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Optimal Control for Linear Discrete Systems with Respect to Probabilistic Criteria

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Abstract—We consider the optimal control problem for a linear discrete stochastic system. The optimality criterion is the probability for the first coordinate of the system to fall into a given neighborhood of zero in time not exceeding a predefined value. The problem reduces to an equivalent stochastic optimal control problem with probabilistic terminal criterion. The latter can be solved analytically with dynamical programming. We give sufficient conditions for which the resulting optimal control turns out to be also optimal with respect to the quantile criterion.

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1. INTRODUCTION

Optimal control problems with probabilistic quality criteria comprise the subject of study for a special section of stochastic optimal control theory. Probabilistic criteria include the probability functional and the quantile functional. The probability functional is the probability that a certain precision functional does not exceed a certain admissible level. Here the precision functional itself characterizes the accuracy of this control system but depends on the trajectory of the stochastic system. One example of such a precision functional is the terminal miss of a homing system. In the optimal control problem setting with quality criterion in the form of a probability functional one usually has to maximize this functional.

The quantile functional is, in a sense, a characteristic inverse to the probability functional. The physical meaning of the quantile functional is that it serves as an upper confidence bound for a precision functional and basically characterizes the control system's accuracy guaranteed in probability. Optimal control problems with criteria in the form of a quantile functional are usually set as minimization problems.

Optimal control problems with probabilistic criteria have been studied in [1–4]. The case of discrete time has been considered in [1], which contains a comprehensive study of optimal control problems for discrete stochastic systems with probabilistic criteria and terminal precision functional. Note also the work [5] that studies the problem of equivalence between optimal control problems with probability and quantile functionals.

In this work, we consider an optimal control problem with a probability functional with unknown but bounded from above end moment. This lets us interpret the control problem as a homing problem. The setting considered below is similar to a continuous control problem for a material point [3], but in [3] there are no restrictions on the time when the control process ends.

2. PROBLEM SETTING

Consider a control problem with regard to random influences in discrete time $k = \overline{0, N+1}$. We describe the system's dynamics with the following system of recurrent equations:

$$\begin{cases} x^1_{k+1} = x^1_k + x^2_k h \\ x^2_{k+1} = x^2_k + x^3_k h \\ \dots\dots\dots\dots\dots \\ x^n_{k+1} = x^n_k + u_k h + \xi_k. \end{cases} \tag{1}$$

Here $x_k = (x^1_k, \dots, x^n_k)^{\mathrm{T}}$ is the state vector; $x_k \in R^n$, $u_k \in R^1$, the control; h, a parameter arising in discretizing a continuous system that represents the discrete time step size; ξ_k, continuous random values whose distribution densities are even functions. Here ξ_k are not necessarily identically distributed. $n \geqslant 2$, $N \geqslant n-1$. Initial conditions $x^1_0, \dots, x^n_0$ for system (1) are known exactly.

Note that structure (1) arises after discretizing a linear continuous system written in canonical controllability form, where a random obstacle models errors in the control channel.

We introduce the probability functional

$$P_\varphi(u(\cdot)) = P\left(\min_{k\in\{0,\dots,N+1\}} |x^1_k| \leqslant \varphi\right), \tag{2}$$

where P is the probability, $\varphi \in R^1$ is a scalar parameter, and $u(\cdot) = (u_0, \dots, u_N)$ is the control. The control strategy on the kth step is represented by a function $u_k(x_0, \dots, x_k)$.

We pose the problem of finding optimal control that maximizes the probabilistic optimality criterion

$$P_\varphi(u(\cdot)) \to \max_{u(\cdot)}. \tag{3}$$

As we have already noted in the introduction, this problem can be interpreted as a homing problem for an object into a given neighborhood of zero in time not exceeding a predefined value. Below we will show that optimal control in problem (3) exists in a class of functions $u_k(x_k)$ that depend only on the current state x_k. To do so, we reduce the problem at hand to an equivalent problem with a fixed end moment.

3. AN EQUIVALENT PROBLEM WITH TERMINAL PRECISION FUNCTIONAL

To use the methods of [1], we extend the phase vector $x_k = (x^1_k, \dots, x^n_k)^{\mathrm{T}}$ and reduce the problem to an equivalent optimal control problem in a space of higher dimension. To do so, we introduce a new coordinate y_k whose dynamics will be given by relation

$$y_{k+1} = \min\left\{y_k, |x^1_k + x^2_k h|\right\}, \tag{4}$$

$y_0 = |x^1_0|$, $k = \overline{0, N+1}$. Note that system (1), (4) is nonlinear.

