) (with control $u(t)$) if

$$x(t) = W(t,0)[x_0 - g(x)] + \int_0^t W(t,s)\,[Bu(s) + F(s, x(r(s)))]\ ds,\ \ t \in [0,T].$$

Definition 2. System (1) is called approximately controllable on the interval $[0,T]$ if for every $x_0 \in X$ the set $\mathcal{R}(T,x_0) :=\{x(T;x_0,u),\ \ u(\cdot) \in L^2([0,T],U)\}$ is dense in X, i.e.,

$$\overline{\mathcal{R}(T,x_0)} = X,$$

where $\mathcal{R}(T,x_0) := \left\{x(T;x_0,u),\ \ u(\cdot) \in L^2([0,T],U)\right\}$.

To use the resolvent condition [6], we introduce the operator

$$\Gamma_T = \int_0^T W(T,s)BB^*W^*(T,s)ds,$$

$$R(\lambda,(-\Gamma_T)) = (\lambda I + \Gamma_T)^{-1},\ \text{for}\ \lambda > 0,$$

where B^* is the operator conjugate to B, and $W^*(t,s)$ is the evolutionary operator conjugate to $W(t,s)$. Since, obviously, operator Γ_T is positive, we see that $R(\lambda,(-\Gamma_T))$ is well defined for all $\lambda > 0$. We will assume that

H_0. $\lambda R(\lambda,(-\Gamma_T)) \to 0$ for $\lambda \to 0^+$ in the strong operator topology.

Condition H_0 relates to approximate controllability of a non-autonomous linear system

$$\begin{cases} \dfrac{dy(t)}{dt} = -A(t)y(t) + Bu(t),\ \ t \in [0,T] \\ y(0) = 0. \end{cases} \tag{8}$$

More precisely, the following theorem holds.

Theorem 1. *The following statements are equivalent.*

i. *Controllable system* (8) *is approximately controllable on* $[0,T]$.

ii. *If* $B^*W^*(t,0)y = 0$ *for all* $t \in [0,T]$ *then* $y = 0$.

iii. *Condition* H_0 *holds.*

Proof of Theorem 1 is similar to the proof of Theorem 4.4.17 from [3] and Theorem 2 from [6], and we do not show it here.

3. APPROXIMATE CONTROLLABILITY OF SYSTEM (1)

We will show that for every $x^{\mathrm{T}} \in X$ for a suitable control u^λ (and any given $\lambda \in (0,1)$) there exists a soft solution $x_*^\lambda(\,\cdot\,; x_0, u) : [0,T] \to D(A^\alpha(t_0))$ of Eq. (1). Then we show that $x_*^\lambda(T) \to x^{\mathrm{T}}$ in X, which precisely means the approximate controllability.

The existence of soft solutions is guaranteed by the following assumptions on system (1).

H_1. $B \in \mathscr{L}(U,X)$, i.e., B is a bounded linear operator from U to X, $\|B\| = \mathrm{N}$, where $\mathrm{N} > 0$ is a constant.

H_2. Function $F : [0,T] \times X_\alpha(t_0) \to X$ is continuous, and there exist constants $L_1 > 0$ and $0 < \gamma < 1$ such that

$$\|F(t,x)\| \leqslant L_1(\|x\|_\alpha^\gamma + 1)$$

for every $(t,x) \in [0,T] \times X_\alpha(t_0)$.

H_3. Function $g : C([0,T]; X_\alpha(t_0)) \to X_1(t_0)$ is continuous, and there exists a constant $L_2 > 0$ such that[1]

$$\|g(x)\|_1 = \|A(t_0)g(x)\| \leqslant L_2\|x(\cdot)\|_{C^\alpha}^{\gamma}$$

for every $x \in C([0,T], X_\alpha(t_0))$, where $\|x(\cdot)\|_{C^\alpha}$ is the supremum norm in the space $C([0,T]; X_\alpha(t_0))$. There also exists $\tau^* = \tau^*(k) \in (0,T)$ such that $g(p) = g(q)$ for every $p, q \in B_k$, $p(s) = q(s)$, $s \in [\tau^*, T]$, where

$$B_k := \{x \in C([0,T]; X_\alpha(t_0)),\ \|x(\cdot)\|_{C^\alpha} \leqslant k\}.$$

H_4. Function $r(\cdot) \in C([0,T]; [0,T])$.

For every $x^{\mathrm{T}} \in X$ and $\lambda \in (0,1)$ we define the control function $u^\lambda(t)$ (we denote it simply as $u(t)$) as follows:

$$u(t) := B^* W^*(T,t) R(\lambda, (-\Gamma_T)) \times \left[x^{\mathrm{T}} - W(T,0)[x_0 - g(x)] - \int_0^T W(T,\tau) F(\tau, x(r(\tau))) d\tau \right]. \tag{9}$$

For this control, the following theorem holds.

