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SHORT COMMUNICATIONS

Functions Almost Universal in the Sense of Signs with Respect to the Trigonometric System and the Walsh System

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This paper is a continuation of the author's papers [1]–[6].

We will use the following notation.

Let $L^0[0,1]$ be the class of all functions almost everywhere finite and measurable on $[0,1]$, and let $M[0,1]$ be the class of all functions measurable on $[0,1]$.

A sequence $\{f_k(x)\}_{k=1}^{\infty} \subset L^0[0,1]$ converges to $f(x)$ in $L^0[0,1]$ (respectively, $M[0,1]$) if $\{f_k(x)\}_{k=1}^{\infty}$ converges to $f(x)$ almost everywhere on $[0,1]$ (accordingly, almost everywhere or with respect to the measure on $[0,1]$).

Assume that $E \subseteq [0,1]$ is a measurable set, $|E|$ is the Lebesgue measure of the measurable set E, and $L^p(E)$ is the class of all functions measurable on E for which $\int_E |f(x)|^p\,dx < \infty$, $p > 0$.

Let $\{\varphi_k(x)\}_{k=0}^{\infty}$ be an orthonormal system.

Let $c_k(f) := \int_0^1 f(x)\varphi_k(x)\,dx$, $k \in \mathbb{N} \cup \{0\}$, be Fourier coefficients, and let

$$S_m(f,\Phi) = \sum_{k=0}^{m} c_k(f)\varphi_k(x), \qquad m \in \mathbb{N} \cup \{0\},$$

be partial sums of the Fourier series $\sum_{k=0}^{\infty} c_k(f)\varphi_k(x)$ with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ of the function $f \in L^1[0,1]$.

Let S be one of the spaces $L^p[0,1]$, $p \in (0,1)$, $L^0[0,1]$, and $M[0,1]$. By $\#(\Omega)$ we denote the number of points in a finite set Ω.

Before proceeding to the formulation of some results, we give the corresponding definition.

Definition 1. Let $\Lambda \subset \Omega \subseteq \mathbb{N}$ be a bounded subset, and let

$$\rho(\Lambda)_\Omega := \lim_{n\to\infty} \sup \frac{\#(\Lambda \cap (0,n))}{\#(\Omega \cap (0,n))}\,.$$

A function $\rho(\Lambda)_\Omega$ is called the (*upper asymptotics*) *density* of a subset Λ with respect to the set Ω. In particular, $\rho(\{N_m\}_{m=1}^{\infty})_{\mathbb{N}}$ is called the *density of the subsequence* $\Lambda = \{N_m\}_{m=1}^{\infty}$ with respect to the set of positive integers.

Definition 2. A function $U \in L^1[a,b]$ for the class S with respect to the orthonormal system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is said to be

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(a) *universal in the sense of signs* if its Fourier series with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is universal in S in the sense of signs, i.e., for each function $f \in S$, there exists a sequence of signs $\{\delta_k = \pm 1\}_{k=0}^{\infty}$ for which the series $\sum_{k=0}^{\infty} \delta_k c_k(U)\varphi_k(x)$ converges to the function $f(x)$ in S;

(b) *almost universal in the sense of signs* if there exists a subsequence of positive integers $\{N_m\}_{m=1}^{\infty}$ with $\rho(\{N_m\}_{m=1}^{\infty})_{\mathbb{N}} = 1$ such that, for every function $f \in S$, there exists a sequence of signs $\{\delta_k = \pm 1\}_{k=0}^{\infty}$ such that the subsequence $\sum_{k=0}^{N_m} \delta_k c_k(U)\varphi_k(x)$ converges to $f(x)$ in S;

(c) *almost universal in the usual sense* if there exists a sequence $\{\delta_k = \pm 1\}_{k=0}^{\infty}$ of signs satisfying $\rho(\Lambda(U))_{\Omega(U)} = 1$ (where $\Lambda(U) = \{k \in \Omega(U) = \mathrm{spec}(U),\ \delta_k = 1\}$) such that the series $\sum_{k=0}^{\infty} \delta_k c_k(U)\varphi_k(x)$ is universal in S in the usual sense, i.e., for each function $f \in S$, there is a subsequence $\{m_j\}_{j=1}^{\infty}$ of increasing positive integers such that $\sum_{k=0}^{m_j} \delta_k c_k(U)\varphi_k(x)$ converges to the function $f(x)$ in S.

