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# Optimal Sub-band Methods for Analysis and Synthesis of Finite-duration Signals

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**Abstract**—The portion of energy of a finite-duration signal hitting the given frequency interval was shown to be suitable as a basis for construction of the optimal methods of their analysis and synthesis. The relations obtained define directly these portions of energy in the signal space and enable one to formulate the variational conditions for their optimal processing. The problems of optimal detection (filtration) of the additive signal components and synthesis of signals featuring the maximal/minimal concentration of energy within the given frequency ranges were formulated and solved.

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## 1. INTRODUCTION

The notion of signal as the function of an argument—most frequently, time—is widely used in cybernetics at control of various processes, information transmission, processing of the location information, and so on. Two main aspects of their processing can be singled out such as (i) analysis of signals with the aim of identifying some distinctions that are essential in terms of the applied problem at hand and (ii) signal synthesis on the basis of some criteria for optimal operation of technical systems, for example, at actions upon the control plants, transmission and processing of information in radio engineering and communication, testing in diverse measurement systems, and so on.

The signal processing procedures rely on various models among which the most commonly encountered are the frequency representations [1, 2] enabling one to use the Fourier transforms for description of the distribution of energy within the frequency domain which is often used at handling the aforementioned problems because it has an important physical sense. For example, high concentration of energy within some frequency interval is usually indicative of the presence of quasi-periodic components in the signal, and at signal synthesis the level of the concentration attained is often used as an optimality criterion, especially in radio engineering and communication (narrow-band and wide-band signals).

Therefore, at signal processing it is advisable to act from the standpoint of certain decomposition of the frequency axis into a totality of frequency intervals (bands) of which some have limited size. In the publications on signal and image processing, such methods are traditionally referred to as the “sub-band” ones [3–5].

It is possible to indicate a wide range of the applied problems of signal processing where the sub-band representations are adequate. It is namely these circumstances that stimulated studies in the sub-band modeling which gave and still are giving rise to the methods of sub-band analysis and synthesis corresponding to one or another conception of the quality of signal processing problems.

Presently, the most important tools of sub-band analysis are represented by various modifications of the discrete Fourier transform (DFT) realizable by fast algorithms and the FIR-filtration which also underlies the methods of wavelet analysis that became popular in the recent years [6] as is testified by the fact that the libraries of mathematical packets of applied programs include the corresponding program modules [7].

Without going into details, we just note that the current sub-band methods for signal processing are far from being optimal in terms of the criteria reflecting the sense of the problems under consideration. For example, most frequently the pulse characteristics of filters are calculated from the criteria of accuracy of approximation of the rectangular frequency characteristics. At the same time, the criterion for accuracy of approximation of the segments of the Fourier transform of the original signal within the given frequency intervals seems to be more natural.

In its turn, DFT represents in essence the coefficients of decomposition with respect to the orthogonal system of basic vectors whose sense, generally speaking, is not quite clear. The same conclusion is valid for the coefficients of decompositions at realization of the wavelet analysis [6, 7],

The present paper established some results on the construction of the optimal methods of sub-band analysis and synthesis of the finite-duration signals on the basis of the energy criteria that are measures of the error of processing. At that, the continuous signals are considered first with the aim of reaching theoretical generality.

## 2. OPTIMAL METHOD FOR ANALYSIS OF THE SIGNAL ENERGY DISTRIBUTION IN FREQUENCY INTERVALS

Let $x(t)$, $t \in [0, T]$, be a continuous signal with limited energy

$$||x||^2 = \int_0^T x^2(t)dt < \infty, \tag{2.1}$$

because there exists a Fourier transform

$$X(\omega) = \int_0^T x(t)\exp(-j\omega t)dt, \ j = \sqrt{-1}, \tag{2.2}$$

and the Parceval equality is valid [7] which from the point of view of the sub-band analysis/synthesis is representable as

$$||x||^2 = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega/2\pi = \sum_{r=0}^{\infty} S_r(x), \tag{2.3}$$

where $S_r(x)$ the portions of energy

$$S_r(x) = \int_{\omega \in \Omega_r} |X(\omega)|^2 d\omega/2\pi, \tag{2.4}$$

hitting the frequency intervals like

$$\Omega_r = [-\Omega_{2r}, \ -\Omega_{1r}) \cup [\Omega_{1r}, \Omega_{2r}). \tag{2.5}$$

We note that decomposition of the frequency axis can be arbitrary, but the conditions for internal boundaries must be satisfied:

$$\Omega_{2r} = \Omega_{1,r+1}; \quad r \geqslant 0; \quad \Omega_{10} = 0. \tag{2.6}$$

It seems only natural to regard the energy portions

$$P_r(x) = S_r(x)/||x||^2 \tag{2.7}$$

as the basic sub-band energy characteristic of the signal. It will be shown below that it can be used to construct optimal methods for signal analysis and synthesis.