We write the equivalent problem with the newly introduced notation:

$$P(y_{N+1} \leqslant \varphi) \to \max_{u(\cdot)}. \tag{5}$$

Problem (5) belongs to the class of problems with a fixed end moment, so there exists a solution of the original problem [1] in the class of Markov strategies $u_k = u_k(x_k)$. It has also been shown in [1] that to solve the equivalent problem (5) one can use dynamical programming.

According to the dynamical programming algorithm we define the payoff function

$$W_k^{\varphi}(x,y) = \sup_{u_k(\cdot),\ldots,u_N(\cdot)} P(y_{N+1} \leqslant \varphi | x_k = x, y_k = y).$$

For $k = N+1$ we have

$$W_{N+1}^{\varphi}(x_{N+1}, y_{N+1}) = \begin{cases} 1 & \text{for } \ y_{N+1} \leqslant \varphi \\ 0 & \text{for } \ y_{N+1} > \varphi. \end{cases} \tag{6}$$

Then for $k = N$ the payoff function is determined by solving a finite-dimensional problem

$$W_N^{\varphi}(x_N, y_N) = \max_{u_N} M\left[W_{N+1}^{\varphi}(x_{N+1}, y_{N+1}) | x_N, y_N\right].$$

Here $M[\cdot]$ denotes expectation. In the right-hand side of the last expression, arguments of function $W_{N+1}^{\varphi}(x_{N+1}, y_{N+1})$ are transformed according to (1), x_N, y_N are fixed. Taking into account (6), we get

$$W_N^{\varphi}(x_N, y_N) = \max_{u_N} P\left(\min\{y_N, |x_N^1 + x_N^2 h|\} \leqslant \varphi\right).$$

The function being maximized in the right-hand side of this expression does not depend on the control u_N. Therefore,

$$W_N^{\varphi}(x_N, y_N) = \begin{cases} 1 & \text{for } \min\{y_N, |x_N^1 + x_N^2 h|\} \leqslant \varphi \\ 0 & \text{for } \min\{y_N, |x_N^1 + x_N^2 h|\} > \varphi. \end{cases} \tag{7}$$

Control at this step can be arbitrary. We now move on to step $k = N-1$. Similar to step $k = N$, we get

$$\begin{aligned} W_{N-1}^{\varphi}(x_{N-1}, y_{N-1}) &= \max_{u_{N-1}} M\left[W_N^{\varphi}(x_N, y_N) | x_{N-1}, y_{N-1}\right] \\ &= \max_{u_{N-1}} P\left(\min\left\{\min\{y_{N-1}, |x_{N-1}^1 + x_{N-1}^2 h|\}, |x_{N-1}^1 + 2x_{N-1}^2 h + x_{N-1}^3 h^2|\right\} \leqslant \varphi\right) \\ &= \begin{cases} 1 \text{ for } \min\left\{\min\left\{y_{N-1}, |x_{N-1}^1 + x_{N-1}^2 h|\right\}, |x_{N-1}^1 + 2x_{N-1}^2 h + x_{N-1}^3 h^2|\right\} \leqslant \varphi \\ 0 \text{ for } \min\left\{\min\left\{y_{N-1}, |x_{N-1}^1 + x_{N-1}^2 h|\right\}, |x_{N-1}^1 + 2x_{N-1}^2 h + x_{N-1}^3 h^2|\right\} > \varphi \end{cases} \\ &= \begin{cases} 1 & \text{for } \min\left\{y_{N-1}, |x_{N-1}^1 + x_{N-1}^2 h|, |x_{N-1}^1 + 2x_{N-1}^2 h + x_{N-1}^3 h^2|\right\} \leqslant \varphi \\ 0 & \text{for } \min\left\{y_{N-1}, |x_{N-1}^1 + x_{N-1}^2 h|, |x_{N-1}^1 + 2x_{N-1}^2 h + x_{N-1}^3 h^2|\right\} > \varphi. \end{cases} \end{aligned}$$

