ISSN 0001-4346, Mathematical Notes, 2024, Vol. 115, No. 6, pp. 1017–1029. © Pleiades Publishing, Ltd., 2024.
Russian Text © The Author(s), 2024, published in Matematicheskie Zametki, 2024, Vol. 115, No. 6, pp. 919–934.

Yu. N. Subbotin's Method in the Problem of Extremal Interpolation in the Mean in the Space $L_p(\mathbb{R})$ with Overlapping Averaging Intervals

V. T. Shevaldin[1*]

[1] N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences, Ekaterinburg, 620108 Russia

Received November 17, 2023; in final form, January 11, 2024; accepted January 15, 2024

Abstract—On a uniform grid on the real axis, we study the Yanenko–Stechkin–Subbotin problem of extremal function interpolation in the mean in the space $L_p(\mathbb{R})$, $1 < p < \infty$, of two-way real sequences with the least value of the norm of a linear formally self-adjoint differential operator $\mathcal{L}_n$ of order n with constant real coefficients. In case of even n, the value of the least norm in the space $L_p(\mathbb{R})$, $1 < p < \infty$, of the extremal interpolant is calculated exactly if the grid step h and the averaging step h_1 are related by the inequality $h < h_1 \le 2h$.

DOI: 10.1134/S0001434624050365

Keywords: *extremal interpolation, spline, uniform mesh, formally self-adjoint differential operator, minimal norm.*

INTRODUCTION

Let D denote the symbol of differentiation, $\mathcal{L}_n = \mathcal{L}_n(D)$, $n \in \mathbb{N}$, be an arbitrary linear differential operator of order n with real constant coefficients, whose leading coefficient is equal to 1. Operator $\mathcal{L}_n$ can be written as

$$\mathcal{L}_n = \mathcal{L}_n(D) = \prod_{s=1}^{k}(D^2 - 2\gamma_s D + \gamma_s^2 + \alpha_s^2) \prod_{j=1}^{n-2k}(D - \beta_j), \tag{0.1}$$

where $\alpha_s, \beta_j, \gamma_s \in \mathbb{R}$, moreover, in the case $k \neq 0$ one can assume that $\alpha_s > 0$. Denote by

$$p_n = p_n(\lambda) = \prod_{s=1}^{k}(\lambda^2 - 2\gamma_s\lambda + \gamma_s^2 + \alpha_s^2) \prod_{j=1}^{n-2k}(\lambda - \beta_j) \tag{0.2}$$

a characteristic polynomial of this differential operator. To a linear differential operator $\mathcal{L}_n$ we put in correspondence a difference operator with step $h > 0$

$$\Delta_h^{\mathcal{L}_n} y_m = \prod_{s=1}^{k}(T^2 - 2Te^{\gamma_s h}\cos\alpha_s h + e^{2\gamma_s h}E) \prod_{j=1}^{n-2k}(T - e^{\beta_j h}E) y_m \tag{0.3}$$

defined on the space of sequences $y = \{y_m\}_{m=-\infty}^{\infty}$. Here $Ty_m = y_{m+1}$ and E is an identity operator. A difference operator $\Delta_h^{\mathcal{L}_n}$ is chosen such that for any solution f of the homogeneous equation $\mathcal{L}_n(D)f = 0$ for any $x \in \mathbb{R}$ the inequality

$$\Delta_h^{\mathcal{L}_n} f(x + mh) = 0$$

*E-mail: Valerii.Shevaldin@imm.uran.ru

holds.

Introduce a class of sequences

$$Y_{h,p} = \big\{y = \{y_m\}_{m=-\infty}^{\infty} \colon \|\Delta_h^{\mathcal{L}_n} y\|_{l_p} \le 1\big\}, \qquad h > 0, \quad 1 \le p \le \infty.$$

The norm on the space of sequences y is defined, as usual, by the equality

$$\|y\|_{l_p} = \begin{cases} \Big(\sum\limits_{m\in\mathbb{Z}} |y_m|^p\Big)^{1/p}, & 1 \le p < \infty, \\ \sup\limits_m |y_m|, & p = \infty. \end{cases}$$

Let AC be a class of locally absolute continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, $L_p(\mathbb{R})$, $1 \le p < \infty$, be a space of functions g absolutely integrable on $\mathbb{R}$ with the norm

$$\|g\|_{L_p(\mathbb{R})} = \left(\int_{\mathbb{R}} |g(t)|^p \, dt\right)^{1/p},$$

and $L_\infty(\mathbb{R})$ be a space of essentially bounded on $\mathbb{R}$ functions with the norm

$$\|g\|_{L_\infty(\mathbb{R})} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |g(x)|.$$

For the operator $\mathcal{L}(D) = D^n$ a problem of connection between the finite differences $\Delta_h^{D^n} = \Delta_h^n$ (as well as divided differences) of order n and the derivative of the function of order n is well studied (see, e.g., [1]). Favard [2] considered this problem in 1940 in the extremal setting on the segment for a uniform grid of nodes and respective divided differences.

The problem of the extremal functional interpolation (*on a uniform grid of nodes*), which was set up in the beginning of 1960s by Yanenko and Stechkin for a particular case of the operator of n-fold differentiation D^n, is as follows.