It is required to represent first of all integrals (2.4) in the form of explicit dependence on the processed signals as time functions, including the sought ones. Only then it becomes possible to formulate the variational optimization conditions.

The desired representation is obtained by substituting definition (2.2) into (2.4) and performing simple transformations

$$S_r(x) = \int_0^T \int_0^T A_r(t_1 - t_2) x(t_1) x(t_2) dt_1 dt_2, \tag{2.8}$$

where $A_r(t)$ is the sub-band kernel given by

$$A_r(t) = \int_{\omega \in \Omega_r} \exp(-jt\omega) d\omega / 2\pi = 2\cos(\omega_r t) \sin(\Delta_r t/2)/\pi t. \tag{2.9}$$

Here and in what follows,

$$\omega_r = (\Omega_{2r} + \Omega_{1r})/2; \quad \Delta_r = \Omega_{2r} - \Omega_{1r}. \tag{2.10}$$

One can readily see that the relation (2.8) allows one to calculate with arbitrary precision integrals like (2.4) without calculating the Fourier transforms with application of the quadrature formulas. It is of interest to generalize this result to some totality $R$ of frequency intervals

$$\Omega_R = \bigcup_{r \in R} \Omega_r, \tag{2.11}$$

to which gets the total part of the signal energy

$$S_R(x) = \sum_{r \in R} S_r(x). \tag{2.12}$$

One can easily see that in this case (2.8) is generalized and goes over to

$$S_R(x) = \int_0^T \int_0^T A_R(t_1 - t_2) x(t_1) x(t_2) dt_1 dt_2, \tag{2.13}$$

where

$$A_R(\cdot) = \sum_{r \in R} A_r(\cdot), \tag{2.14}$$

identical arguments being put in the parentheses.

Relations (2.8) and (2.13) define, obviously, the precision-optimal method for solution of the problem of calculation of the portions of signal energy getting into the given frequency intervals or in their not necessarily solid totality.

## 3. SUB-BAND SIGNAL OPTIMIZATION

We notice that the sub-band kernel (2.9) generalizes the kernel given by

$$A_{r0}(t) = \sin(\Delta_r t/2)/\pi t \tag{3.1}$$

and considered in the works on the eigenfunctions of the Fourier transform (see, for example, [1, 2]). In the present paper, this generalization was established on the basis of definitions (2.4) (or (2.12)) and (2.2), which allows one to make use of its properties not only for construction of bases that are full in the sense of the basis space $L_2$, but also to solve other problems.

First of all, we notice that the positive definiteness of the kernel (2.9) follows directly from definition (2.4) and representation (2.8). It also satisfies conditions for expansion into a uniformly converging series [8]

$$A_r(t_1 - t_2) = \sum_{k=1}^{\infty} \lambda_k^r g_k^r(t_1) g_k^r(t_2), \tag{3.2}$$

where $\lambda_k^r$ and $g_k^r(t)$ are, respectively, the eigenvalues and functions of the kernel (integral operator) such that the following conditions are satisfied:

$$\lambda_k^r g_k^r(t_1) = \int_0^T A_r(t_1 - t_2) g_k^r(t_2) dt_2; \tag{3.3}$$

$$(g_k^r, g_i^r) = \int_0^T g_k^r(t) g_i^r(t) dt = \delta_{ik}, \tag{3.4}$$

where $\delta_{ik}$ is the Kronecker symbol, and

$$\lambda_k^r > 0. \tag{3.5}$$

It is assumed below without loss of generality that the eigenvalues are arranged in the descending order

$$\lambda_k^r > \lambda_{k+1}^r. \tag{3.6}$$

Obviously, the conditions for existence of the Fourier transform of the eigenfunctions

$$G_k^r(z) = \int_0^T g_k^r(t) \exp(-jzt) dt$$

are met. According to the property of orthonormality of (3.4), it follows at that from the Parseval equality that the relations

$$\int_{-\infty}^{\infty} |G_k^r(z)|^2 dz/2\pi = 1$$

are valid.

Therefore, with regard for (2.9) the important inequality

$$\lambda_k^r = \int_{\omega \in \Omega_r} |G_k^r(\omega)|^2 d\omega/2\pi \leqslant 1 \tag{3.7}$$

is obtained directly from definition (3.3).

The eigenvalues are, thus, equal numerically to the portions of energy of the corresponding eigenfunctions hitting the frequency interval and do not exceed 1, although can be close to it, which is important for getting the highest concentration of the frequency energy.