Theorem 2. *If assumptions H_1–H_4 are satisfied, then for every $0 < \lambda < 1$ Eq.* (1) *admits a soft solution on* $[0,T]$.

We begin by proving two lemmas.

Lemma 1. *If conditions of Theorem 2 hold then for every natural number n and every $0 < \lambda < 1$ system*

$$\begin{cases} \dfrac{d}{dt}x(t) = -A(t)x(t) + Bu(t) + F(t, x(r(t))),\ t \in [0,T] \\ x(0) + W\left(\dfrac{1}{n}, 0\right) g(x) = x_0 \in X_\alpha(t_0) \end{cases} \tag{10}$$

admits a soft solution on $[0,T]$.

We define sets of solutions:

$$D = \left\{x_n^\lambda \in C([0,T], X_\alpha(t_0)) : x_n^\lambda = Q_n^\lambda x_n^\lambda,\ n \geqslant 1\right\},$$
$$D(t) = \left\{x_n^\lambda(t) : x_n^\lambda \in D,\ n \geqslant 1\right\},\ t \in [0,T].$$

Lemma 2. *Suppose that conditions of Theorem 2 hold for every $t \in (0,T]$. Then the set D is compact in $X_\alpha(t_0)$, and $D(t)$ is uniformly continuous on $(0,T]$.*

Proofs of the lemmas and Theorem 2 are given in the Appendix.

Theorem 3. *Suppose that assumptions H_1–H_4 of Theorem 2 are verified by function $F(\cdot,\cdot)$, $g(\cdot)$ is uniformly bounded; then system* (1) *is approximately controllable in* $[0,T]$.

[1] $\|\cdot\|_1$ is the norm in the space of measurable functions L^1.

4. EXAMPLE

To demonstrate an application of Theorem 3, consider the system

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x) + a(t) z(t,x) + Bu(t) + h\left(t, z(t\cos t, x), \dfrac{\partial}{\partial x} z(t,x)\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant \pi \\ z(t,0) = z(t,\pi) = 0, \qquad\qquad\qquad\qquad\qquad 0 \leqslant t \leqslant T \\ z(0,x) + \sum\limits_{i=1}^{p} \sin\left(\int\limits_0^\pi b(x,y) z(t_i,y) dy\right) = z_0(x), \qquad 0 \leqslant x \leqslant \pi, \end{cases} \tag{11}$$

where $T \leqslant \pi$, $p \in \mathbb{N}$, $0 < t_1 < t_2 < \cdots < t_p < T$, $z_0(x) \in X := L^2([0,\pi])$. Function $a(t) : [0,T] \to \mathbb{R}^-$ is Hölder continuous with parameter $0 < \delta < 1$, $b(\cdot,\cdot)$ is a function from C^2, and $u(\cdot) : [0,T] \to X$ is a control function.

Suppose that the family of operators $A(t)$ is defined as follows:

$$A(t) f = -f'' - a(t) f$$

with domain

$$D(A) = \{ f(\cdot) \in X : f, f' \text{ are absolutely continuous, } f'' \in X,\ f(0) = f(\pi) = 0 \}.$$

Then it is easy to check that $A(t)$ satisfies assumptions B_1–B_4 and generates an evolutionary operator $W(t,s)$ of the form

$$W(t,s) = T(t-s) \exp\left(\int\limits_s^t a(\tau) d\tau \right),$$

where $T(t)$ is a compact analytic semigroup generated by operator $(-A)$, $Af = -f''$, $f \in D(A)$. It is easy to compute that A has a discrete spectrum with n^2 eigenvalues, $n \in \mathbb{N}$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Thus, for $f \in D(A)$ it holds that

$$-A(t) f = \sum_{n=1}^{\infty} (-n^2 + a(t)) \langle f, z_n \rangle z_n.$$

We can see that the general domain coincides with the domain of operator A. Further, we can define $A^\alpha(t_0)$ ($t_0 \in [0,T]$) for the self-conjugate operator $A(t_0)$, and with the classical spectral theorem it is easy to derive that

$$A^\alpha(t_0) f = \sum_{n=1}^{\infty} (n^2 - a(t_0))^\alpha \langle f, z_n \rangle z_n$$

in the domain of $D(A^\alpha) = \{ f(\cdot) \in X,\ \sum_{n=1}^\infty (n^2 - a(t_0))^\alpha \langle f, z_n \rangle z_n \in X \}$. In particular,

$$A^{\frac{1}{2}}(t_0) f = \sum_{n=1}^{\infty} \sqrt{n^2 - a(t_0)} \langle f, z_n \rangle z_n.$$

Hence for every $f \in X$

$$W(t,s) f = \sum_{n=1}^{\infty} e^{-n^2(t-s) + \int_s^t a(\tau) d\tau} \langle f, z_n \rangle z_n$$

and

$$A^{\alpha}(t_0)W(t,s)f = W(t,s)A^{\alpha}(t_0)f = \sum_{n=1}^{\infty}(n^2 - a(t_0))^{\alpha} e^{-n^2(t-s)+\int_s^t a(\tau)d\tau}\langle f, z_n\rangle z_n.$$

We now let $U = X$ and $B = I$.