Let $h_1 \ge 0$. For any sequence $y \in Y_{h,p}$ consider a class of functions

$$F_{h,h_1,p}(y) = \bigg\{f\colon f^{(n-1)} \in AC,\ \mathcal{L}_n(D)f \in L_p(\mathbb{R}),\\ \frac{1}{h_1}\int_{-h_1/2}^{h_1/2} f(mh+t)\,dt = y_m,\ m \in \mathbb{Z}\bigg\}$$

(for $h_1 = 0$ we set $f(mh) = y_m$). For every sequence $y \in Y_{h,p}$ it is required to construct a function $f \in F_{h,h_1,p}(y)$ and compute (or efficiently estimate from below and above) the following quantity

$$A_p = A_p(\mathcal{L}_n, h, h_1) = \sup_{y\in Y_{h,p}} \inf_{f\in F_{h,h_1,p}(y)} \|\mathcal{L}_n(D)f\|_{L_p(\mathbb{R})}. \tag{0.4}$$

This problem is commonly called the *Yanenko–Stechkin–Subbotin problem*, and a significant number of works (see some early works of Subbotin [3]–[5] on the operator $\mathcal{L}_n = D^n$ and the survey [6]) are devoted to it (and its generalizations).

In this work, only the problem of interpolation in the mean is studied, i.e. the case $h_1 \ne 0$. In that case, a solution to problem (0.4) turned out to be quite difficult, and there are few results in this direction. For the operator $\mathcal{L}_n = D^n$ quantity (0.4) for $1 \le p \le \infty$, $0 < h < \infty$, $0 \le h_1 \le 2h$ was computed by Subbotin [1]–[3], [7]–[9]. It is worth noting that the time interval between Subbotin's first and last works on this topic is 32 years, and the case $h < h_1 \le 2h$ (overlapping averaging intervals) turned out to be the most difficult to study. Subbotin proved that for any $r \in \mathbb{N}$ the equality $A_p(D^n, h, 2rh) = +\infty$ holds. In 1983, in [10] quantity (0.4) was computed for $1 \le p \le \infty$, $0 < h < h_0 = \pi/(\max \alpha_s)$ for an arbitrary linear differential operator of form (0.1) for nonoverlapping averaging intervals, i.e. in case $0 \le h_1 \le h$.

Later in 1998 in [11], the quantity $A_p(\mathcal{L}_n, h, h_1)$ was found for $0 < h < h_0$, $h < h_1 \le 2h$ (overlapping averaging intervals) also in the general case for an arbitrary linear differential operator $\mathcal{L}_n$ of form (0.1), but only for $p = \infty$. Moreover, it turned out that the quantity $A_\infty(\mathcal{L}_n, h, 2h) = \infty$, and interest in the problem of extremal interpolation subsided for a long time. Recently, the author (see [12]) managed, using the ideas of previous works by Subbotin and works [10], [11], to generalize the result of [11] to the case $1 < p < \infty$, but only under the additional assumptions that the operator $\mathcal{L}_n$ is formally self-adjoint and n is odd. Later in [13], for such differential operators, quantity (0.4) was also computed for $p = 1$ for all positive integer numbers $n \in \mathbb{N}$.

In the present article, we indicate the value of the quantity $A_p(\mathcal{L}_n, h, h_1)$ for $0 < h < h_0, h < h_1 \le 2h$, if the number n is even and the operator $\mathcal{L}_n$ is formally self-adjoint (i.e., satisfies the condition $\mathcal{L}_n(-D) = \mathcal{L}_n(D)$). Let us explain how the proof of the supremum given below for the quantity $A_p(\mathcal{L}_n, h, h_1)$ (this is the main content of the present work) differs in the case when the number n is even. The auxiliary polynomials considered below are the same ones that appeared in the author's previous works [10]–[13]. On their basis, more general characteristic polynomials of difference equations are constructed, depending on the "gluing" parameter of the subsequent extremal interpolation $\mathcal{L}$-splines and their generalizations. Since this parameter is chosen differently depending on the parity of the number n (This choice for the n-fold differentiation operator was first made by Subbotin [3] in 1965), the properties of these polynomials for different n are significantly different. For even n, an additional study of the zeros (namely, the proof of their negativity and simplicity) of the characteristic polynomials constructed below was required, which in the present work is carried out using Subbotin method [7]–[9]. For the operator $\mathcal{L}_n(D) = D^n$, he also divided his proof into two cases depending on the parity of n. Note that for the even n it turned out to be more complicated (see Lemmas 1, 2 and 4–9 below).

To formulate the main statement of this work, we introduce auxiliary functions from the papers [10], [11]. Let

$$H_n(t) = C(\mathcal{L}_n, h) \sum_{s \in \mathbb{Z}} \frac{e^{i(2s+1)\pi(1-t)} \sin(((2s+1)\pi h_1)/(2h))}{\pi(2s+1)h_1 p_n(((2s+1)\pi i)/h)}, \qquad (0.5)$$

$$C(\mathcal{L}_n, h) = 2(-1)^{n+1} h \prod_{s=1}^{k} (1 + 2e^{\gamma_s h} \cos \alpha_s h + e^{2\gamma_s h}) \prod_{j=1}^{n-2k} (1 + e^{\beta_j h}),$$

where $p_n = p_n(\lambda)$ is a characteristic polynomial of the operator $\mathcal{L}_n$ (see (0.2)).