By assuming that

$$\alpha_k^r = (g_k^r, x) = \int_0^T g_k^r(t)x(t)dt \tag{3.8}$$

and substituting (3.2) into (2.8), one can readily reduce (2.8) to

$$S_r(x) = \sum_{k=1}^{\infty} \lambda_k^r (\alpha_k^r)^2, \tag{3.9}$$

the representation

$$x(t) = \sum_{k=1}^{\infty} \alpha_k^r g_k^r(t) \tag{3.10}$$

being valid for any limited-energy signal.

By assuming that $t_1 = t_2 = t$ in (3.2) and integrating with respect to time and due regard for definition (2.9), we get an equality for the sum of eigenvalues

$$\sum_{k=1}^{\infty} \lambda_k^r = 2T\Delta_r, \tag{3.11}$$

from which it follows that only their minor portion will be other than zero. Therefore, in (3.9) one can leave a finite number of terms

$$S_r(x) = \sum_{k=1}^{J_r} \lambda_k^r (\alpha_k^r)^2, \tag{3.12}$$

and, as the computer experiments demonstrated, assume that

$$J_r = 2[T\Delta_r/2\pi] + 4, \tag{3.13}$$

because the equalities

$$\lambda_k^r = 0, \quad k > J_r \tag{3.14}$$

are satisfied with high degree of precision.

It is easy to understand that all the considered properties of the sub-band kernels including (3.7) and representation (3.12) are valid also for the generalization defined by (2.12), (2.13), and (2.14), that is,

$$S_R(x) = \sum_{k=1}^{J_{R_r}} \lambda_k^R (\alpha_k^R)^2, \tag{3.15}$$

the number of terms in (3.15) obeying the equality

$$J_R = \sum_{r \in R} J_r.$$

This allows one to formulate the following assertion.

**Assertion 1.** *Solution of the variational problem of seeking the optimal signal* $y(t)$, $t \in [0, T]$ *satisfying the condition*

$$S_R(y) = \max S_R(x) \ \forall ||x||^2 = c^2, \tag{3.16}$$

*is given by*

$$y(t) = cg_1^R(t), \quad t \in [0, T], \tag{3.17}$$

*the portion of energy getting in the united frequency interval* (2.11) *being equal to the corresponding eigenvalue of the sub-band kernel.*

Validity of Assertion 1 follows from representation (3.15) and the conditions for descending orderliness of the eigenvalues and orthonormality of the eigenfunctions of the total kernels.

In some cases such as solution of the electromagnetic compatibility problems, the result opposite in a sense, namely Assertion 2, is of interest.

**Assertion 2.** *A signal like*

$$z(t) = cg_k^R(t), \quad t \in [0, T] \tag{3.18}$$

*have close to zero portion of energy in the united frequency interval* (2.11) *if the condition* $\lambda_k^r \approx 0$ *is satisfied.*

## 4. OPTIMAL SUB-BAND SEPARATION OF THE ADDITIVE COMPONENTS (FREQUENCY FILTRATION)

We consider the problem of dividing the signal into components

$$x(t) = y_R(t) + u_R(t), \quad t \in [0, T], \tag{4.1}$$

extraction of a component whose Fourier transform satisfies the condition

$$\begin{gathered} Y_R(\omega) = \int_0^T y_R(t) \exp(-j\omega t) dt = X(\omega), \quad \omega \in \Omega_R, \\ Y_R(\omega) \equiv 0, \quad \omega \notin \Omega_R \end{gathered} \tag{4.2}$$

being regarded as an ideal.

It is clear that this "ideal" condition cannot be satisfied under finite duration of the signal. Therefore, it is possible to use only a certain approximation whose precision can be naturally represented as the functional

$$\begin{aligned} F_R(x, y_R, w) = w & \int_{\omega \in \Omega_R} |X(\omega) - Y_R(\omega)|^2 d\omega / 2\pi \\ & + (1 - w) \int_{\omega \notin \Omega_R} |Y_R(\omega)|^2 d\omega / 2\pi, \end{aligned} \tag{4.3}$$

where the parameter $w$ satisfying the inequality

$$0 < w < 1 \tag{4.4}$$

defines the weights of the contributions of the components of the error of execution of (4.2).

Obviously, optimization lies in minimizing the error functional (4.3) which can be rearranged in

$$F_R(x, y_R, w) = wS_R(x - y_R) + (1 - w)(||y_R||^2 - S_R(y_R))$$

using a representation like (2.13).