Since the control together with the random error only occur in the relation for the nth coordinate of the phase vector, the control will be arbitrary on the last $n-1$ steps. We proceed in this way until the control together with the random error come up in the expression for the payoff function. This happens at step $k = N-n+1$. Let us write the payoff function on this step.

$$\begin{aligned} & W_{N-n+1}^{\varphi}(x_{N-n+1}, y_{N-n+1}) \\ &= \max_{u_{N-n+1}} M\left[W_{N-n+2}^{\varphi}\left(x_{N-n+2}, y_{N-n+2}\right) | x_{N-n+1}, y_{N-n+1}\right] \\ &= \max_{u_{N-n+1}} P\Bigg(\min\Bigg\{y_{N-n+1}, |x_{N-n+1}^1 + x_{N-n+1}^2 h|, \\ & |x_{N-n+1}^1 + 2x_{N-n+1}^2 h + x_{N-n+1}^3 h^2|, \ldots, \left|\sum_{i=0}^{n-1} C_{n-1}^i h^i x_{N-n+1}^{i+1}\right|, \\ & \left|\sum_{i=0}^{n-1} C_n^i h^i x_{N-n+1}^{i+1} + h^n u_{N-n+1} + h^{n-1}\xi_{N-n+1}\right|\Bigg\} \leqslant \varphi\Bigg), \end{aligned} \tag{8}$$

and the derivation of (8) is given in the Appendix.

We denote

$$U_k(x_k, y_k) = \min\left\{y_k, |x_k^1 + hx_k^2|, \ldots, \left|\sum_{i=0}^{n-1} C_{n-1}^i h^i x_k^{i+1}\right|\right\},$$

$$V_k(x_k, u_k) = \sum_{i=0}^{n-1} C_n^i h^i x_k^{i+1} + h^n u_k.$$

In this notation, the payoff function on step $k = N-n+1$ takes the form

$$W_{N-n+1}^{\varphi}(x_{N-n+1}, y_{N-n+1}) = \max_{u_{N-n+1}} P\Big(\min\Big\{U_{N-n+1}(x_{N-n+1}, y_{N-n+1}), \Big|V_{N-n+1}(x_{N-n+1}, u_{N-n+1}) + \xi_{N-n+1}h^{n-1}\Big|\Big\} \leqslant \varphi\Big). \tag{9}$$

Lemma. *Let ξ_{N-n+1} have a distribution whose density is an even function. Then problem* (9) *is equivalent in its solution to the following problem:*

$$|V_{N-n+1}(x_{N-n+1}, u_{N-n+1})| \to \min_{u_{N-n+1}}. \tag{10}$$

Proof of lemma is given in the Appendix. Problem (10) is a deterministic equivalent for the stochastic programming problem (9). Constructions of deterministic equivalents for stochastic programming problems with probabilistic criteria have been presented in [6, 7]. The optimal control on the $N-n+1$th step is easy to find from (10):

$$u_{N-n+1} = -\frac{1}{h^n}\sum_{i=0}^{n-1} C_n^i h^i x_{N-n+1}^{i+1}, \tag{11}$$

and the payoff function on this step is

$$W_{N-n+1}^{\varphi}(x_{N-n+1}, y_{N-n+1}) = \begin{cases} 1 & \text{for } U_{N-n+1}(x_{N-n+1}, y_{N-n+1}) \leqslant \varphi \\ P(|\xi_{N-n+1}h^{n-1}| \leqslant \varphi) & \text{for } U_{N-n+1}(x_{N-n+1}, y_{N-n+1}) > \varphi. \end{cases}$$

We denote

$$p_k(\varphi) = P(|\xi_k h^{n-1}| \leqslant \varphi)$$

and write the payoff function on step $N-n$ with the full expectation formula:

$$\begin{aligned} W_{N-n}^{\varphi}(x_{N-n}, y_{N-n}) &= \max_{u_{N-n}}\Big[P\big(\min\{U_{N-n}(x_{N-n}, y_{N-n}), |V_{N-n}(x_{N-n}, u_{N-n}) + \xi_{N-n}h^{n-1}|\} \leqslant \varphi\big) \\ &+ p_{N-n+1}(\varphi)(1 - P\big(\min\{U_{N-n}(x_{N-n}, y_{N-n}), |V_{N-n}(x_{N-n}, u_{N-n}) + \xi_{N-n}h^{n-1}|\} \leqslant \varphi)\big)\Big] \\ &= \max_{u_{N-n}}\Big[P\big(\min\{U_{N-n}(x_{N-n}, y_{N-n}), |V_{N-n}(x_{N-n}, u_{N-n}) \\ &+ \xi_{N-n}h^{n-1}|\} \leqslant \varphi\big)(1 - p_{N-n+1}(\varphi)) + p_{N-n+1}(\varphi)\Big]. \end{aligned}$$