The main result of the work is

Theorem 1. *Let number n be even, and $\mathcal{L}_n = \mathcal{L}_n(D)$ be an arbitrary linear differential operator of form* (0.1) *satisfying the condition*

$$\mathcal{L}_n(-D) = \mathcal{L}_n(D).$$

Then for any $1 < p < \infty$, $0 < h < h_0 = \pi/(\max \alpha_s)$, $h < h_1 < 2h$, the equality

$$A_p(\mathcal{L}_n, h, h_1) = (\|H_n\|_{L_q[0;1]})^{-1}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

holds.

Remark 1. Note that in [10], [11] it was proven that for any $0 < h < h_0$, $0 < h_1 < 2h$ the equality

$$A_\infty(\mathcal{L}_n, h, h_1) = (\|H_n\|_{L_1[0;1]})^{-1}$$

holds for an arbitrary linear differential operator of form (0.1) (without the requirement of its formal self-adjointness). As mentioned earlier, Theorem 1 for odd n was proved in [12].

1. PROPERTIES OF AUXILIARY FUNCTIONS

With the operator $\mathcal{L}_n$ in (0.1) we associate the operator $\mathcal{L}^0_{n+1}(D) = D\mathcal{L}_n(D)$. Let $\varphi_n = \varphi_n(t)$ (respectively, $\varphi^0_{n+1} = \varphi^0_{n+1}(t)$) be the unique solution of the equation $\mathcal{L}_n(D)f = 0$ (respectively, $\mathcal{L}^0_{n+1}(D)f = 0$) with the condition $\varphi_n^{(j)}(0) = \delta_{j,n-1}$ (respectively, $(\varphi^0_{n+1})^{(j)}(0) = \delta_{j,n}$). Here $\delta_{j,n-1}$ and $\delta_{j,n}$ are the Kronecker deltas. With the differential operator $\mathcal{L}^0_{n+1}$ we associate a difference operator

$$\Delta_h^{\mathcal{L}^0_{n+1}} = (T - E)\Delta_h^{\mathcal{L}_n}$$

(see (0.3)), which is defined on the space of sequences of the form $y = \{y_m\}_{m=-\infty}^{\infty}$.

The operators $\Delta_h^{\mathcal{L}_n}$ and $\Delta_h^{\mathcal{L}_{n+1}^0}$ can be easily reduced to the form

$$\Delta_h^{\mathcal{L}_n} y_m = \sum_{s=0}^{n} (-1)^{n-s} \mu_s y_{m+s}, \qquad \Delta_h^{\mathcal{L}_{n+1}^0} y_m = \sum_{s=0}^{n+1} (-1)^{n+1-s} \mu_s^0 y_{m+s},$$

where $\mu_s = \mu_s(\mathcal{L}_n, h) > 0$, $\mu_s^0 = \mu_s^0(\mathcal{L}_{n+1}^0, h) > 0$ and do not depend on y_m, moreover, $\mu_s^0 = \mu_s + \mu_{s-1}$, $s = 0, 1, \dots, n+1$ (the numbers μ_{-1} and μ_{n+1} are set to zero). Coefficients $\{\mu_s\}$ and $\{\mu_s^0\}$, functions φ_n and φ_{n+1}^0 are employed in the definition of all further auxiliary functions. Following [10], for $0 \le t \le 1$, $h_1 > 0$, $0 < h < \pi/(\max \alpha_s)$ we define the functions

$$\begin{aligned}
P_n(t) &= \sum_{j=0}^{n} (-1)^j \sum_{s=j}^{n} (-1)^{n-s} \mu_s \varphi_n((s - j + 1 - t)h), \\
P_{n+1}^0(t) &= \sum_{j=0}^{n+1} (-1)^j \sum_{s=j}^{n+1} (-1)^{n+1-s} \mu_s^0 \varphi_{n+1}^0((s - j + 1 - t)h), \\
a_{j,n}(t, h, h_1) &= \frac{h^2}{h_1} \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{n} (-1)^{n-l} \mu_l \varphi_n((l + x + 1 - j - t)_+ h)\, dx, \\
H_n(t) &= \sum_{j=0}^{n+1} (-1)^j a_{j,n}(t, h, h_1).
\end{aligned} \tag{1.1}$$

Here, as usual, u_+ stands for $\max\{0, u\}$.