Using a decomposition in kernel eigenfunctions like (2.14)

$$x(t) = \sum_{k=1}^{\infty} \alpha_k^R g_k^R(t),$$
$$y_R(t) = \sum_{k=1}^{\infty} \beta_k^R g_k^R(t), \tag{4.5}$$

after tedious but evident transformations we get the necessary and sufficient conditions for minimum of the error functional (4.3)

$$w \sum_{k=1}^{\infty} \lambda_k^R \alpha_k^R g_k^R(t) \equiv \sum_{k=1}^{\infty} \left(1 - w + (2w - 1)\lambda_k^R\right) \beta_k^R g_k^R(t), \quad t \in [0, T].$$

Relations for the coefficients of the series to the right and left of the equality sign are established from the uniform convergence of these series:

$$w\lambda_k^R \alpha_k^R = \left(1 - w + (2w - 1)\lambda_k^R\right) \beta_k^R, \quad k = 1, 2, \ldots. \tag{4.6}$$

We first notice that if equal weights are used for the components of the error functional (4.4) where

$$w = 0.5, \tag{4.7}$$

then the equalities

$$\lambda_k^R \alpha_k^R = \beta_k^R, \quad 1 \leqslant k \tag{4.8}$$

follow from (4.6).

Additionally, it is advisable to use these equalities for the totality $1 \leqslant k \leqslant K$ of all eigenfunctions corresponding to the eigenvalues that are very close to one where

$$\lambda_k^R = 1 - \varepsilon, \quad \varepsilon/\lambda_k^R \ll 1, \quad 1 \leqslant k \leqslant K. \tag{4.9}$$

In the general case, therefore, validity of Assertion 3 follows from (4.6).

**Assertion 3.** *The desired signal component which is optimal in the sense of the minimum of functional* (4.3) *is given by*

$$y_R(t) = w \sum_{k=1}^{\infty} \lambda_k^R \alpha_k^R / (1 - w + (2w - 1)\lambda_k^R) \bullet g_k^K(t).$$

With regard for the property of (3.15), one can suggest a sufficiently precise approximation

$$y_R(t) = w \sum_{k=1}^{J_R} \lambda_k^R \alpha_k^R / \left(1 - w + (2w - 1)\lambda_k^R\right) \bullet g_k^R(t) \tag{4.10}$$

which is used in what follows. Here,

$$\alpha_k^R = \int_0^T g_k^R(t)x(t)dt.$$

In the special case of (39), we obtain with allowance for (40) that

$$y(t) = \sum_{k=1}^{J_R} \lambda_k^R \alpha_k^R \bullet g_k^R(t), \tag{4.11}$$

or correspondingly

$$y_R(t_1) = \int_0^T A_R(t_1 - t_2)x(t_2)dt_2, \tag{4.11$'$}$$

which does without calculating the eigenvalues and eigenfunctions.

By substituting (2.14), we obtain with regard for definition (2.9) that

$$y_R(t_1) = \sum_{r \in R} \int_{\omega \in \Omega_r} \exp(j\omega t_1)X(\omega)d\omega/2\pi.$$

Therefore, the component defined by (4.11$'$) depends only on the segments of the Fourier transform of the original signal from the totality of the frequency intervals as (2.11). This property seems to be especially essential for extraction of sub-band low-energy components in the presence of components with exceeding energy within the adjacent frequency intervals because other filters exert influence on the results of filtration owing to the transient bands.

Of interest is also the relation defining the minimum of the functional like (4.3) which can be obtained using representation (4.10)

$$\min F(x, y_R, w) = w(1-w)\sum_{k=1}^{J_R} \lambda_k^R(1-\lambda_k^R)(\alpha_k^R)^2/(1-w+(2w-1)\lambda_k^R). \tag{4.12}$$

It is evident that here the minimal value of the right side is reached if either $w = 0$ (in the absence of eigenvalues satisfying (4.9)) or $w = 1$. Both cases are senseless because in the former case (4.5) gives zero (degenerate case) and the latter one corresponds to filtration ($x_1(t) \equiv x(t)$, $t \in [0, T]$). That is why they are disregarded (see (4.4)).

It is also clear that it is difficult to determine the weight if the right side of (4.12) reaches the maximal value (maximin) because it depends on the projections of the signal on the eigenfunctions of the total kernel. At the same time, it deserves noting that the coefficient before the sum in the right side of (4.11) reaches maximum in the case of equal weight of (4.7). At that we get a relation for the values of the error functional which are close to the maximum

$$\min F(x, y_R, 0.5) = 0.5\sum_{k=1}^{J_R} \lambda_k^R(1-\lambda_k^R)(\alpha_k^R)^2. \tag{4.13}$$

It also deserves noting that the weight of squared projections on the eigenfunctions in the sum of (4.13) does not exceed 0.25 and reaches maximum for $\lambda_k^R = 0.5$.