We let $\alpha = \frac{1}{2}$ and assume that system (11) satisfies the following hypothesis.

H_5. Function $c : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a constant L_1 such that

$$\left(\int_0^{\pi} \left|c\,(t,\ z_1,\ z_2)\right|^2 dx\right)^{\frac{1}{2}} \leqslant L_1$$

for every $z_1,\ z_2 \in X_{\frac{1}{2}}(t_0)$.

We define functions $F\colon [0,T] \times X_{\frac{1}{2}}(t_0) \to X_{\frac{1}{2}}(t_0)$ and $g\colon C([0,T], X_{\frac{1}{2}})(t_0) \to D(A)$ as

$$F(t,z)(x) = h\left(t, z(t\cos t, x), \frac{\partial}{\partial x} z(t,x)\right),$$

$$g(z(t,x)) = \sum_{i=1}^{p} \sin\left(\int_0^{\pi} b(x,y) z(t_i, y) dy\right).$$

Thus, system (11) can be rewritten in the form (1). Since by assumption the function $b(\cdot,\cdot)$ belongs to C^2, it is easy to see that $g(\cdot)$ satisfies condition H_3 and the uniform boundedness condition, while H_3 ensures that function $F(\cdot,\cdot)$ is continuous and uniformly bounded.

Finally, to establish approximate controllability of system (11) it remains to check that condition H_0 holds. To do that, we define the operator

$$\Gamma_T = \int_0^T W(T,s)BB^*W^*(T,s)ds = \int_0^T W(T,s)W^*(T,s)ds$$

and show that $W^*(T,0)y = 0$ implies $y = 0$. Indeed, if $W^*(T,0)y = 0$ then

$$\int_0^T \|W^*(T,0)y\|^2 ds = \int_0^T \|W(T,0)y\|^2 ds = 0.$$

Consequently,

$$\sum_{n=1}^{\infty} \int_0^T e^{-2n^2T - 2\int_0^T a(\tau)d\tau} ds \langle y,\ z_n\rangle^2 = 0,$$

which immediately implies that $\langle y, z_n\rangle^2 = 0$ for every $n \geqslant 0$, so $y = 0$. Theorem 1 implies that operator $\lambda(\lambda I + \Gamma_T)^{-1} \to 0$ in the strong topology for $\lambda \to 0$, so H_0 holds. Consequently, by Theorem 3 system (11) is approximately controllable on the interval $[0,T]$.

5. CONCLUSION

Controllability of non-autonomous evolutionary systems is a hard and important problem in control theory. In this work, we study the approximate controllability of semilinear non-autonomous evolutionary systems with nonlocal conditions by methods of functional analysis and theory of evolutionary operators, fractional powers, and α-norms. We formulate and prove sufficient approximate controllability conditions (see Theorem 3). Our result is based on the uniform boundedness of nonlinear terms in the considered system and approximate controllability of the corresponding linear systems. However, it is not so easy to check these sufficient conditions, since it is hard to get an exact expression for the evolutionary operator for a non-autonomous linear system. The example shown in this work relates to a special case.

ACKNOWLEDGMENTS

This work was supported by the China National Natural Science Foundation, grant no. 11171110 and Shanghai Leading Academic Discipline Project, grant no. B407.

APPENDIX

Proof of Lemma 1. For brevity we denote

$$M_1 := \|A^{-1}(t_0)\| \quad \text{and} \quad M_2 := \|A^{\alpha-1}(t_0)\|.$$

Control function $u_n(t)$ corresponding to state $x(\cdot)$ of system (10) has the form

$$u_n(t) := B^*W^*(T,t)R(\lambda,(-\Gamma_T)) \times \left\{x^{\mathrm{T}} - W(T,0)\left[x_0 - W\left(\frac{1}{n},0\right)g(x)\right] - \int\limits_0^T W(T,\tau)F(\tau,x(r(\tau)))d\tau\right\}. \quad \text{(A.1)}$$

Suppose that B_k is defined as in H_3, then it is a nonempty, bounded, closed, and convex subset of the space $C([0,T];X_\alpha(t_0))$. Let Q_n^λ be an operator on B_k:

$$(Q_n^\lambda x)(t) := W(t,0)\left[x_0 - W\left(\frac{1}{n},0\right)g(x)\right] + \int\limits_0^t W(t,s)[Bu_n(s) + F(s,x(r(s)))]ds.$$

Obviously, the fixed point of operator Q_n^λ is a soft solution for Eq. (10) with control $u_n(\cdot)$ from (A.1). We will show that Q_n^λ has a fixed point in B_k by Schauder fixed point theorem, which will mean that soft solutions exist for (10). We will show that Q_n^λ maps B_k to itself for some $k > 0$, and Q_n^λ is a fully continuous operator.