Due to the equalities $P_n(1) = -P_n(0)$, $P_{n+1}^0(1) = -P_{n+1}^0(0)$, $H_n(1) = -H_n(0)$, these functions can be continued to the entire real axis by the formulas

$$P_n(t+1) = -P_n(t), \qquad P_{n+1}^0(t+1) = -P_{n+1}^0(t), \qquad H_n(t+1) = -H_n(t).$$

In [10] it was proved that for these continuations the inclusions $P_n \in C^{n-2}(\mathbb{R})$, $P_{n+1}^0 \in C^{n-1}(\mathbb{R})$, $H_n \in C^{n-1}(\mathbb{R})$ take place. Moreover, the following equalities hold:

$$(P_{n+1}^0(t))' = 2h P_n(t), \qquad \frac{h_1}{h^2} H_n(t) = \frac{1}{2h}\left(P_{n+1}^0\left(t + \frac{h_1}{2h}\right) - P_{n+1}^0\left(t - \frac{h_1}{2h}\right)\right).$$

Since the constructed function $H_n(t)$ is 2-periodic, it can be expanded into the Fourier series, and at the same time representation (0.5) is valid (see [10], [11]). The properties of the functions $P_n(t)$, $P_{n+1}^0(t)$ and $H_n(t)$ are described in [10, Lemmas 1, 7], [11, Lemmas 1, 5], [12, Lemmas 1–5 and Remark 4] (see also [14], [15]). Let us continue to study the properties of the function $H_n(t)$ in the case of overlapping averaging intervals, i.e. in the case $h < h_1 < 2h$. These properties are useful for us in the next sections. In all subsequent statements we also assume the operator $\mathcal{L}_n$ to be formally self-adjoint (i.e. we assume that it satisfies the condition $\mathcal{L}_n(-D) = (-1)^n \mathcal{L}_n(D)$) and the condition $0 < h < h_0$ to hold. Under these assumptions, from [14] it follows that for even $n = 2r$ the function $\varphi_{2r}(t)$ is odd, and for odd $n = 2r + 1$ the function $\varphi_{2r+1}(t)$ is even.

Lemma 1. *Let* $1 < q < \infty$, $n = 2r$, $r \in \mathbb{N}$, *and*

$$\Phi_{2r}(t) = \left|H_{2r}\left(t - \frac{h_1}{2h}\right)\right|^{q-1} - \left|H_{2r}\left(t + \frac{h_1}{2h}\right)\right|^{q-1}.$$

The following inequalities hold:

$$\text{a) } \Phi_{2r}(t) \le 0 \qquad \textit{for} \quad 0 \le t \le \frac{h_1}{(2h)} - \frac{1}{2},$$

$$\text{b) } \Phi_{2r}(t) \ge 0 \qquad \textit{for} \quad \frac{3}{2} - \frac{h_1}{(2h)} \le t \le 1.$$

Proof. It follows from [Lemma 5 and Remark 4][12] that the function $|H_{2r}(t)|$ is monotonically decreasing on the segment $[0; 0.5]$, moreover, $H_{2r}(0.5) = 0$, and it is monotonically increasing on the segment $[0.5; 1]$, moreover, $|H_{2r}(1)| = |H_{2r}(0)|$. In addition, $|H_{2r}(t+1)| = |H_{2r}(t)|$, $|H_{2r}(1-t)| = |H_{2r}(t)|$ for all $t \in \mathbb{R}$. To prove Lemma 1 we consider two cases:

1) $1 < h_1/h \le 3/2$,

2) $3/2 < h_1/h < 2$.

In the first case, due to the properties of the function $H_{2r}(t)$, the function $|H_{2r}(t - h_1/(2h))|^{q-1}$ is monotonically decreasing on the segment $[0; h_1/(2h) - 1/2]$, and the function $|H_{2r}(t + h_1/(2h))|^{q-1}$ is monotonically increasing on this segment, since for $1 < h_1/h \le 3/2$ the inequality $h_1/h - 1/2 \le 1$ holds and

$$\Phi_{2r}(0) = \left|H_{2r}\left(-\frac{h_1}{2h}\right)\right|^{q-1} - \left|H_{2r}\left(\frac{h_1}{2h}\right)\right|^{q-1} = 0,$$

$$\Phi_{2r}\left(\frac{h_1}{2h} - \frac{1}{2}\right) = -\left|H_{2r}\left(\frac{h_1}{h} - \frac{1}{2}\right)\right|^{q-1} < 0.$$

This implies the validity of the first inequality in Lemma 1 for $0 \le t \le h_1/(2h) - 1/2$.

In case 2), we divide the segment $[0; h_1/(2h) - 1/2]$ into two subsegments:

$$\left[0; 1 - \frac{h_1}{2h}\right] \qquad \text{and} \qquad \left[1 - \frac{h_1}{2h}; \frac{h_1}{2h} - \frac{1}{2}\right].$$

On the first subsegment, the previous proof is completely valid, since the function $|H_{2r}(t - h_1/(2h))|^{q-1}$ is monotonically decreasing, and the function $|H_{2r}(t + h_1/(2h))|^{q-1}$ is monotonically increasing. On the second subsegment, we use the fact the function $H_{2r}(t)$ is 2-periodic, namely, we use the equality

$$\left|H_{2r}\left(t + \frac{h_1}{2h}\right)\right|^{q-1} = \left|H_{2r}\left(t + \frac{h_1}{2h} - 2\right)\right|^{q-1}.$$