## 5. SUB-BAND PROCESSING OF DISCRETE SIGNALS

All previous relations are readily reproduced for the discrete finite-duration signals (finite-dimension vectors) $\vec{x} = (x_1, \ldots, x_N)'$ where the prime stands for transposition. It is assumed here for simplicity that the vector components represent the equidistant readings of some continuous signal. Therefore, in what follows we imply the so-called normalized frequencies. One can easily prove [9] that the Fourier transform of the discrete signal

$$X(v) = \sum_{k=1}^{N} x_k \exp(-jv(k-1)) \tag{5.1}$$

is a periodic function with the period $2\pi$ of the circular frequency $v$ normalized to the discretization frequency. That is why the sub-band analysis and synthesis of signals is realizable only within the basic segment of the frequency axis $-\pi \leqslant v < \pi$ and having in mind the frequency intervals like

$$V_r = [-V_{2r}, \ -V_{1r}) \cup [V_{1r}, V_{2r}), \quad 0 \leqslant V_{1r} < V_{2r} \leqslant \pi. \tag{5.2}$$

It is also clear that if nonoverlapping intervals are used, then the number of intervals covering the entire segment is finite.

In what follows, some of the above notation are retained wherever this does not give rise to contradictions. We assume that

$$S_R(\vec{x}) = \int\limits_{\omega \in V_R} |X(\omega)|^2 d\omega / 2\pi, \tag{5.3}$$

where as in (2.11) consideration is given to the union of a finite number of nonoverlapping frequency intervals

$$V_R = \bigcup_{r \in R} V_r \tag{5.4}$$

of the total width

$$\Delta_R(x) = \sum_{r \in R} \Delta_r = \sum_{r \in R} (V_{2r} - V_{1r}).$$

Then, the part of the energy of signal (5.3) that hits this union of frequency intervals obeys the quadratic form

$$S_R(\vec{x}) = \sum_{r \in R} S_r(\vec{x}) = \vec{x}' A_R \vec{x}, \tag{5.5}$$

where $A_R$ is the square matrix of obvious dimension

$$A_R = \sum_{r \in R} A_r; \tag{5.6}$$

$$A_r = \{a_{ik}^r\}, \quad a_{ik}^r = \int\limits_{v \in V_r} \exp(-jv(i-k)) dv / 2\pi, \quad i, k = 1, \ldots, N. \tag{5.7}$$

It is only natural to call the addends in the right side of (5.6) the sub-band matrices.

It is clear that the quadratic form (5.5) can be calculated with arbitrary accuracy.

In view of its positive definiteness and symmetricity, matrix (5.6) has the full system of orthonormalized eigenvectors [10] corresponding to the positive eigenvalues of which only

$$J_R = 2[N\Delta_R/2\pi] + 4 \tag{5.8}$$

can be regarded as other than zero, whereas for the rest of them the equalities

$$\lambda_k^R = 0, \quad k > J_R \tag{5.9}$$

are satisfied with high precision, bearing in mind that the eigenvalues are arranged in the descending order.

From the definitions of the eigenvalues and eigenvectors, one can easily determine with regard for (5.7) an analogue of inequality (3.7) and the above Assertions 1 and 2 formulated for the discrete signals.

Assertion 3 can be reformulated in turn in a form doing without the eigenvalues.

**Assertion 4.** *In the decomposition*

$$\vec{x} = \vec{y}_R + \vec{u}_R,$$

*the vector*

$$\vec{y}_R = w((1-w)I + (2w-1)A_R)^{-1}A_R\vec{x}. \tag{5.10}$$

*is optimal in the sense of minimal error functional*

$$F_R(\vec{x}, \vec{y}_R, w) = wS_R(\vec{x} - \vec{y}_R) + (1-w)(||\vec{y}_R||^2 - S_R(\vec{y}_R)). \tag{5.11}$$

*Here $I$ is the identity matrix of the corresponding dimension and $||\cdot||$ denotes the Euclidean norm of the corresponding vector.*

We note that for a changeless combination of the frequency intervals like (5.4) the matrices in (5.10) can be generated in advance, which allows one to reduce the computational overhead arising at repeated use of filtration.

Obviously, the relation for the desired component

$$\vec{y}_R = A_R\vec{x} \tag{5.12}$$

can be obtained by satisfying condition (4.7).

Then, with regard for definition (5.7) one can establish an analog of (4.13) suggesting that the obtained vector is defined completely by the segments of the Fourier transform of the original signal from the totality of the frequency intervals like (5.4).

Of interest is the following property of the extracted components which is somewhat unique for the frequency filtration.