We first show that for every $0 < \lambda < 1$ there exists $k = k(\lambda) > 0$ such that $Q_n^\lambda(B_k) \subset B_k$. If it is not true then for every $k > 0$ there exists $x_k(\cdot) \in B_k$ such that $Q_n^\lambda(x_k) \notin B_k$, i.e., there exists $t = t(k) \in [0,T]$ such that $\|(Q_n^\lambda x_k)(t)\|_\alpha > k$. Then taking into account (d) and H_1–H_3, without loss of generality we assume that $\|R(\lambda,(-\Gamma_T))\| < \frac{1}{\lambda}$ for $\lambda \in (0,1)$,

$$\begin{aligned}\|u_n(t)\| &= \Bigg\| B^*W^*(T,t)R(\lambda,(-\Gamma_T)) \\ &\times \left\{x^{\mathrm{T}} - W(T,0)\left[x_0 - W\left(\frac{1}{n},0\right)g(x_k)\right] - \int\limits_0^T W(T,\tau)F(\tau,x_k(r(\tau)))d\tau\right\}\Bigg\| \\ &\leqslant \frac{1}{\lambda}MN\left(\|x^{\mathrm{T}}\| + M\|x_0\| + M\|g(x_k)\| + M\int\limits_0^T \|F(\tau,x_k(r(\tau)))\|d\tau\right) \\ &\leqslant \frac{1}{\lambda}MN\left[\|x^{\mathrm{T}}\| + M\|x_0\| + MM_1L_2k^\gamma + ML_1T(k^\gamma+1)\right],\end{aligned}$$

so (4)–(7) imply that

$$k < \|(Q_n^\lambda x_k)(t)\|_\alpha \leqslant \left\| W(t,0)\left[x_0 - W\left(\frac{1}{n},0\right) g(x_k)\right]\right\|_\alpha$$

$$+ \left\| \int_0^t W(t,s)\left[Bu_n(s) + F(s, x_k(r(s)))\right] ds \right\|_\alpha$$

$$\leqslant M\left[\|x_0\|_\alpha + M\left\|A^{\alpha-1}(t_0)\right\| \|g(x_k)\|_1\right]$$

$$+ \int_0^t \|A^\alpha(t_0)A^{-\beta}(t)A^\beta(t)W(t,s)\| \, \|Bu_n(s) + F(s, x_k(r(s)))\| ds$$

$$\leqslant M\Big[\|x_0\|_\alpha + MM_2L_2k^\gamma\Big] + \int_0^t C_{\alpha,\beta}\frac{C_\beta}{(t-s)^\beta}\left[N\|u(s)\| + \|F(s, x_k(r(s)))\|\right] ds$$

$$\leqslant \left[M^2M_2L_2 + \frac{C_{\alpha,\beta}C_\beta}{1-\beta}T^{1-\beta}\Big(\frac{M^2N^2}{\lambda}M_1L_2 + \frac{M^2N^2}{\lambda}L_1T + L_1\Big)\right]k^\gamma + K,$$

where $\alpha < \beta < 1$ and

$$K = M\|x_0\|_\alpha + \frac{C_{\alpha,\beta}C_\beta}{1-\beta}T^{1-\beta}\left(\frac{MN^2}{\lambda}\|x^{\mathrm{T}}\| + \frac{M^2N^2}{\lambda}\|x_0\| + \frac{M^2N^2}{\lambda}L_1T + L_1\right).$$

This is impossible for $k \to \infty$ since $0 < \gamma < 1$. Thus, there exists $k = k(\lambda) > 0$ such that Q_n^λ maps B_k to itself for every $0 < \lambda < 1$.

To prove that operator Q_n^λ is fully continuous, let us show that it is continuous on B_k.

