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The Bubnov–Galerkin Method in Control Problems for Bilinear Systems

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Abstract—We suggest to apply the Bubnov–Galerkin method to solving control problems for bilinear systems. We reduce the solution of a control problem to a finite-dimensional system of linear problem of moments. We show a specific example of applying this procedure and give its numerical solution.

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1. INTRODUCTION

A number of applied problems, including optimization of structure and topology of designs [1–4], certain processes in quantum mechanical systems, ecology, medicine, mathematical economics, engineering and so on [5], can be mathematically modeled by bilinear systems, i.e., systems whose state equations are linear with respect to both functions in question. Bilinear systems are the simplest nonlinear systems that describe a number of real world processes in very different fields of science that would be incorrectly modeled in the framework of linear theory.

The notion of a bilinear system has been introduced to control theory in the 1960s, and since then numerous works have appeared on this topic. A relatively comprehensive list of references can be found in the book [5]. Other control problems for bilinear systems described by ordinary differential equations can be found, e.g., in [6, 7]; partial differential equations, in [8–13].

At present, several exact and approximate methods for solving control problems for bilinear systems have been developed. To solve control problems for bilinear systems, the work [5] systematically uses the theory of matrix Lie groups. The same method has been applied in [8] to prove full controllability by a bilinear control of bending oscillations of plates when the control in question depends on all independent variables. In [13], the required control function occurs in the coefficients of the state function of Schrödinger equation, while in [9, 12] it occurs in that of the first derivative of the state function of the wave equation. An interesting control problem for the coefficient of a Korteweg–de Vries equation which is nonlinear in the state function but linear in the control function has been considered in [11]. In [1], classical variational calculus methods are used to study different problems of the theory of elasticity, when state equations are linear with respect to the state function and nonlinear with respect to the control function. In the studies of structural and typological optimization problems, researchers often employ the method of finite elements together with the Fourier method of separating variables [2–4].

In the present work, we propose a novel approximate method for solving control problems for bilinear equations which is mathematically founded on the Bubnov–Galerkin procedure [14]. We demonstrate this approach with an important example where the required control function does not depend explicitly on one of the independent variables. In such situations we propose to apply the generalized Butkovsky's finite control method to find the control in question [15].

2. THE BUBNOV–GALERKIN PROCEDURE FOR BILINEAR CONTROL SYSTEMS

The control problem for a bilinear partial differential system usually requires one to optimize a given functional $\kappa[u]$ by choosing a function u from a given set $\mathcal{U}$ of admissible controls under differential constraints

$$\mathcal{D}_u[w] = \mathcal{N}(x,t), \quad x \in \Omega \subset \mathbb{R}^3, \quad t > 0. \tag{1}$$

A solution of (1) satisfies given linear boundary conditions

$$\mathcal{B}[w] = w_\partial(t), \quad x \in \partial\Omega, \quad t > 0, \tag{2}$$

and certain initial conditions. Here $\mathcal{D}_u[w]$ is a differential operator defined in region $\Omega \times \mathbb{R}^+$ and containing the product of state function w and control function u or their derivatives, $\mathcal{B}[\cdot]$ is a given linear operator defined in the region $\partial\Omega \times \mathbb{R}^+$. As usual, $\partial\Omega$ denotes the boundary of region Ω, $\boldsymbol{n}$ is the vector of its external normal. Examples of operator $\mathcal{D}_u[\cdot]$ can be found, for instance, in [1, 4, 5, 8–13].

The purpose of a control problem can be to provide for solutions of the boundary problem (1) and (2) with required final conditions for fixed $t = T$. Final conditions are often assumed to be equal to zero.

In the present work, we propose to use the Bubnov–Galerkin procedure [14] to solve this control problem. If we are able to construct a system of linear independent basis (approximating) functions $\{\varphi_k(x,t)\}_{k=0}^n$ for the boundary value problem (1), (2) then the residue obtained by substituting approximate solutions

$$w_n(x,t) = \varphi_0(x,t) + \sum_{k=1}^{n} \alpha_k \varphi_k(x,t) \tag{3}$$

into Eq. (1) will be

$$\mathcal{R}_n(x,t) \equiv \mathcal{D}_u[w_n] - \mathcal{N}(x,t), \quad x \in \overline{\Omega}, \quad t > 0. \tag{4}$$

According to the Bubnov–Galerkin method, the coefficients α_k are determined from orthogonality conditions on basis functions $\{\varphi_k(x,t)\}_{k=0}^n$ to the residue (4) [14]:

$$\int_0^T \int_\Omega \mathcal{R}_n(x,t)\varphi_k(x,t)\,dx\,dt = 0, \quad k = \overline{1;n}. \tag{5}$$

If for some $n_0 \in \mathbb{N}$ the residue (4) is identically zero, $\mathcal{R}_n(x,t) \equiv 0$, then the corresponding approximation $w_{n_0}(x,t)$ (3) will be an exact solution of the boundary problem (1), (2). Otherwise, increasing the number n of the terms in (3), we can approximate the solution in question up to a given accuracy. Then the limit case $w_\infty(x,t)$ will be an exact solution of problem (1), (2).