We remind that $0 \leqslant p_{N-n+1}(\varphi) \leqslant 1$. Then the optimization problem on step $k = N-n$ takes the form

$$P\Big(\min\Big\{U_{N-n}(x_{N-n}, y_{N-n}), |V_{N-n}(x_{N-n}, u_{N-n}) + \xi_{N-n}h^{n-1}|\Big\} \leqslant \varphi\Big) \to \max_{u_{N-n}}, \tag{12}$$

and similar to the previous step, by lemma we write a deterministic equivalent for (12)

$$|V_{N-n}(x_{N-n}, u_{N-n})| \to \min_{u_{N-n}}$$

and optimal control

$$u_{N-n} = -\frac{1}{h^n} \sum_{i=0}^{n-1} C_n^i h^i x_{N-n}^{i+1}.$$

The payoff function on step $k = N - n$ is given by

$$W_{N-n}^{\varphi}(x_{N-n}, y_{N-n}) = \begin{cases} 1, & U_{N-n}(x_{N-n}, y_{N-n}) \leqslant \varphi \\ p_{N-n}(\varphi) + p_{N-n+1}(\varphi)(1 - p_{N-n}(\varphi)), & U_{N-n}(x_{N-n}, y_{N-n}) > \varphi. \end{cases}$$

Proceeding in this fashion, it is easy to see that on every step k, $k = \overline{0, N-n+1}$, we have to solve the same optimization problem

$$P\big(\min\{U_k(x_k, y_k), |V_k(x_k, u_k) + \xi_k h^{n-1}|\} \leqslant \varphi\big) \to \max_{u_k}.$$

Reduction of such optimization problems to their deterministic equivalents

$$|V_k(x_k, u_k)| \to \min_{u_k}$$

according to lemma lets us find optimal control on step k as

$$u_k = -\frac{1}{h^n} \sum_{i=0}^{n-1} C_n^i h^i x_k^{i+1}.$$

We denote

$$P_0^{\varphi} = p_0(\varphi) + p_1(\varphi)(1 - p_0(\varphi)) + \ldots + p_{N-n+1}(\varphi)(1 - p_{N-n}(\varphi)) \ldots (1 - p_0(\varphi)),$$

and consequently, the payoff function on step $k = 0$ is written as

$$W_0^{\varphi}(x_0) = \begin{cases} 1, & U_0(x_0) \leqslant \varphi \\ P_0^{\varphi}, & U_0(x_0) > \varphi. \end{cases}$$

In what follows we call it the optimal payoff function since according to dynamical programming

$$W_0^{\varphi}(x_0) = \max_{u(\cdot)} P_{\varphi}(u(\cdot)).$$

Note that $W_0^{\varphi}(x_0)$ and $U_0(x_0)$ do not depend on y_0 due to the initial conditions $y_0 = |x_0^1|$.

4. PROBLEMS WITH QUANTILE FUNCTIONALS

We define the quantile criterion as [1]

$$\Phi_{\alpha}(u) = \min\{\varphi | P_{\varphi}(u) \geqslant \alpha\}, \tag{13}$$

where $\alpha \in (0, 1)$ is a given confidence probability. The quantile functional (13) is a level of the above-mentioned precision functional guaranteed with a given probability, i.e., an upper confidence bound for it.