In this short paper, we discuss the existence and description of the structure of functions whose Fourier series are almost universal in the sense of signs with respect to the trigonometric system (with respect to the Walsh system) in the class of measurable functions.

From the Talalyan–Arutyunyan theorem [7] (a series in the Walsh system cannot converge to infinity on a set of positive measure) and from the Konyagin theorem [8] (a trigonometric series cannot converge to infinity on a set of positive measure), it follows that there is no integrable function that is universal in the sense of signs with respect to the Walsh system and with respect to the trigonometric system for the class $M[0,1]$ in the case of convergence almost everywhere.

At the same time, the following statements are true:

(1) there exists a function $U \in L^1[0,1]$ that is universal in the sense of signs with respect to the Walsh system for the class $M[0,1]$ in the case of convergence with respect to the measure (see [9]);

(2) there is a function $U \in L^1[0,1]$ that is universal in the sense of signs with respect to the Walsh system for the classes $L^p[0,1]$ for $p \in (0,1)$ and $L^0[0,1]$ (see [10] and [11]).

The following assertion holds.

Theorem 1. *There exists an integrable function $U \in L^1[0,1]$ with Fourier series converging with respect to the $L^1[0,1]$-norm with monotone decreasing Fourier–Walsh coefficients that is universal in the sense of signs with respect to the Walsh system for the class $M[0,1]$ in the case of convergence with respect to the measure and almost universal in the sense of signs with respect to the Walsh system for the class $L^0[0,1]$ in the case of convergence almost everywhere.*

The following questions immediately arise to which no answers are known.

Question 1. Does there exist an orthonormal system of bounded functions $\{\varphi_k(x)\}_{k=0}^{\infty}$ and a function $U \in L^1[0,1]$ universal in the sense of signs with respect to the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ for the class $M[0,1]$ in the case of convergence almost everywhere?

Question 2. Does there exist a function $U \in L^1[0,1]$ universal in the sense of signs with respect to the trigonometrical system for the class $M[0,1]$ in the case of convergence with respect to the measure?

Question 3. Is Theorem 1 true for the trigonometric system and for the Vilenkin system?

As was noted above, there exists a function $U \in L^1[0,1]$ which, for the class $L^0[0,1]$, is universal in the sense of signs with respect to Walsh system. We do not know whether there exists a function $U \in L^1[0,1]$ which, for the class $L^0[0,1]$, is universal in the sense of signs with respect to the trigonometric system.

It should be noted that there does not exist a function $U \in L^1[0,1]$ universal in the sense of signs for the class $L^0[0,1]$ with respect to the system $\{f_k(x)\}_{k=1}^{\infty}$ constructed by Kashin in [12] (he constructed a complete in $L^2[0,1]$ orthonormal uniformly bounded system of functions $\{f_k\}_{k=1}^{\infty}$ such that the convergence almost everywhere on $[0,1]$ of the series $\sum_{k=1}^{\infty} a_k f_k(x)$ implies that $\sum_{k=1}^{\infty} a_k^2 < \infty$).

We also note that the problems of existence were studied in [3], [9] for the functions that are universal in the sense of signs for the classes $L^p[0,1]$ for $p \in (0,1)$, with respect to the trigonometric system (accordingly, with respect to Walsh system), and in [13], the functions were constructed that are almost universal in the usual sense for the classes $L^p[0,1]$ for $p \in (0,1)$ with respect to the trigonometric system (accordingly, with respect to the multiple trigonometric system) (also see [15]).

It should be noted that the existence of functions and series that are universal in a different sense was studied by many mathematicians who worked in the theory of functions of both real and complex variables.

The first examples of universal functions were constructed by Birkhoff [16] in the framework of complex analysis, in which the entire functions were represented in any circle by uniformly converging shifts of a universal function; as well as by Marcinkiewicz [17] in the framework of real analysis, in which any measurable function was represented as the limit almost everywhere of some sequence of difference relations of the universal function. The notion of universal series goes back to Men'shov [18] and Talalyan [19]. The most general results were obtained by them and their students.

It turns out that any finite function measurable almost everywhere, by changing its values on a set of arbitrarily small measure, can be turned into an almost universal function in the sense of signs with respect to the trigonometric system, as well as with respect to the Walsh system for the class M[0, 1]. $M[0,1]$.