Let $\{x^m\}_{m\in\mathbb{N}^+}$ be a sequence in B_k such that $x^m \to x$ $(m \to +\infty)$; then

$$\|A(t_0)\left[g(x^m) - g(x)\right]\| \to 0 \ (m \to +\infty),$$

$$\|F(s, x^m(r(s))) - F(s, x(r(s)))\| \to 0 \ (m \to +\infty)$$

and

$$\|F(s, x^m(r(s))) - F(s, x(r(s)))\| \leqslant 2L_1(k^\gamma + 1).$$

Lebesgue's theorem on dominated convergence implies that

$$\|Q_n^\lambda x^m - Q_n^\lambda x\|_\alpha \leqslant M^2\left\|A^{\alpha-1}(t_0)\right\| \|g(x^m) - g(x)\|_1$$

$$+ \int_0^t C_{\alpha,\beta}\frac{C_\beta}{(t-s)^\beta}\|u_n^m(s) - u_n(s)\| ds$$

$$+ \int_0^t C_{\alpha,\beta}\frac{C_\beta}{(t-s)^\beta}\|F(s, x^m(r(s))) - F(s, x(r(s)))\| ds \to 0 \quad (\text{ for } m \to +\infty),$$

where $u_n^m(\cdot)$ is a control function corresponding to $x^m(\cdot)$, and, obviously, $\|u_n^m(s) - u_n(s)\| \to 0$ for $m \to +\infty$. Thus, Q_n^λ is continuous.

Let us now show that for an arbitrary $0 < \lambda < 1$ operator Q_n^λ maps B_k into a relatively compact subset of the set $C([0,T], X_\alpha(t_0))$. We first show that the set $V(t) = \left\{(Q_n^\lambda x)(t), x \in B_k\right\}$ is relatively compact in $X_\alpha(t_0)$ for every $t \in [0,T]$. Indeed, the case when $t = 0$ is trivial since $\left(Q_n^\lambda x(0)\right) = x_0 - W\left(\frac{1}{n}, 0\right) g(x)$, where $g(x)$ is bounded on B_k and $W\left(\frac{1}{n}, 0\right)$ is compact. We now fix $t \in (0,T]$ and note that for $0 < \alpha < \alpha_1 < \beta < 1$

$$\|(Q_n^\lambda x)(t)\|_{\alpha_1} \leqslant \left\|A^{\alpha_1}(t_0)W(t,0)\left[x_0 - W\left(\frac{1}{n}, 0\right) g(x)\right]\right\|$$

$$+ \left\|A^{\alpha_1}(t_0)\int\limits_0^t W(t,s)\left[Bu_n(s) + F(s, x(r(s)))\right] ds\right\|$$

$$\leqslant M\|x_0\|_{\alpha_1} + M^2\left\|A^{\alpha_1 - 1}(t_0)\right\| L_2 k^\gamma + \frac{C_{\alpha_1,\beta}C_\beta}{1-\beta}T^{1-\beta}\left[N\|u_n(\cdot)\| + L_1(k^\gamma + 1)\right],$$

which implies that $V(t)$ is bounded in $X_{\alpha_1}(t_0)$. Therefore, $V(t)$ is relatively compact in $X_\alpha(t_0)$, and as a consequence the operator $A^{-\alpha_1}(t_0) : X \to X_\alpha(t_0)$ is compact (because $X_{\alpha_1}(t_0) \hookrightarrow X_\alpha(t_0)$ is compact). Consequently, for every $t \in [0,T]$ $V(t)$ is relatively compact in $X_\alpha(t_0)$.

It remains to show that the family of functions $V = \left\{P^\lambda(z)(\cdot) : z \in B_k\right\}$ is uniformly continuous on the interval $[0,T]$. This is obvious for $t = 0$. Let $0 < t_1 < t_2 \leqslant T$, then

$$\left\|(Q_n^\lambda x)(t_2) - (Q_n^\lambda x)(t_1)\right\|_\alpha$$

$$\leqslant \left\|[W(t_2,0) - W(t_1,0)]\left[x_0 - W\left(\frac{1}{n}, 0\right) g(x)\right]\right\|_\alpha$$

$$+ \left\|\int\limits_0^{t_1}[W(t_2,s) - W(t_1,s)][Bu_n(s) + F(s, x(r(s)))]ds\right\|_\alpha$$

$$+ \left\|\int\limits_{t_1}^{t_2} W(t_2,s)[Bu_n(s) + F(s, x(r(s)))]ds\right\|_\alpha.$$

Obviously, the first term in the right-hand side tends to zero for $t_2 \to t_1$ since $W\left(\frac{1}{n}, 0\right)$ is compact and $g(x)$ is bounded on B_k. For the third term we get

$$\left\|\int\limits_{t_1}^{t_2} W(t_2,s)[Bu_n(s) + F(s, x(r(s)))]ds\right\|_\alpha$$

$$\leqslant \int\limits_{t_1}^{t_2} C_{\alpha,\beta}\frac{C_\beta}{(t-s)^\beta}\left[N\|u_n\| + \|F(s, x(r(s)))\|\right] ds$$

$$\leqslant \frac{C_{\alpha,\beta}C_\beta}{1-\beta}\left[N\|u_n\| + L_1(k^\gamma + 1)\right](t_2 - t_1)^{1-\beta} \;\to\; 0 \quad \text{for } t_2 \to t_1.$$