Let the conditions

$$V_{1r} = V_{2,r-1}, \quad V_{10} = 0, \quad V_{2M} = \pi.$$

be satisfied for the frequency intervals (5.2). Then, it is easy to establish on the basis of definition (5.7) that the sum of the sub-band matrices is equal to the identity matrix (property of additivity of the sub-band matrices)

$$\sum_{r=0}^{M} A_r = I,$$

whence it follows that the equality

$$\vec{x} = \sum_{r=0}^{M} \vec{y}_r = \sum_{r=0}^{M} A_r\vec{x}$$

defining a simple method of restoring the original vector from the extracted components is valid.

## 6. COMPUTER EXPERIMENTS

Figures 1–4 demonstrate the dependencies of the optimal components like (5.12) only on the segments of the Fourier transform from the corresponding frequency interval acquired as the result of the computer experiments. At that, processed were the model data, the results of the band FIR filtration within the same frequency intervals were used for comparison.

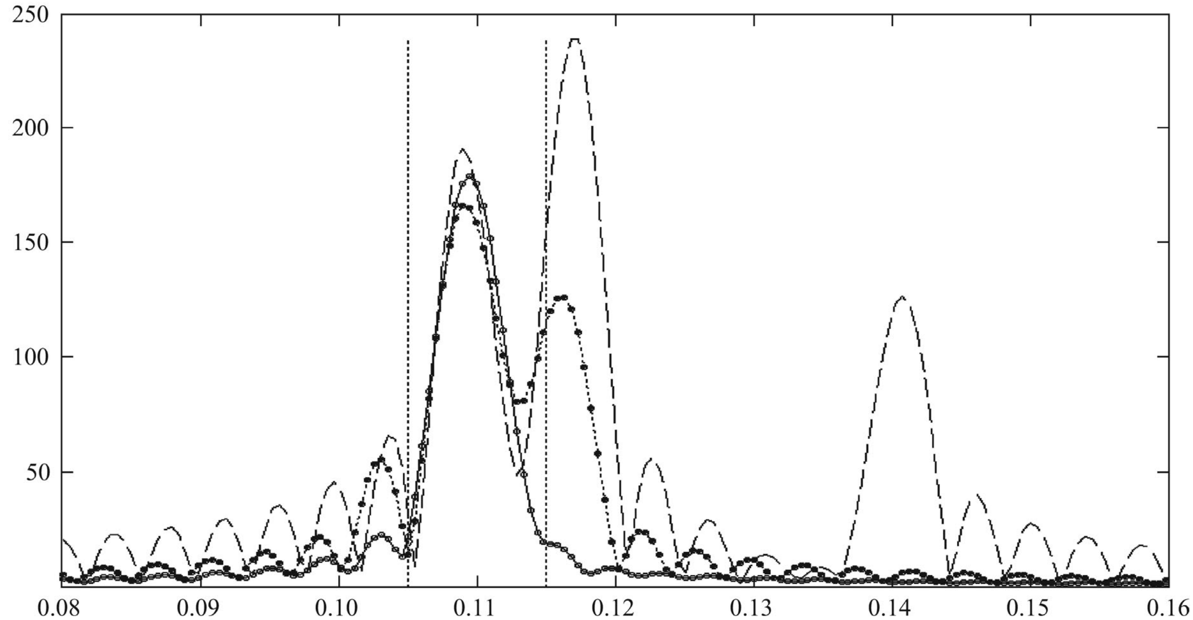


**Fig. 1.** Modules of the Fourier transforms (axis of ordinates) of the original signal (broken line) and output sequences of the FIR filter (line marked by "point") and optimal filter (line marked by "circle") vs. the normalized frequency (abscissa) within the frequency range $v_1 = 0.105\pi$; $v_2 = 0.115\pi$ (vertical broken lines).