Consider a minimization problem for the quantile criterion:

$$\Phi_\alpha(u) \to \min_{u(\cdot)}. \tag{14}$$

Since there is a probabilistic constraint $P_\varphi(u) \geqslant \alpha$, it is hard to apply dynamical programming to problem (14). The approach used in [1] for discrete stochastic systems only lets us get approximate (the so-called guaranteeing) solutions for the quantile problem in the class of controls that depend on all previous system states. The problem of whether there exist optimal controls that depend only on the current state remains open. In the previous section we have obtained an analytic solution for the optimal control problem with probabilistic criterion. In [5], a method was proposed to transform this solution into a solution of problem (14). In order to formulate sufficient conditions for the applicability of this method, we introduce functions of optimal values of the considered functionals

$$F(\varphi) = \sup_{u(\cdot)} P_\varphi(u),$$
$$G(\alpha) = \inf_{u(\cdot)} \Phi_\alpha(u).$$

Definition [5]. Let $f(x)$ be a nondecreasing function of scalar argument. A point x_0 such that

$$f(x_0 - \epsilon) \leqslant 0 \leqslant f(x_0 + \epsilon)$$

for every $\epsilon > 0$ is called a generalized root of equation $f(x) = 0$.

The following theorem [5] establishes an equivalence between probabilistic and quantile optimization problems.

Theorem 1 [5]. *Let φ_α be the only generalized root of equation $F(\varphi) = \alpha$. Then $G(\alpha) = \varphi_\alpha$. Moreover, if for $\varphi = \varphi_\alpha$ there exists a solution u_φ of problem*

$$P_\varphi(u) \to \max_{u(\cdot)}$$

and it holds that $F(\varphi_\alpha) \geqslant \alpha$, then u_φ is a solution of problem (14).

Let us check that sufficient conditions for the equivalence of probabilistic and quantile optimization problems hold for the problem from the previous section. Note that the definition of the optimal payoff function implies that

$$F(\varphi) = W_0^\varphi(x_0).$$

It is easy to check that in case when random values ξ_k are identically distributed the optimal payoff function takes the form

$$F(\varphi) = \begin{cases} 1, & U_0(x_0) \leqslant \varphi \\ p(\varphi) \sum\limits_{j=0}^{N-n} (1 - p(\varphi))^j, & U_0(x_0) > \varphi, \end{cases} \tag{15}$$

where

$$p(\varphi) = P(|\xi_0 h^{n-1}| \leqslant \varphi).$$

In order for the conditions of Theorem 1 to hold it suffices for function $p(\varphi)$ to be strictly increasing in its argument.

Theorem 2. *Let $p(\varphi)$ be strictly increasing in φ. Then the optimal control problem with a criterion in the form of probability functional* (2) *is equivalent to the optimal control problem with quantile criterion* (14), *and the control that solves problem* (2) *is also optimal for problem* (14).

Proof of Theorem 2 is given in the Appendix.

5. EXAMPLE

As an example, consider a model for a perturbed one-dimensional motion of a material point. Acceleration plays the role of control, with random errors with a Gaussian distribution acting on it. Then equations describing this system's dynamics will have the form

$$\begin{cases} x_{k+1} = x_k + v_k h \\ v_{k+1} = v_k + u_k h + \xi_k, \end{cases} \tag{16}$$

x_k, v_k are respectively the coordinate and velocity of the material point at the kth time moment, $\xi_k \sim N(0, \sigma^2)$, $k = \overline{0, N}$, and the criterion is

$$P_\varphi(u(\cdot)) = P\left(\min_{k \in \{0,\ldots,N+1\}} |x_k| \leqslant \varphi\right) \to \max_{u(\cdot)}. \tag{17}$$

Note that the distribution density of random values ξ_k belongs to the family of densities that are symmetric and unimodal with respect to the expectation. According to the methodology of Section 3, we introduce a new coordinate for the phase vector

$$y_{k+1} = \min\{y_k, |x_k + v_k h|\},$$

$y_0 = |x_0|$ are the initial conditions. An equivalent problem has the form

$$P(y_{N+1} \leqslant \varphi) \to \max_{u(\cdot)}.$$

We should note that the control occurs in the second equation that describes system dynamics. Therefore, on the last step the control will be arbitrary.