After we find coefficients α_k from the system of linear algebraic Eqs. (5) and substitute them into the approximate solution (3), we take into account that at time moment T given final conditions must hold, for finding the function in question we get a system of constraints of the form

$$\int_0^T \int_\Omega u\mathcal{K}_k(x,t)\,dx\,dt = \mathcal{M}_k, \quad k = \overline{1;n}, \tag{6}$$

where kernels $\mathcal{K}_k(x,t)$ and constants $\mathcal{M}_k$ depend on the parameters of system (1), (2).

System (6) lets us find the function in question with different methods, including an approach proposed in [16] where such systems are considered as problem of moments [16, 17]. The convenience of this approach is that we not only are able to construct an explicit form of the control function but also establish conditions for its existence [15–19].

Solving control problems for bilinear systems can be thus reduced to a problem of moments, which in this case, unlike [15], is finite-dimensional.

3. DAMPING A BEAM UNDER MOVING INFLUENCE

Elastic beams subject to moving loads represent the simplest models of railroad bridges on which trains are moving. Thus, damping bending oscillations in such beams over a finite period time has immediate and useful applications. The works [20–25] consider numerous applied problems of damping the oscillations of a system subject to moving loads with dynamic dampers of oscillations. Dampers are considered in either viscous or viscoelastic model, and the main parameter for optimization is the shift of dampers for a given configuration and location of dampers under the beam [20–23]. A viscous model of dampers lets us replace their influence on the beam with a force proportional to some degree of the velocity of the beam's points, while the influence of viscoelastic dampers on the beam is replaced by a force proportional to a combination of the velocity and movement of the beam's points. Lately, in order to improve the seismic resistance of constructions researchers have also studied optimal placements and the number of viscoelastic dampers of oscillations [24, 25]. The main technique for solving these problems is the method of separating the variables since for a given location of dampers the problem is mathematically formulated as differential equations with constant coefficients.

As an example of applying the above procedure consider the optimal distribution problem for viscous dampers of oscillations under an elastic homogeneous beam of finite length $2l$ which is simply supported at the ends $x_* = -l$ and $x_* = l$. Suppose that the beam is bended under the influence of a constant load P_* which uniformly moves along the beam with velocity v_* (see figure).

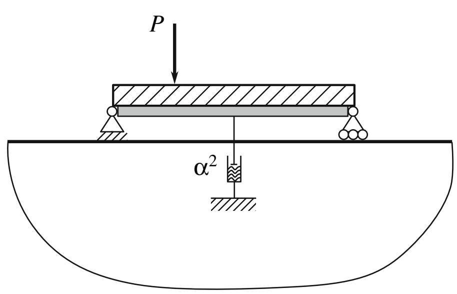

Illustration of a beam with viscous damper.

We assume a linear–viscous model of dampers, i.e., we assume that the influence of dampers on the beam is proportional to the first degree of the velocity with which points of the beam move. Assuming that the load separates from a beam at a given time moment $\tau_* > 0$, we need to damp the oscillations of a beam at a fixed (given) time moment $T > \tau_*$. The problem can be mathematically formulated as the following bilinear differential equation with respect to a dimensionless deflection of the beam [21]:

$$\frac{\partial^4 w}{\partial x^4} + \beta^2 \frac{\partial^2 w}{\partial t^2} + \alpha^2 u(x) \frac{\partial w}{\partial t} \tag{7}$$
$$= P\delta(x + 1 - v(t + \pi))[\theta(t + \pi) - \theta(t + \pi - \tau)], \quad x \in (-1, 1), \quad t \in (-\pi, \pi),$$

whose solution satisfies boundary conditions of simple support:

$$w(\pm 1, t) = \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=\pm 1} = 0, \quad t \in (-\pi, \pi). \tag{8}$$