The following theorems are true.

Theorem 2. *For any $\varepsilon > 0$, there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\varepsilon$ and a function $U \in L^1[0,1]$ with the following properties*:

(1) *$U(x)$ is almost universal in the sense of signs with respect to the trigonometric system for the class $M[0,1]$,*

(2) *for each function $f \in L^1[0,1]$, there exists a function $\widetilde{f} \in L^1[0,1]$ such that $\widetilde{f}(x) = f(x)$ on E and $\widetilde{f}(x)$ is almost universal in the sense of signs with respect to the trigonometric system for the class $M[0,1]$ in the case of convergence almost everywhere and, in addition,*

$$\left|\int_0^1 \widetilde{f}(t)e^{-i2\pi kt}\,dt\right| = \left|\int_0^1 U(t)e^{-i2\pi kt}\,dt\right|, \qquad k = 0, \pm 1, \pm 2, \dots .$$

Theorem 3. *For any $\varepsilon > 0$, there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\varepsilon$ and a function $U \in L^1[0,1]$ with the following properties*:

(1) *the function $U(x)$ is almost universal in the sense of signs with respect to the Walsh system $\{W_k(t)\}_{k=1}^{\infty}$ for the class $M[0,1]$ in the case of convergence almost everywhere*;

(2) *the function $U(x) = 0$ on $[\varepsilon, 1]$*;

(3) *the Fourier–Walsh series of the function $U(x)$ converges in $L^1[0,1]$ and everywhere on $[0,1)$*;

(4) *the Fourier–Walsh coefficients of the function $U(x)$ are positive and monotone decreasing*;

(5) *for each function* $f \in L^1[0,1]$, *there exists a function* $\widetilde{f} \in L^1[0,1]$ *such that* $\widetilde{f}(x) = f(x)$ *on* E, $\widetilde{f}(x)$ *is almost universal in the sense of signs with respect to the Walsh system for the class* $M[0,1]$ *in the case of convergence almost everywhere and, in addition,*

$$\left|\int_0^1 \widetilde{f}(t)W_k(t)\,dt\right| = \int_0^1 U(t)W_k(t)\,dt, \qquad k = 0, 1, 2, \dots .$$

Remark. It should be noted that from the obtained results it is clear that the existence of universal functions depends on the type of universality, on the system, on the meaning of convergence, and on space (questions in this direction are very capacious).

To prove Theorem 2, we will use the following lemma proved in [6].

Lemma 1. *Let the numbers* $N_0 \in N$, $\varepsilon, \eta \in (0,1)$ *and a step function* $f(x)$ *be given. Then there exists a function* $g(x)$, *a measurable set* $E \subset [0,1]$, *and polynomials in the trigonometric system*

$$H(x) = \sum_{N_0 \le |k| \le N} d_k e^{i2\pi kx}, \qquad Q(x) = \sum_{N_0 \le |k| \le N} \delta_k\, d_k e^{i2\pi kx}, \quad \delta_k = \pm 1,$$

satisfying the following conditions:

$$1)\ \int_0^1 |g(x)|\,dx < 4\int_0^1 |f(x)|\,dx, \qquad 2)\ g(x) = f(x), \quad x \in E, \qquad 3)\ |E| > 1 - \varepsilon,$$

$$4)\ \int_0^1 |H(x)|\,dx < \eta, \qquad 5)\ \int_0^1 |f(x) - g(x)|\,dx < \eta.$$

We note that a similar lemma also holds for the Walsh system: in this case, one can construct a function $g(x)$, a set E, and polynomials

$$H(x) = \sum_{k=N_0}^{N} a_k W_k(x) \qquad \text{and} \qquad Q(x) = \sum_{k=N_0}^{N} \delta_k a_k W_k(x)$$

with respect to the Walsh system so that, together with properties (1)–(5), the following relations are also satisfied (see [2]):

$$6)\ 0 < a_{k+1} \le a_k < \eta, \quad k \in [N_0, N),$$

$$7)\ \max_{N_0 \le M < N} \int_0^1 \left|\sum_{k=N_0}^{M} \delta_k a_k W_k(x)\right| dx < 4\int_0^1 |f(x)|\,dx,$$

$$8)\ \max_{N_0 \le M \le N} \int_0^1 \left|\sum_{k=N_0}^{M} a_k W_k(x)\right| dx < \varepsilon.$$

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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