Based on the results of Section 3, we define optimal control on steps $k = \overline{0, N-1}$

$$u_k = -\frac{x_k}{h^2} - \frac{2v_k}{h}$$

and the payoff function

$$W_k^\varphi(x_k, v_k, y_k) = \begin{cases} 1 & \text{for } \min\{y_k, |x_k + v_k h|\} \leqslant \varphi \\ 2\Phi_0\left(\dfrac{\varphi}{\sigma}\right) \displaystyle\sum_{j=0}^{N-(k+2)} \left(1 - 2\Phi_0\left(\dfrac{\varphi}{\sigma}\right)\right)^j & \text{for } \min\{y_k, |x_k + v_k h|\} > \varphi, \end{cases}$$

where $2\Phi_0(\frac{\varphi}{\sigma})$ is the probability that random value ξ_k falls into the interval $(-\frac{\varphi}{\sigma}, \frac{\varphi}{\sigma})$, $\Phi_0(x)$ is the Laplace function,

$$\Phi_0(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^x e^{-\frac{(t-m)^2}{2\sigma^2}}\, dt,$$

$m = M[\xi_k]$. Further, we write the optimal payoff function as

$$F(\varphi) = \begin{cases} 1, & \min\{|x_0|, |x_0 + v_0 h|\} \leqslant \varphi \\ 2\Phi_0\left(\dfrac{\varphi}{\sigma}\right) \displaystyle\sum_{j=0}^{N-2} \left(1 - 2\Phi_0\left(\dfrac{\varphi}{\sigma}\right)\right)^j, & \min\{|x_0|, |x_0 + v_0 h|\} > \varphi. \end{cases} \tag{18}$$

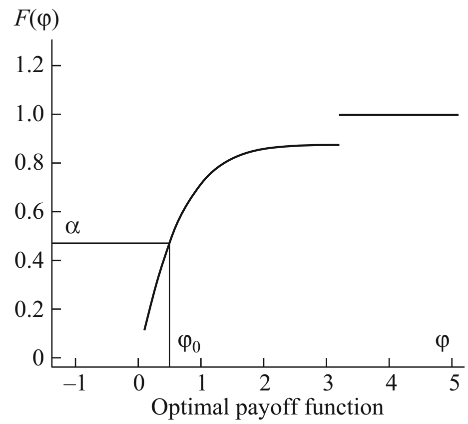

Figure.

Theorem 2 implies that the maximization problem for the probability function (17) can be reduced to a minimization problem for the quantile function

$$\Phi_\alpha(u(\cdot)) \to \min_{u(\cdot)}, \tag{19}$$

and the optimal control found in problem (17) is also optimal in problem (19). The figure shows a plot of the optimal payoff function (18).

The function $F(\varphi)$ was plotted with the GNUPLOT environment with a φ step of 0.1 and with parameters $\sigma = 1$, $N = 3$. The plot also shows the level α, which according to Theorem 2 means that u_φ is a solution of problem (19) for $\varphi = \varphi_\alpha$.

6. CONCLUSION

We have considered the optimal control problem for a linear discrete stochastic system with respect to the probabilistic criterion with free but bounded from above end time. By extending the phase vector, we have obtained an equivalent optimal control problem of higher dimension but with a fixed end moment. Solution for the equivalent has been obtained in analytic form with dynamical programming. We have derived sufficient conditions that let us get from this solution a solution of an equivalent problem with quantile quality criterion.

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APPENDIX

Proof of Lemma.

$$P\left(\min\left\{U_{N-n+1}(x_{N-n+1}, y_{N-n+1}), \left|V_{N-n+1}(x_{N-n+1}, u_{N-n+1}) + \xi_{N-n+1}h^{n-1}\right|\right\} \leqslant \varphi\right)$$
$$= 1 - P\left(\min\left\{U_{N-n+1}(x_{N-n+1}, y_{N-n+1}), \left|V_{N-n+1}(x_{N-n+1}, u_{N-n+1}) + \xi_{N-n+1}h^{n-1}\right|\right\} > \varphi\right)$$
$$= 1 - P\left(\left\{U_{N-n+1}(x_{N-n+1}, y_{N-n+1}) > \varphi\right\}\left\{\left|V_{N-n+1}(x_{N-n+1}, u_{N-n+1}) + \xi_{N-n+1}h^{n-1}\right| > \varphi\right\}\right).$$