Then the arguments of both functions $|H_{2r}(t - h_1/(2h))|^{q-1}$ and $|H_{2r}(t + h_1/(2h) - 2)|^{q-1}$ belong to the segment $[-1; -0.5]$, moreover, $t + h_1/(2h) - 2 < t - h_1/(2h)$, and hence, due to the fact that the function $|H_{2r}(t)|^{q-1}$ is monotonically decreasing on the segment $[0; 0.5]$, it implies the first statement of Lemma 1. The second statement of Lemma 1 follows by the first one, if we make a change of variable $1 - t = t'$ and use the properties of the function $|H_{2r}(t)|$ mentioned in the beginning of the proof of Lemma 1. Proof of Lemma 1 is completed. □

Corollary 1. *For $q > 1$, $-1 \le x \le 1$, $3/2 - h_1/(2h) \le t \le 1$ the inequality*

$$\psi_{2r}(x,t) = \left|H_{2r}\left(t - \frac{h_1}{2h}\right)\right|^{q-1} - x^2\left|H_{2r}\left(t + \frac{h_1}{2h}\right)\right|^{q-1} \ge 0 \tag{1.2}$$

holds.

Proof. The validity of corollary 1 follows by the inequality

$$\psi_{2r}(x,t) \ge \Phi_{2r}(t) \ge 0$$

for $3/2 - h_1/(2h) \le t \le 1$. □

Lemma 2. *For $q > 1$ the function*

$$S_{2r}(t) = \left|\frac{H_{2r}(t - h_1/(2h))}{H_{2r}(t + h_1/(2h))}\right|^{q-1}$$

monotonically decreases from 1 to 0 on the segment $[0; h_1/(2h) - 1/2]$.

Proof. In the cases described in Lemma 1

1) $1 < h_1/h \le 3/2$,

2) $3/2 < h_1/h < 2, 0 \le t \le 1 - h_1/(2h)$

the statement of Lemma 2 is in fact proved in the mentioned lemma, since the numerator of the fraction $S_{2r}(t)$ is monotonically decreasing with rising t, and the denominator is monotonically increasing.

It only remains to consider the case

3) $3/2 < h_1/(2h) < 2, 1 - h_1/(2h) \le t \le h_1/(2h) - 1/2$.

Since in this case $1 \le t + h_1/(2h) \le 3/2$, $1 \le t + 2 - h_1/(2h) \le 3/2$, then to prove Lemma 2, due to the properties of the function $H_{2r}(t)$, it is sufficient to establish that the derivative of the function

$$\widetilde{S}_{2r}(t) = \frac{H_{2r}(t + 2 - h_1/(2h))}{H_{2r}(t + h_1/(2h))}$$

is nonpositive on the segment $[1 - h_1/(2h); h_1/(2h) - 1/2]$. In that case, without loss of generality, one can assume that the numerator and denominator of the fraction $\widetilde{S}_{2r}(t)$ are nonnegative. In view of [12, Lemma 5 and Remark 4], we have

$$\widetilde{S}'_{2r}(t) = \left(H_{2r}\left(t + \frac{h_1}{2h}\right)\right)^{-2}\left(\widetilde{H}_{2r-1}\left(2 + t - \frac{h_1}{2h}\right)H_{2r}\left(t + \frac{h_1}{2h}\right) - H_{2r}\left(2 + t - \frac{h_1}{2h}\right)\widetilde{H}_{2r-1}\left(t + \frac{h_1}{2h}\right)\right),$$

where

$$\widetilde{H}_{2r-1}(t) = H'_{2r}(t) = P_{2r}\left(t + \frac{h_1}{2h}\right) - P_{2r}\left(t - \frac{h_1}{2h}\right)$$

(see (1.1)). Note that both points $t + h_1/(2h)$ and $t + 2 - h_1/(2h)$ belong to the interval $[1; 3/2]$ and $t + h_1/(2h) < t + 2 - h_1/(2h)$. Since the function $H_{2r}(t)$ is monotone decreasing on this interval, we obtain the inequality

$$H_{2r}\left(t + \frac{h_1}{2h}\right) > H_{2r}\left(t + 2 - \frac{h_1}{2h}\right) \ge 0. \tag{1.3}$$

From [11, Lemma 5] (see also [12, Lemma 5 and Remark 4]), in the case of formally self-adjoint operator $\mathcal{L}_n$, due to the fact that on the segment $[1; 3/2]$ the function H_{2r} is nonnegative, monotonically decreasing, and, moreover, the equality $\widetilde{H}_{2r-1}(t) = H'_{2r}(t)$ holds, we get that the function $\widetilde{H}_{2r-1}(t)$ on this segment is monotonically decreasing and nonpositive. Since $t + h_1/(2h) < t + 2 - h_1/(2h)$,

$$0 \ge \widetilde{H}_{2r-1}\left(t + \frac{h_1}{2h}\right) > \widetilde{H}_{2r-1}\left(t + 2 - \frac{h_1}{2h}\right). \tag{1.4}$$

(1.3) and (1.4) imply that $\widetilde{S}'_{2r}(t) \le 0$ for $1 - h_1/(2h) \le t \le h_1/(2h) - 1/2$. The proof of Lemma 2 is complete. □

2. PROPERTIES OF SOME ALGEBRAIC POLYNOMIALS

In Sec. 3, the problem of obtaining an upper bound for the quantity $A_p(\mathcal{L}_n, h, h_1)$ is reduced to studying the zeros of the characteristic polynomial of an infinite-difference equation. So we need to study the properties of some algebraic polynomials.