We next consider the second term. Now (3) implies that

$$\left\| \int_0^{t_1} [W(t_2,s) - W(t_1,s)][Bu_n(s) + F(s, x(r(s)))]ds \right\|_\alpha$$

$$\leqslant \int_0^{t_1-\varepsilon} \left\| A^\alpha(t_0) \left[e^{-(t_2-s)A(t_2)} - e^{-(t_1-s)A(t_1)} \right] \right\| [N\|u_n\| + L_1(k^\gamma + 1)]\, ds$$

$$+ \int_0^{t_1-\varepsilon} \left\| A^\alpha(t_0) \left[\int_s^{t_2} e^{-(t_2-\tau)A(\tau)} \Phi(\tau,s)d\tau - \int_s^{t_1} e^{-(t_1-\tau)A(\tau)} \Phi(\tau,s)d\tau \right] \right\| [N\|u_n\| + L_1(k^\gamma + 1)]ds$$

$$+ \int_{t_1-\varepsilon}^{t_1} \|A^\alpha(t_0)[W(t_2,s) - W(t_1,s)]\| [N\|u_n\| + L_1(k^\gamma + 1)]ds$$

$$\leqslant C_{\alpha,1} \int_0^{t_1-\varepsilon} \left\| A(t_0) \left[e^{-(t_2-s)A(t_2)} - e^{-(t_1-s)A(t_1)} \right] \right\| [N\|u_n\| + L_1(k^\gamma + 1)]ds$$

$$+ CC_{\alpha,1} \int_0^{t_1-\varepsilon} \left\| \int_s^{t_1} A(t_0) \left[e^{-(t_2-\tau)A(\tau)} - e^{-(t_1-\tau)A(\tau)} \right] (\tau - s)^{\delta-1} d\tau \right\| [N\|u_n\| + L_1(k^\gamma + 1)]ds$$

$$+ \int_0^{t_1-\varepsilon} \left\| \int_{t_1}^{t_2} A^\alpha(t_0) e^{-(t_2-\tau)A(\tau)} \Phi(\tau,s)d\tau \right\| [N\|u_n\| + L_1(k^\gamma + 1)]ds$$

$$+ \int_{t_1-\varepsilon}^{t_1} \|A^{-\alpha}(t_0)[W(t_2,s) - W(t_1,s)]\| [N\|u_n\| + L_1(k^\gamma + 1)] := I_1 + I_2 + I_3 + I_4,$$

where $\varepsilon > 0$ is sufficiently small. The operator function $A(t)e^{-\tau A(s)}$ is uniformly continuous with respect to (t, τ, s) for $0 \leqslant t \leqslant T$, $z \leqslant \tau \leqslant T$ and $0 \leqslant s \leqslant T$, where z is any positive number (see [27, 28]); therefore, $(I_1 + I_2)$ tends to zero for $t_2 - t_1 \to 0$. For the two latter terms we get

$$I_3 \leqslant \int_0^{t_1-\varepsilon} \int_{t_1}^{t_2} CC_{\alpha,\beta} \frac{C_\beta}{(t_2-\tau)^\beta} (\tau - s)^{\delta-1} [N\|u_n\| + L_1(k^\gamma + 1)] d\tau ds$$

$$\leqslant CC_{\alpha,\beta} C_\beta \frac{2T^\delta}{\delta} \int_{t_1}^{t_2} (t_2 - \tau)^{-\beta} ds [N\|u_n\| + L_1(k^\gamma + 1)]$$

$$\leqslant CC_{\alpha,\beta} C_\beta \frac{2T^\delta}{\delta(1-\beta)} (t_2 - t_1)^{1-\beta} [N\|u_n\| + L_1(k^\gamma + 1)]$$

and

$$I_4 \leqslant \frac{C_{\alpha,\beta} C_\beta}{1-\beta} \left[(t_2 - t_1)^{1-\beta} + (t_2 - t_1 + \varepsilon)^{1-\beta} + \varepsilon^{1-\beta} \right] [N\|u_n\| + L_1(k^\gamma + 1)].$$

All of the derivations above imply that $V = \{(Q_n^\lambda x)(\cdot), x \in B_k\}$ is uniformly continuous on $[0, T]$. Thus, the infinite-dimensional version of the Arzela–Ascoli theorem tells us that $Q_n^\lambda x$ is a fully continuous operator on $C([0, T]; X_\alpha(t_0))$.

This lets us conclude (by Schauder's theorem) that there exists a fixed point $x_n^\lambda(\cdot)$ for Q_n^λ on B_k, which represents a soft solution for Eq. (10) on $[0, T]$. This completes the proof of Theorem 1.