Note that $U_{N-n+1}(x_{N-n+1}, y_{N-n+1})$ is a deterministic function, so we consider the case when $U_{N-n+1}(x_{N-n+1}, y_{N-n+1}) > \varphi$. Then the right-hand side of the latter expression equals

$$\begin{aligned} &1 - P\left(\left|V_{N-n+1}(x_{N-n+1}, u_{N-n+1}) + \xi_{N-n+1}h^{n-1}\right| > \varphi\right) \\ &= P\Big(-\varphi - V_{N-n+1}(x_{N-n+1}, u_{N-n+1}) \leqslant \xi_{N-n+1}h^{n-1} \\ &\qquad \leqslant \varphi - V_{N-n+1}(x_{N-n+1}, u_{N-n+1})\Big) \\ &= \int\limits_{-\varphi - V_{N-n+1}(x_{N-n+1}, u_{N-n+1})}^{\varphi - V_{N-n+1}(x_{N-n+1}, u_{N-n+1})} f_\xi(t)dt, \end{aligned}$$

where $f_\xi(t)$ is the distribution density of random values $\xi_{N-n+1}h^{n-1}$. Note that the latter probability functional falls into the so-called class of additive loss functions described in [6]. We write the deterministic equivalent obtained in [6]:

$$\left|M\left[\xi_{N-n+1}h^{n-1}\right] + V_{N-n+1}(x_{N-n+1}, u_{N-n+1})\right| \to \min_{u_{N-n+1}}.$$

The latter problem is equivalent to problem (9) if the distribution density for random values $\xi_{N-n+1}h^{n-1}$ is symmetric and unimodal with respect to $M[\xi_{N-n+1}h^{n-1}]$. But since, by assumption, function $f_\xi(t)$ is even, we have

$$|V_{N-n+1}(x_{N-n+1}, u_{N-n+1})| \to \min_{u_{N-n+1}}.$$

This completes the proof of the lemma.

Deriving Relation (8)

The payoff function on step $k = N - 1$ has the form

$$W^\varphi_{N-1}(x_{N-1}, y_{N-1}) = \begin{cases} 1, & \min\left\{y_{N-1}, |x^1_{N-1} + x^2_{N-1}h|, |x^1_{N-1} + 2x^2_{N-1}h + x^3_{N-1}h^2|\right\} \leqslant \varphi \\ 0, & \min\left\{y_{N-1}, |x^1_{N-1} + x^2_{N-1}h|, |x^1_{N-1} + 2x^2_{N-1}h + x^3_{N-1}h^2|\right\} > \varphi. \end{cases}$$

We write the payoff function on step $k = N - 2$:

$$\begin{aligned} &W^\varphi_{N-2}(x_{N-2}, y_{N-2}) = \max_{u_{N-2}} M\left[W^\varphi_{N-1}(x_{N-1}, y_{N-1})|x_{N-2}, y_{N-2}\right] \\ &= \max_{u_{N-2}} P\Big(\min\Big\{\min\Big\{\min\Big\{y_{N-2}, \left|x^1_{N-2} + x^2_{N-2}h\right|\Big\}, \left|x^1_{N-2} + 2x^2_{N-2}h + x^3_{N-2}h^2\right|\Big\}, \\ &\qquad \left|x^1_{N-2} + 3x^2_{N-2}h + 3x^3_{N-2}h^2 + x^4_{N-2}h^3\right|\Big\} \leqslant \varphi\Big) \\ &= \max_{u_{N-2}} P\Big(\min\Big\{y_{N-2}, \left|x^1_{N-2} + x^2_{N-2}h\right|, \left|x^1_{N-2} + 2x^2_{N-2}h + x^3_{N-2}h^2\right|, \\ &\qquad \left|x^1_{N-2} + 3x^2_{N-2}h + 3x^3_{N-2}h^2 + x^4_{N-2}h^3\right|\Big\} \leqslant \varphi\Big) \\ &= P\Big(\min\Big\{y_{N-2}, \left|x^1_{N-2} + x^2_{N-2}h\right|, \left|x^1_{N-2} + 2x^2_{N-2}h + x^3_{N-2}h^2\right|, \\ &\qquad \left|x^1_{N-2} + 3x^2_{N-2}h + 3x^3_{N-2}h^2 + x^4_{N-2}h^3\right|\Big\} \leqslant \varphi\Big) \\ &= P\left(\min\left\{y_{N-2}, \left|\sum_{i=0}^{1} C^i_1 h^i x^{i+1}_{N-2}\right|, \left|\sum_{i=0}^{2} C^i_2 h^i x^{i+1}_{N-2}\right|, \left|\sum_{i=0}^{3} C^i_3 h^i x^{i+1}_{N-2}\right|\right\} \leqslant \varphi\right), \end{aligned}$$