Let $\mathcal{L}_n$ be an arbitrary linear differential operator of form (0.1) and $0 < h < h_0$, $h < h_1 < 2h$, $q > 1$. For $0 \le t \le 1$, consider a polynomial of degree n in variable x

$$R_n^0(x,t) = \sum_{l=0}^{n} c_l x^l, \qquad c_l = \sum_{s=0}^{l} (-1)^{n-s} \mu_s^0 \varphi_{n+1}^0((s-l-t)h).$$

Lemma 3 [10, Lemmas 3 and 5]. 1. *For the coefficients of the polynomial $R_n^0(x,t)$ for $0 < t < 1$, the inequalities $c_0 > 0$, $c_n > 0$ hold.*

2. *Let $0 < t < 1$ and $\eta_j(t) < 0$, $j = 1, 2, \dots, n$, be zeros of the polynomial $R_n^0(x,t)$ written in descending order, and $\eta_1 = 0$, $\eta_j < 0$, $j = 2, 3, \dots, n$, be zeros of the polynomial $R_n^0(x,0)$ written in descending order. Then the following assertions hold*:

1) *for any $j = 1, 2, \dots, n$, the function $\eta_j(t)$ is strictly decreasing on the interval* (0;1)*, and for $0 < t < 1$ the inequalities*

$$\eta_{j+1} < \eta_j(t) < \eta_j, \quad j = 1, 2, \dots, n-1, \qquad -\infty < \eta_n(t) < \eta_n$$

hold;

2) *the following equalities hold*:

$$\operatorname{sign} R_n^0(\eta_j(t), u) = \begin{cases} (-1)^j, & 0 \le u < t, \\ (-1)^{j+1}, & t < u < 1, \end{cases}$$

$$\operatorname{sign} R_n^0(\eta_j, t) = (-1)^{j+1}, \qquad 0 < t < 1.$$

In what follows, we will need some more properties of the polynomial $R_n^0(x,t)$ ialignn the case where the operator $\mathcal{L}_n$ is formally self-adjoint and $n = 2r$, $r \in \mathbb{N}$.

Lemma 4. *Let $0 < h < h_0$, and the operator $\mathcal{L}_{2r}$ be formally self-adjoint. Then the following assertions hold*:

1) $\operatorname{sign} R_{2r}^0(-1,t) = (-1)^r$*, $0 < t < 1$,*

2) *the polynomial $R_{2r}^0(x,0)/x$ is self-reciprocal* (*i.e. if x_0 is its root, then $1/x_0$ is its root as well*)*;*

3) $R_{2r}^0(-1,0) = 0$*, i.e. $\eta_{r+1} = -1$.*

Proof. From the definitions of functions $P_{n+1}^0(t)$ and $R_n^0(x,t)$ for $x = -1$ (see [14], [15]) equality $R_{2r}^0(-1,t) = -P_{2r+1}^0(t)$ follows, $0 < t < 1$. Moreover, $\operatorname{sign} P_{2r+1}^0(t) = (-1)^r$, $1/2 < t < 1$, by [14, Eq. (1.19)]. This implies the first statement of Lemma 4.

To prove the second statement, we first note that at $t = 0$ we have $c_0 = 0$ (therefore $\eta_1 = 0$), and we need to prove the equality $c_{2r+1-l} = c_l$, $l = 1, 2, \dots, 2r$. This fact follows, e.g., from [14, Eq. (2.9)].

The third statement follows by the second one and Lemma 3. Proof of Lemma 4 is completed. □

Lemma 5. *The nonlinear system of equations*

$$\begin{cases} R_{2r}^0(x,u) = 0, \\ \psi_{2r}(x,u) = 0 \end{cases}$$

in the region $-1 < x < 0$, $0 \le u \le h_1/(2h) - 1/2$ has a unique solution $(\overline{x}, \overline{u})$ consisting of exactly r points: $\overline{x} = \{\eta_j(u_j)\}_{j=1}^r$, $\overline{u} = \{u_j\}_{j=1}^r$.

Proof. The function $x = -\sqrt{S_{2r}(u)}$ (see (1.2) and Lemma 2) with rising of argument u from 0 to $h_1/(2h) - 1/2$ monotonically increases from -1 to 0, and the functions $\eta_j(u)$, $j = 1, 2, \dots, 2r$, for $0 < u < 1$, due to Lemma 3, monotonically decrease. At the same time, the inequalities

$$\eta_j < \eta_j(u) < \eta_{j-1}, \qquad j = 2, 3, \dots, r+1,$$

hold, where $\eta_1 = 0$ and $\eta_{r+1} = -1$ (see Lemma 4). Therefore, each nonlinear system of equations

$$\begin{cases} R^0_{2r}(\eta_j(u), u) = 0, \\ x = -\sqrt{S_{2r}(u)}\,, \end{cases} \qquad j = 1, 2, \dots, r,$$

has a unique solution; denote it by $(\eta_j(u_j), u_j)$. Thus, the system of equations in Lemma 5 has exactly r pairwise distinct solutions $(\overline{x}, \overline{u})$, where $\overline{x} = \{\eta_j(u_j)\}_{j=1}^r$, $\overline{u} = \{u_j\}_{j=1}^r$, for which the inequalities