Here $u(x)$ is the dimensionless control function that characterizes the distribution of dampers under the beam, $\theta(t)$ is the unit Heaviside step function, $\delta(x)$ is the impulse Dirac function [26],

$$w = \frac{w_*}{l}, \quad x = \frac{x_*}{l}, \quad t = \frac{2t_* - T}{T}\pi, \quad \tau = \frac{2\pi}{T}\tau_*, \quad P = \frac{P_* l^2}{EJ},$$
$$\alpha^2 = \frac{2\pi}{T}\frac{\alpha_*^2 l^4}{EJ}, \quad \beta^2 = \frac{4\pi^2}{T^2}\frac{\rho S l^4}{EJ}, \quad v = \frac{T}{2\pi}\frac{v_*}{l} = \frac{T}{\pi\tau_*},$$

EJ is beam's bending stiffness, and α_*^2 is the viscosity coefficient of the dampers. A characteristic feature of this example is that the control function does not depend explicitly on the independent variable t.

The main goal of our study is to dampen the oscillations of the beam by choosing an admissible control function $u^o \in \mathcal{U}$ that minimizes the functional [16]

$$\kappa[u] = \int_{-1}^{1} u(x)dx, \quad u \in \mathcal{U}. \tag{9}$$

Already at this point we could use the procedure outlined above, but due to the characteristic features of this specific example we can do it slightly differently. First we write Eq. (7) and conditions (8) for all real t. Introducing the operator [10, 18, 19] $\mathcal{A}_\pi[f] = [\theta(t+\pi) - \theta(t-\pi)]f(t) \equiv f_1(t)$ defined on the entire real axis, we write problem (7), (8) in generalized functions [26]:

$$\frac{\partial^4 w_1}{\partial x^4} + \beta^2\frac{\partial^2 w_1}{\partial t^2} + \alpha^2 u(x)\frac{\partial w_1}{\partial t} = P\delta(x+1-v(t+\pi))[\theta(t+\pi) - \theta(t+\pi-\tau)]$$
$$+\alpha^2 u(x) w_0(x)\delta(t+\pi) + \beta^2\left[w_0(x)\delta'(t+\pi) + \dot{w}_0(x)\delta(t+\pi)\right], \quad x \in (-1,1), \quad t \in \mathbb{R}, \tag{10}$$

$$w_1(\pm 1, t) = \left.\frac{\partial^2 w_1(x,t)}{\partial x^2}\right|_{x=\pm 1} = 0, \quad t \in \mathbb{R}, \tag{11}$$

$w_0(x)$ and $\dot{w}_0(x)$ are initial functions, and the derivative of the delta function, $\delta'(t)$, is understood in the generalized sense [26]. Note that the function $\dot{w}_0(x)$ is only a symbolic notation and does not represent a derivative.

It is clear that the introduced function $w_1(x,t)$ is concentrated in the region $[-1,1]\times[-\pi,\pi]$, i.e., supp $w_1 \subseteq [-1,1]\times[-\pi,\pi]$, where supp $f = \overline{\{x \in \mathbb{R} : \ f(x) \neq 0\}}$ denotes the support of function f. Judging by physical consideration, we assume that the set $\mathcal{U}$ of admissible controls consists of real nonnegative functions $u(x)$ for which supp $u \subseteq [-1,1]$.

We now apply to system (10), (11) a real generalized integral Fourier transform with respect to variable t [26]:

$$\frac{d^4\overline{w}_1}{dx^4} - \sigma^2\left[\beta^2 + i\frac{\alpha^2}{\sigma}u(x)\right]\overline{w}_1 = \frac{P}{v}[\theta(x+1) - \theta(x+1-v\tau)]e^{i\sigma\left(\frac{x+1}{v}-\pi\right)}$$
$$+\alpha^2 e^{-i\sigma\pi}u(x)w_0(x) + \beta^2 e^{-i\sigma\pi}\left[\dot{w}_0(x) - i\sigma w_0(x)\right], \quad x \in (-1,1), \quad \sigma \in \mathbb{R}, \tag{12}$$

$$\overline{w}_1(\pm 1, \sigma) = \left.\frac{d^2\overline{w}_1(x,\sigma)}{dx^2}\right|_{x=\pm 1} = 0, \quad \sigma \in \mathbb{R}, \tag{13}$$

where

$$\mathcal{F}[f] \equiv \overline{f}(\sigma) = \int\limits_{-\infty}^{\infty} f(t)e^{i\sigma t}dt, \quad \sigma \in \mathbb{R},$$

is the Fourier image of function $f(t)$, σ is the spectral parameter of the Fourier transform. In the derivation of (12) we have used well-known relations $\mathcal{F}[\delta(t-t_0)] = e^{i\sigma t_0}$, $|\lambda|\delta(t) = \delta(|\lambda|t)$, $\theta(|\lambda|t) = \theta(t)$, $\lambda \neq 0$ [26].