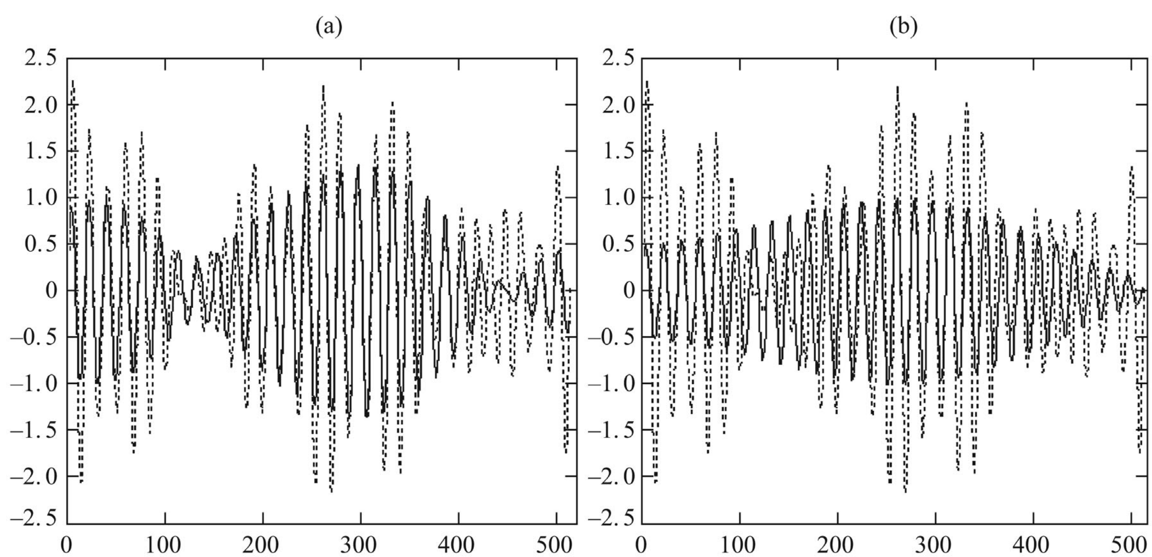


**Fig. 2.** Original signal (broken line, axis of ordinates) and the output sequences (solid line) of (a) FIR filter and (b) optimal filter vs. the numbers of readings (abscissa) within the frequency interval $v_1 = 0.105\pi$; $v_2 = 0.115\pi$.

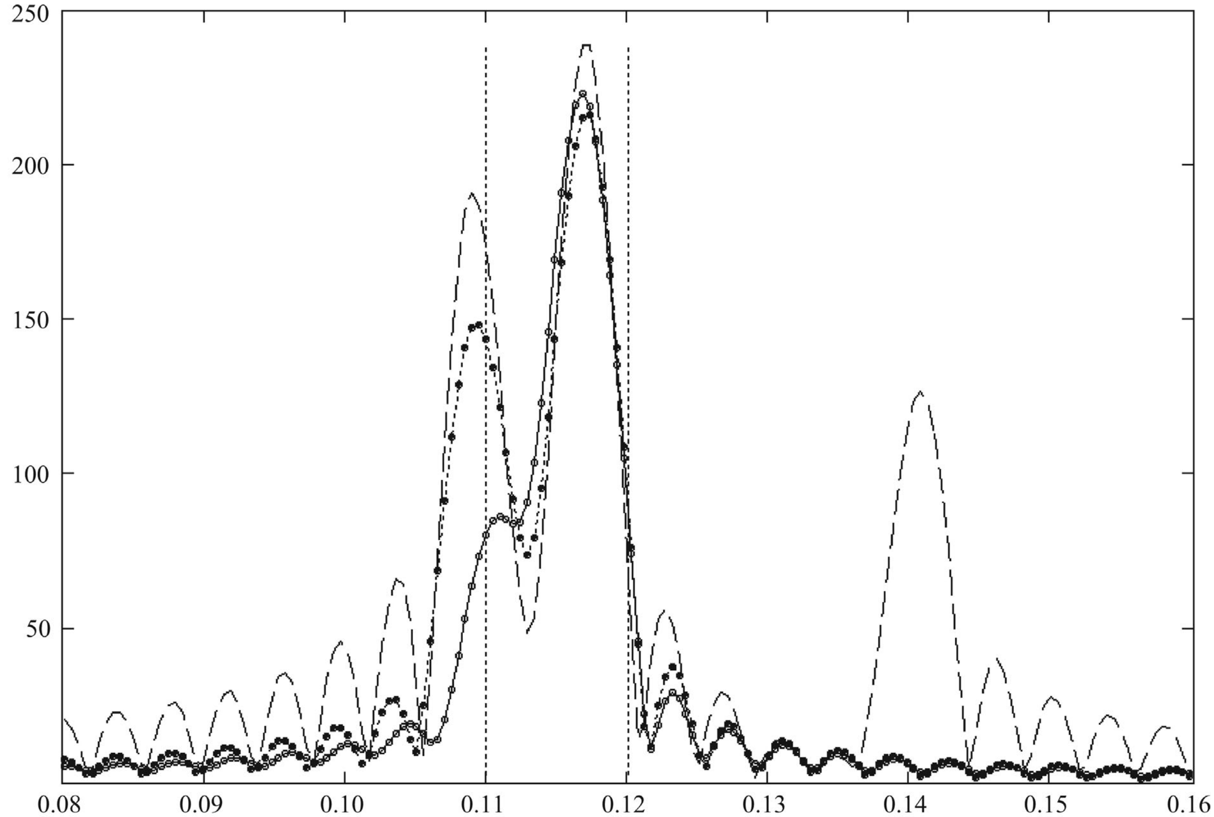


**Fig. 3.** Modules of the Fourier transform (axis of ordinates) of the original signal (broken line) and output sequences of the FIR filter (line marked by "point") and optimal filter (line marked by "circle") vs. the normalized frequency (abscissa) within the frequency interval $v_1 = 0.11\pi$; $v_2 = 0.125\pi$ (vertical broken lines).