$$0 < u_r < \dots < u_2 < u_1 < \frac{h_1}{2h} - \frac{1}{2}\,, \qquad -1 < \eta_r(u_r) < \dots < \eta_2(u_2) < \eta_1(u_1) < 0$$

hold. Proof of Lemma 5 is completed. □

3. LOWER AND UPPER BOUNDS FOR $A_p(\mathcal{L}_{2r}, h, h_1)$

Let us proceed to the proof of Theorem 1. First, we obtain a precise upper bound for the quantity $A_p(\mathcal{L}_{2r}, h, h_1)$. Let $0 < h < h_0$, $h < h_1 < 2h$, $1 < p < \infty$, $1/p + 1/q = 1$ and $\mathcal{L}_n(D)$ be an arbitrary linear differential operator of order n of form (0.1). Any solution to the linear nonhomogeneous differential equation $\mathcal{L}_n(D)f = u$, where $u \in L_p(\mathbb{R})$, can be written as

$$f(x) = \sum_{j=1}^n C_j v_j(x) + \int_0^x \varphi_n(x-t)u(t)\,dt. \tag{3.1}$$

Here $\{C_j\}_{j=1}^n$ are arbitrary constants, the function φ_n is defined in Sec. 1, and $\{v_j(x)\}_{j=1}^n$ is an arbitrary linearly independent system of functions from the kernel $\operatorname{Ker}\mathcal{L}_n$ of the operator $\mathcal{L}_n$. Let

$$y_m = \frac{1}{h_1}\int_{-h_1/2}^{h_1/2} f(mh+t)\,dt, \qquad m \in \mathbb{Z}.$$

In [10] for $\Delta_h^{\mathcal{L}_n} y_m$ (see (0.3)) the following equality was proved:

$$\Delta_h^{\mathcal{L}_n} y_m = \frac{h^2}{h_1}\int_0^1 \sum_{j=0}^{n+1} u((t+m-1+j)h)\,dt$$
$$\times \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^n (-1)^{n-l}\mu_l \varphi_n((l+x+1-j-t)_+ h)\,dx. \tag{3.2}$$

Further in this section we assume that $n = 2r$ and the operator $\mathcal{L}_{2r}$ is formally self-adjoint. Let us construct a generalized $\mathcal{L}$-spline $f \in F_{h,h_1,p}(y)$ for any sequence $y \in Y_{h,p}$ using formula (3.1), setting

$$u(t) = \mathcal{L}_{2r}(D)f(t) = Z_m\left|H_{2r}\left(\frac{t}{h}\right)\right|^{q-1}, \quad (m-0.5)h \le t < (m+0.5)h, \qquad m \in \mathbb{Z}, \tag{3.3}$$

where the function H_{2r} is defined by equality (0.5) for $n = 2r$. From (3.3) it follows that the "gluing" nodes of $\mathcal{L}$-spline of f are uniform (moreover, they are also the zeros of the function H_{2r}), and they are shifted for half a step compared to the interpolation nodes. The real numbers $\{Z_m\}_{m=-\infty}^{\infty}$ are still to be determined. From (3.2) and (3.3) we obtain a difference equation for the numbers $\{Z_m\}_{m=-\infty}^{\infty}$

$$\Delta_h^{\mathcal{L}_{2r}} y_m = \frac{h^2}{h_1}\sum_{j=0}^{2r+2} Z_{m-1+j}B_j, \qquad m \in \mathbb{Z}, \tag{3.4}$$

where

$$B_0 = \int_0^{0.5} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_l \varphi_{2r}((l+z+1-t)_+h)\,dz,$$

$$B_j = \int_0^{0.5} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_l \varphi_{2r}((l+z+1-j-t)_+h)\,dz$$
$$+ \int_0^{0.5} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_l \varphi_{2r}((l+z+2-j-t)_+h)\,dz,$$
$$j = 1,2,\dots,2r+1,$$

$$B_{2r+2} = \int_{0.5}^{1} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_l \varphi_{2r}((l+z+2r-t)_+h)\,dz.$$

The characteristic polynomial of the difference equation (3.4) can be written as

$$U_{2r+2}(x) = \sum_{j=0}^{2r+2} B_j x^j. \tag{3.5}$$

Lemma 6. *The polynomial $U_{2r+2}(x)$ is self-reciprocal.*

Proof. We have to prove that $B_{2r+2-j} = B_j$, $j = 0,1,\dots,2r$. For $j = 1,2,\dots,2r-1$, after the changes $1-t = t'$, $z = -z'$ (and then after omitting the primes), with the oddness of the function $\varphi_{2r}(t)$ (see Sec. 1) and the properties of the function $H_{2r}(t)$ taken into account, we have

$$B_j = \int_{0.5}^{1} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_l (\varphi_{2r}((l-z-j+t)h))_+\,dz$$
$$+ \int_0^{0.5} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_l (\varphi_{2r}((l-z-j+1-t)h))_+\,dz.$$