Lemma 2 is easy to show in a similar way.

Proof of Theorem 2. First, we have to prove that the set of solutions D is relatively compact in $C([0,T],X_\alpha)$. Obviously, it suffices to show that $D(0)$ is relatively compact in $X_\alpha(t_0)$ and D is uniformly continuous in $t=0$ by Lemma 2. For $x_n^\lambda \in D$, $n \geqslant 1$ we let

$$\bar{x}_n^\lambda(t) := \begin{cases} x_n^\lambda(\delta), & t \in [0,\tau^*) \\ x_n^\lambda(t), & t \in [\tau^*,T], \end{cases}$$

where $\tau^* = \tau^*(k)$ from H_3. Then $g\left(x_n^\lambda\right)(t) = g\left(\bar{x}_n^\lambda\right)(t)$ by H_3.

Lemma 2 implies that $D|_{[\tau^*,T]}$ is relatively compact in $C([\tau^*,T],X_\alpha(t_0))$, so without loss of generality we can assume that $\bar{x}_n^\lambda(t) \to \bar{x}^\lambda(\cdot)$ $(n\to\infty)$ for some $\bar{x}^\lambda(\cdot) \in C([\tau^*,T],X_\alpha(t_0))$. Then

$$\left\|x_n^\lambda(0) - \left[x_0 + g\left(\bar{x}^\lambda\right)\right]\right\|_\alpha = \left\|W\left(\frac{1}{n},0\right)g\left(x_n^\lambda\right) - g\left(\bar{x}^\lambda\right)\right\|_\alpha$$
$$\leqslant \left\|W\left(\frac{1}{n},0\right)g(x_n^\lambda) - W\left(\frac{1}{n},0\right)g\left(\bar{x}^\lambda\right)\right\|_\alpha$$
$$+\left\|W\left(\frac{1}{n},0\right)g\left(\bar{x}^\lambda\right) - g\left(\bar{x}^\lambda\right)\right\|_\alpha = \left\|W\left(\frac{1}{n},0\right)\left[g\left(\bar{x}_n^\lambda\right) - g\left(\bar{x}^\lambda\right)\right]\right\|_\alpha$$
$$+\left\|\left[W\left(\frac{1}{n},0\right) - I\right]g\left(\bar{x}^\lambda\right)\right\|_\alpha \to 0 \quad \text{for} \quad n\to\infty,$$

i.e., $D(0)$ is relatively compact in $X_\alpha(t_0)$.

On the other hand, for $t \in (0,T]$ we have

$$\left\|x_n^\lambda(t) - x_n^\lambda(0)\right\|_\alpha = \left\|W(t,0)\left[x_0 + W\left(\frac{1}{n},0\right)g(x_n^\lambda)\right]\right.$$
$$\left. + \int_0^t W(t,s)[Bu_n(s) + F(s,x_n^\lambda(r(s)))]ds - \left[x_0 + W\left(\frac{1}{n},0\right)g(x_n^\lambda)\right]\right\|_\alpha$$
$$\leqslant \|W(t,0)x_0 - x_0\|_\alpha + \left\|[W(t,0)-I]W\left(\frac{1}{n},0\right)g(x_n^\lambda)\right\|_\alpha$$
$$+\left\|\int_0^t W(t,s)[Bu_n(s) + F(s,x_n^\lambda(r(s)))]ds\right\|_\alpha \to 0 \quad \text{uniformly with respect to } n \text{ for } t\to 0^+.$$

This, together with Lemma 1, means that D is relatively compact in $C([0,T],X_\alpha(t_0))$. Consequently, there exists $x_*^\lambda \in C([0,T],X_\alpha(t_0))$ such that $x_n^\lambda \to x_*^\lambda$ for $n\to\infty$ without loss of generality.

By the definition of a soft solution (10),

$$x_n^\lambda(t) = W(t,0)\left[x_0 - W\left(\frac{1}{n},0\right)g(x_n^\lambda)\right]$$
$$+\int_0^t W(t,s)\left[Bu_n(s) + F(s,x_n^\lambda(r(s)))\right]ds, \quad 0 \leqslant t \leqslant T.$$

Passing to the limit on both sides for $n\to\infty$, we immediately get

$$x_*^\lambda(t) = W(t,0)\left[x_0 - W\left(\frac{1}{n},0\right)g(x_*^\lambda)\right] + \int_0^t W(t,s)\left[Bu_*(s) + F(s,x_*^\lambda(r(s)))\right]ds, \quad 0 \leqslant t \leqslant T,$$

where

$$u_*(t) := B^*W^*(T,t)R(\lambda,(-\Gamma_T))$$
$$\times \left\{ x^{\mathrm{T}} - W(T,0)\left[x_0 - g(x_*^\lambda)\right] - \int_0^T W(T,\tau)F(\tau, x_*^\lambda(r(\tau)))d\tau \right\}.$$

This shows that Eq. (1) has a soft solution $x_*^\lambda(\cdot)$ on $[0,T]$ for $\lambda \in (0,1)$.