It is characteristic for ordinary differential Eq. (12) that the control function occurs not only in its coefficients but also in the right-hand side. However, introducing the auxiliary function $\overline{w}(x,\sigma) = i\sigma e^{i\sigma\pi}\overline{w}_1(x,\sigma) + w_0(x)$, the control function can be excluded from the right-hand side:

$$\frac{d^4\overline{w}}{dx^4} - \sigma^2\left[\beta^2 + i\frac{\alpha^2}{\sigma}u(x)\right]\overline{w} = \Pi(x,\sigma), \quad x \in (-1,1), \quad \sigma \in \mathbb{R},$$

$$\overline{w}(\pm 1,\sigma) = \left.\frac{d^2\overline{w}(x,\sigma)}{dx^2}\right|_{x=\pm 1} = 0, \quad \sigma \in \mathbb{R}, \tag{14}$$

$$\Pi(x,\sigma) = i\sigma\left[\frac{Pe^{i\sigma\left(\frac{x+1}{v}-\pi\right)}}{v}[\theta(x+1) - \theta(x+1-v\tau)] + \beta^2\dot{w}_0(x)\right] + w_0^{IV}(x).$$

In the derivation of boundary conditions (14) we have used conjugation conditions for the boundary conditions with initial and final conditions.

We now apply the Bubnov–Galerkin procedure. As approximating functions we take the orthonormal (in $[-1,1]$) system $\{\sin(\pi kx)\}_{k=1}^n$ and get

$$\sum_{k=1}^{n}\Lambda_{km}(\sigma)\alpha_k(\sigma) = \Omega_m(\sigma), \quad m = \overline{1;n}, \tag{15}$$

where $\Lambda_{km}(\sigma) = \left[(\pi k)^4 - \beta^2\sigma^2\right]\delta_k^m - i\sigma\alpha^2 J_{km}[u]$,

$$\delta_k^m = \delta_m^k = \int\limits_{-1}^{1}\sin(\pi kx)\sin(\pi mx)dx = \begin{cases}1, & k = m\\ 0, & k \neq m,\end{cases}$$

is the Kroneker symbol,

$$J_{km}[u] = \int\limits_{-1}^{1}u(x)\sin(\pi kx)\sin(\pi mx)dx, \quad \Omega_m(\sigma) = \int\limits_{-1}^{1}\Pi(x,\sigma)\sin(\pi mx)dx.$$

Representing the solution of the system of linear algebraic Eqs. (15) as

$$\alpha_k(\sigma) = \frac{\Delta_k(\sigma)}{\Delta(\sigma)}, \quad k = \overline{1;n}, \tag{16}$$

where Δ is the main and Δ_k is the auxiliary determinant of system (15), and taking into account that $\operatorname{supp} w_1 \subseteq [-1,1]\times[-\pi,\pi]$, to find the function in question we can use a method that has already become traditional [15]. Namely, according to the well-known Wiener–Paley–Schwartz theorem [15, p. 198; 26, p. 125] function $\overline{w}(x,z) = ize^{iz\pi}\overline{w}_1(x,z) + w_0(x)$, $z \in \mathbb{C}$, is a entire function of exponential type, and, consequently, at points where the denominator of (16) continued to the entire complex plane is zero

$$\Delta(z) = 0 \tag{17}$$

for all $k = \overline{1;n}$, the numerators of the fraction (16) extended to the entire complex plane must also turn to zero:

$$\Delta_k(z) = 0, \quad k = \overline{1;n}. \tag{18}$$

It is easy to show that Eqs. (18) under (17) hold for all $k = \overline{1;n}$ simultaneously [15], i.e., to find the function in question we only have to consider one of them.

The form of the main determinant $\Delta(\sigma)$ implies that Eq. (17) holds only at a finite number of points because decomposing this determinant with respect to σ we get a polynomial of finite degree $2n$. Moreover, together with $z = \sigma + i\varsigma$ Eq. (17) will also satisfy $z = -\sigma + i\varsigma$.

Equations (18) for (17) yield necessary constraints of equality types (6) on the functionals $J_{km}[u]$, and hence the control function can be found explicitly. It is known [15–19] that the function in question $u^o \in \mathcal{U}$ satisfying the resulting problem of moments and minimizing the integral functional (9) is

$$u^o(x) = \sum_{j=1}^{N} \delta(x - x_j^o), \quad x \in (-1, 1), \tag{19}$$

corresponding to the pointwise distribution of dampers under the beam, and points in the support of the function in question $-1 < x_j^o < x_{j+1}^o < 1$ correspond to locations of dampers and depend on parameters $\alpha^2, \beta^2, v, \tau, P, w_0, \dot{w}_0, T$. Note that since the optimal solution (19) is not unique [16–19], the number of defining points N from inclusion condition $\{x_j^o\}_{j=1}^N \subset (-1, 1)$ cannot be uniquely determined in the general case. It can be defined in the problem setting.