where

$$C_n^i = \frac{n!}{i!(n-i)!}.$$

Then by induction we can write the payoff function on step $k = N - n + 2$ as

$$\begin{gathered} W^{\varphi}_{N-n+2}(x_{N-n+2}, y_{N-n+2}) \\ = \max_{u_{N-n+2}} P\left(\min\left\{ y_{N-n+2}, \left|\sum_{i=0}^{1} C_1^i h^i x^{i+1}_{N-n+2}\right|, \left|\sum_{i=0}^{2} C_2^i h^i x^{i+1}_{N-n+2}\right|, \ldots, \right.\right. \\ \left.\left. \left|\sum_{i=0}^{n-1} C_{n-1}^i h^i x^{i+1}_{N-n+2}\right|\right\} \leqslant \varphi \right) \\ = P\left(\min\left\{ y_{N-n+2}, |x^1_{N-n+2} + x^2_{N-n+2} h|, |x^1_{N-n+2} \right.\right. \\ \left.\left. +2x^2_{N-n+2}h + x^3_{N-n+2}h^2|, \ldots, \left|\sum_{i=0}^{n-1} C_{n-1}^i h^i x^{i+1}_{N-n+2}\right|\right\} \leqslant \varphi \right). \end{gathered}$$

Further, similarly,

$$\begin{gathered} W^{\varphi}_{N-n+1}(x_{N-n+1}, y_{N-n+1}) \\ = \max_{u_{N-n+1}} P\left(\min\left\{ y_{N-n+1}, |x^1_{N-n+1} + x^2_{N-n+1} h|, \right.\right. \\ \left|x^1_{N-n+1} + 2x^2_{N-n+1}h + x^3_{N-n+1}h^2\right|, \ldots, \\ \left.\left. \left|\sum_{i=0}^{n-1} C_{n-1}^i h^i x^{i+1}_{N-n+1}\right|, \left|\sum_{i=0}^{n-1} C_n^i h^i x^{i+1}_{N-n+1} + h^n u_{N-n+1} + h^{n-1}\xi_{N-n+1}\right|\right\} \leqslant \varphi \right). \end{gathered}$$

Proof of Theorem 2. We use Theorem 1. To check that its conditions hold, since $\alpha \in (0,1)$ it suffices to check that the function in the second branch of (15) is strictly increasing. We have

$$p(\varphi)\sum_{j=0}^{N-n}(1-p(\varphi))^j = p(\varphi)\frac{(1-p(\varphi))^{N-n}-1}{1-p(\varphi)-1} = 1-(1-p(\varphi))^{N-n}.$$

The right-hand side of this expression due to $p(\varphi) \in (0,1)$ strictly increases as a function of φ since it is a superposition of strictly increasing functions ($p(\varphi)$ is strictly increasing by the conditions of the theorem).

This completes the proof of the theorem.

REFERENCES

1. Malyshev, V.V. and Kibzun, A.I., *Analiz i sintez vysokotochnogo upravleniya letatel'nymi apparatami* (Analysis and Synthesis of High Accuracy Control for Flying Vehicles), Moscow: Mashinostroenie, 1987.

2. Zubov, V.I., *Lektsii po teorii upravleniya* (Lectures in Control Theory), Moscow: Nauka, 1975.

3. Afanas'ev, V.N., Kolmanovskii, V.B., and Nosov, V.R., *Matematicheskaya teoriya konstruirovaniya sistem upravleniya* (Mathematical Theory of Constructing Control Systems), Moscow: Vysshaya Shkola, 2003.

4. Krasovskii, N.N., On Optimal Control under Random Perturbations, *Prikl. Mat. Mekh.*, 1960, vol. 24, no. 1, pp. 64–79.

5. Kan, Yu.S., Control Optimization by the Quantile Criterion, *Autom. Remote Control*, 2001, vol. 62, no. 5, pp. 746–757.

6. Vishnaykov, V.B. and Kibzun, A.I., Deterministic Equivalents for the Problems of Stochastic Programming with Probabilistic Criteria, *Autom. Remote Control*, 2006, vol. 67, no. 6, pp. 945–961.

7. Kan, Yu.S. and Kibzun, A.I., *Zadachi stokhasticheskogo programmirovaniya s veroyatnostnymi kriteriyami* (Stochastic Programming Problems with Probabilistic Criteria), Moscow: Fizmatlit, 2009.

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