Proof of Theorem 3. Let $x_*^\lambda(\cdot)$ be a soft solution of Eq. (1) in $[0,T]$ with control

$$u_*^\lambda(t) = B^*W^*(T,t)R(\lambda,(-\Gamma_T))$$
$$\times \left[x^{\mathrm{T}} - W(T,0)(x_0 + g(x_*^\lambda)) - \int_0^T W(T,\tau)F(\tau, x_*^\lambda(r(\tau)))d\tau \right].$$

Then

$$x_*^\lambda(T) = W(T,0)\left[x_0 - g\left(x_*^\lambda\right)\right] + \int_0^T W(T,s)\left[Bu_*^\lambda(s) + F(s, x_*^\lambda(r(s)))\right]ds$$
$$= x^{\mathrm{T}} + (\Gamma_T R(\lambda,(-\Gamma_T)) - I)\left[x^{\mathrm{T}} - W(T,0)\left(x_0 - g(x_*^\lambda)\right) - \int_0^T W(T-s)F(s, x_*^\lambda(r(s)))ds\right]$$
$$= x^{\mathrm{T}} - \lambda R(\lambda,(-\Gamma_T))\left[x^{\mathrm{T}} - W(T,0)\left(x_0 - g(x_*^\lambda)\right) - \int_0^T W(T,s)F(s, x_*^\lambda(r(s)))ds\right]. \quad \text{(A.2)}$$

Assumptions of Theorem 3 imply that sequences $\{g(x_*^\lambda) : \lambda \in (0,1)\}$ and $\{F(s, x_*^\lambda(r(s))) : \lambda \in (0,1)\}$ are uniformly bounded with respect to λ in spaces $X_1(t_0)$ and X respectively. Consequently, $\left\{g(x_*^\lambda) : \lambda \in (0,1)\right\}$ is also uniformly bounded in the X-norm, and therefore, since $W(T,0)$ is compact and by condition (H_0), we conclude that there exists a subsequence, which we denote by $g(x_*^\lambda)$, such that

$$\left\|\lambda R(\lambda,(-\Gamma_T))W(T,0)g(x_*^\lambda)\right\| \to 0$$

for $\lambda \to 0^+$. On the other hand, there exists a subsequence, which we denote by $F(s, x_*^\lambda(r(s)))$, that weakly converges, for instance, to $f(s)$ in the space X for every $s \in [0,T]$. Then, since $W(T,s)$ $(0 \leqslant s < T)$ is compact, we get that

$$\left\|W(T,s)\left[F(s,(s, x_*^\lambda(r(s))) - f(s)\right]\right\| \to 0 \quad \text{for all} \quad s \in [0,T),$$

and, consequently,

$$\left\|\int_0^T W(T,s)\left[F(s,(s, x_*^\lambda(r(s)))) - f(s)\right]ds\right\| \to 0$$

for $\lambda \to 0^+$. Using (A.2), we get

$$\|x_*^\lambda(T) - x^{\mathrm{T}}\| \leqslant \left\|\lambda R(\lambda, (-\Gamma_T))[x^{\mathrm{T}} - W(T,0)x_0]\right\| + \left\|\lambda R(\lambda, (-\Gamma_T))W(T,0)g(x_*^\lambda)\right\|$$
$$+ \left\|\lambda R(\lambda, (-\Gamma_T))\int_0^T W(T,s)F(s,(s,x_*^\lambda(r(s))))ds\right\| \leqslant \left\|\lambda R(\lambda, (-\Gamma_T))\left[x^{\mathrm{T}} - W(T,0)x_0\right]\right\|$$
$$+ \left\|\lambda R(\lambda, (-\Gamma_T))W(T,0)g(x_*^\lambda)\right\| + \left\|\lambda R(\lambda, (-\Gamma_T))\int_0^T W(T,s)f(s)ds\right\|$$
$$+ \left\|\lambda R(\lambda, (-\Gamma_T))\int_0^T W(T,s)[F(s,(s,x_*^\lambda(r(s)))) - f(s)]ds\right\| \to 0 \quad \text{for} \quad \lambda \to 0^+. \quad \text{(A.3)}$$

Hence $x_*^\lambda(T) \to x^{\mathrm{T}}$ in X, and consequently, we get approximate controllability of system (1). This completes the proof of Theorem 3.

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This paper was recommended for publication by V.I. Gurman, a member of the Editorial Board