Table 1. $\alpha^2 = 0.01$

P	v	τ	x_j^o
0.5	0.25	2.5	$x_1^o = 0.3864$, $x_2^o = 0.4498$
0.5	1	5	$x_1^o = -0.8534$, $x_2^o = -0.2306$, $x_3^o = 0.2101$
1	0.5	5	$x_1^o = -0.308$, $x_2^o = -0.2601$, $x_3^o = 0.2436$
1	1	6	$x_1^o = -0.43$, $x_2^o = -0.125$, $x_3^o = 0.88$
5	0.75	3	$x_1^o = -0.5$, $x_2^o = -0.1853$, $x_3^o = 0.1879$
5	1	6	$x_1^o = -0.7203$, $x_2^o = -0.1684$, $x_3^o = 0.6869$

Table 2. $\alpha^2 = 0.5$

P	v	τ	x_j^o
0.5	0.25	3	$x_1^o = -0.1301$, $x_2^o = 0.2885$, $x_3^o = 0.6523$
0.5	1	5	$x_1^o = -0.5$, $x_2^o = 0.4853$, $x_3^o = 0.6122$
1	0.25	3	$x_1^o = -0.8143$, $x_2^o = -0.141$
1	1	5	$x_1^o = -0.4812$, $x_2^o = 0.2431$
5	0.5	3	$x_1^o = -0.8452$, $x_2^o = -0.54$, $x_3^o = 0.03$
5	1	6	$x_1^o = -0.6227$, $x_2^o = -0.2184$, $x_3^o = 0.5613$, $x_4^o = 0.6164$

Table 3. $\alpha^2 = 4$

P	v	τ	x_j^o
0.5	0.25	3	$x_1^o = 0.05$
0.5	1	5	$x_1^o = -0.6534$
1	0.25	5	$x_1^o = -0.1941$, $x_2^o = 0.261$
1	1	6	$x_1^o = -0.7$, $x_2^o = -0.0616$
5	0.5	3	$x_1^o = -0.0644$
5	1	6	$x_1^o = -0.7815$, $x_2^o = -0.0412$, $x_3^o = 0.5984$

The problem of computing points $\{x_j^o\}_{j=1}^N$ can be reduced to a nonlinear programming problem by substituting (19) into the resulting moment equations. As a results, with respect to the points $\{x_j^o\}_{j=1}^N \in (-1, 1)$ we get a system of nonlinear constraints of equality type to which we add constraints of inequality type $-1 < x_j^o < x_{j+1}^o < 1$. The resulting system can be solved by efficient numerical methods of nonlinear programming [27].

We have performed computations in the case of $n = 3$, $T = 2\pi$, $\beta^2 = \pi^4$, $w_0(x) = \sin(\pi x)$, $\dot{w}_0(x) = 0$, for different values of dimensionless parameters α^2, P, v, and τ; our main results are shown in Tables 1–3.

These tables show that the number of defining points decreases as the parameter α_*^2 that characterizes the dampers' viscosity increases. We can also note that the number of defining points increases as parameters P_* and v_* that characterize respectively the absolute value and velocity of the moving impulse increase for the same values of parameter τ_* that characterizes the moment when the load is separated.

4. CONCLUSION

In this work, we apply the Bubnov–Galerkin method to a study of bilinear control systems. Using final conditions, we reduce the problem to solving a finite-dimensional problem of moments (see (6)). As an application of the proposed approach, we have studied the problem of damping bending oscillations of a finite beam subject to moving load influence. The damping is done by choosing an optimal distribution function for viscous dampers under the beam. We have formulated this problem mathematically as bilinear differential partial differential equations of order four (see (7)). We have found that the optimal in the sense of a control function integral (see (9)) is a discrete distribution of dampers (see (19)) which is defined by specifying defining points that characterize the location of dampers and are computed from nonlinear constraints of equality type (see (18)). We give a numerical example that has shown that as the viscosity coefficient of the dampers increases the number of defining points decreases, and it increases as the intensity and speed of the impulses' motion increases.

The proposed procedure can be applied to solve control problems for bilinear systems of dimension $\geqslant 2$ and systems that are nonlinear with respect to the control function but linear with respect to the state function.

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