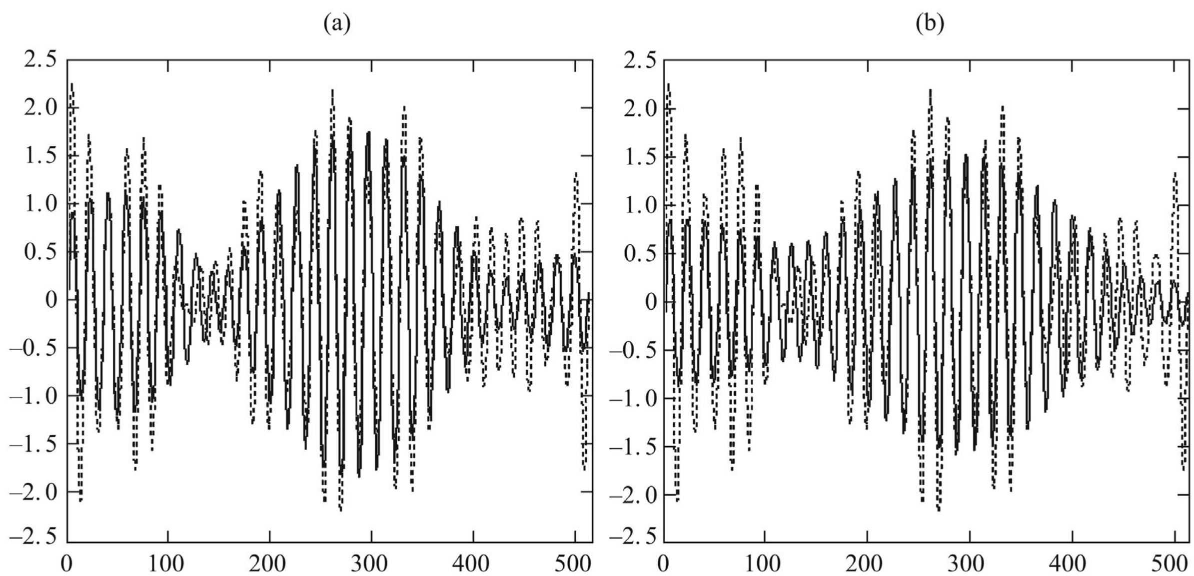


**Fig. 4.** Original signal (broken line, axis of ordinates) and the output sequences (solid line) of (a) FIR filter and (b) optimal filter vs. the numbers of readings (abscissa) writhing the frequency interval $v_1 = 0.105\pi$; $v_2 = 0.115\pi$.

The values of the model signal were generated using the relation

$$x(k) = 0.8\sin(\omega_1 k) + \sin(\omega_2 k) + 0.5\sin(\omega_3 k),$$

where $\omega_1 = 0.3461$; $\omega_2 = 0.3682$; $\omega_3 = 0.4418$.

These values of frequencies enabled one to study various aspects of filtration and, first of all, the impact of the energy of the original signal on the results of filtration beyond the frequency interval. Dimensions (durations) of the segments of the filtered $\vec{x}$ and resulting $\vec{y}$ vectors are the same and equal to $N = 512$. Durations of the pulse characteristic of the FIR filters were selected as 1024 values, which enabled one to reach a very high degree of approximation to the rectangular frequency characteristic. Some of the most descriptive results are depicted in Figs. 1–4.

In the case at hand, the impact of the additional energy (right boundary of the frequency interval) on the result of FIR filtration (Fig. 1) manifests itself in beats (Fig. 2).

In this case, the impact of energy from the neighbor band (left boundary of the frequency interval) on the result of FIR filtration (Fig. 3) also manifests itself in a somewhat greater beats than those of the output sequence of the optimal filter (Fig. 4).

If there are no powerful components in the adjacent frequency intervals, the differences in the results of filtration are negligible.

## 7. CONCLUSIONS

The present paper demonstrated that the portion of the finite-duration signal energy hitting the given totality of the frequency intervals can be used as an optimization criterion for development of the processing methods. Representations of this characteristic were obtained directly in the signal definition domain, which enables one to formulate and solve the corresponding variational problems. Formulated and solved were the problem of optimal signal processing such as calculation of the precise values of the energy portions hitting the given totality of the frequency intervals, design of signals with maximal or minimal concentration of energies in the totality of frequency intervals and determination of the additive components with the Fourier transforms having a minimal quadratic deviation from the Fourier transforms of the original signals within the given totality of the frequency intervals and from zero beyond the frequency totality.

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