Now we use the equality $u_+ + u_- = u$ and the fact that $\mu_{2l-r} = \mu_l$, $l = 0,1,\dots,2r$, since the operator $\mathcal{L}_{2r}$ is formally self-adjoint. In addition, we use the fact that the difference operator $\Delta_h^{\mathcal{L}_{2r}}$ annihilates any function from the kernel of the differential operator $\mathcal{L}_{2r}$, taken on a uniform grid with step h. We get

$$B_j = -\int_0^{0.5} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_{2r-l} (\varphi_{2r}((l-z-j+1+t)h))_-\,dz$$
$$- \int 0.5^1 |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_{2r-l} (\varphi_{2r}((l-z-j+t)h))_-\,dz.$$

Let us apply the equality $(-u)_- = -u_+$ and make the change $2r-l = l'$. Then from the previous equality we obtain (the primes are omitted)

$$B_j = \int_0^{0.5} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_l \varphi_{2r}((l-2r+z+j-1-t)_+h))\,dz$$
$$+ \int_{0.5}^{1} |H_{2r}(t)|^{q-1}\,dt \int_{-h_1/(2h)}^{h_1/(2h)} \sum_{l=0}^{2r} (-1)^{2r-l}\mu_l \varphi_{2r}((l-2r+z+j-t)_+h))\,dz$$
$$= B_{2r+2-j}.$$

The equality $B_0 = B_{2r+2}$ can be proved by similar reasoning. Proof of Lemma 6 is completed. ☐

Let us transform the characteristic polynomial $U_{2r+2}(x)$ of degree $2r+2$ of difference equation (3.4). Let

$$Q_{2r+1}(x,z-t)=\begin{cases} x^2R_{2r-1}(x,-z+t+1), & 0\le z-t<1,\\ xR_{2r-1}(x,-z+t), & -1\le z-t<0,\\ R_{2r-1}(x,-z+t-1), & -2\le z-t<-1,\end{cases}$$

where the function $R_{2r-1}(x,t)$ has the form

$$R_{2r-1}(x,t)=\sum_{l=0}^{2r-1}x^l\sum_{s=0}^{l}(-1)^{2r-1-s}\mu_s\varphi_{2r}((s-l-t)h),\qquad 0\le t\le 1.$$

Then formula (3.5) can be rewritten as

$$U_{2r+2}(x)=\int_0^{0.5}|H_{2r}(t)|^{q-1}\,dt\int_{-t-h_1/(2h)}^{-t+h_1/(2h)}Q_{2r+1}(x,z)\,dz$$
$$+\int_{0.5}^{1}|H_{2r}(t)|^{q-1}\,dt\int_{-t-h_1/(2h)}^{-t+h_1/(2h)}xQ_{2r+1}(x,z)\,dz.$$

In [10] the following formulas were proved:

$$(R^0_{2r}(x,t))'_t=h(1-x)R_{2r-1}(x,t),\quad 0\le t\le 1,\qquad xR^0_{2r}(x,t+1)=R^0_{2r}(x,t),\qquad t\in\mathbb{R}.$$

Changing the order of integration in the formula for $U_{2r+2}(x)$ and using the mentioned formulas, we obtain

$$U_{2r+2}(x)=\int_0^{1-h_1/(2h)}|H_{2r}(t)|^{q-1}\bigg[\int_{-t-h_1/(2h)}^{0}xR_{2r-1}(x,-z)\,dz$$
$$+\int_0^{-t+h_1/(2h)}x^2R_{2r-1}(x,-z+1)\,dz\bigg]\,dt$$
$$+\int_{1-h_1/(2h)}^{0.5}|H_{2r}(t)|^{q-1}\bigg[\int_{-t-h_1/(2h)}^{-1}R_{2r-1}(x,-z-1)\,dz$$
$$+\int_{-1}^{0}xR_{2r-1}(x,-z)\,dz+\int_0^{-t+h_1/(2h)}x^2R_{2r-1}(x,-z+1)\,dz\bigg]\,dt$$
$$+\int_{0.5}^{h_1/(2h)}|H_{2r}(t)|^{q-1}\bigg[\int_{-t-h_1/(2h)}^{-1}xR_{2r-1}(x,-z-1)\,dz$$
$$+\int_{-1}^{0}x^2R_{2r-1}(x,-z)\,dz+\int_0^{-t+h_1/(2h)}x^3R_{2r-1}(x,-z+1)\,dz\bigg]\,dt$$
$$+\int_{h_1/(2h)}^{1}|H_{2r}(t)|^{q-1}\bigg[\int_{-t-h_1/(2h)}^{-1}xR_{2r-1}(x,-z-1)\,dz$$
$$+\int_{-1}^{-t+h_1/(2h)}x^2R_{2r-1}(x,-z)\,dz\bigg]\,dt$$
$$=\frac{1}{h(1-x)}\bigg\{\int_0^{1-h_1/(2h)}|H_{2r}(t)|^{q-1}\bigg[xR^0_{2r}\Big(x,t+\frac{h_1}{2h}\Big)-x^2R^0_{2r}\Big(x,t+1-\frac{h_1}{2h}\Big)\bigg]\,dt$$
$$+\int_{1-h_1/(2h)}^{0.5}|H_{2r}(t)|^{q-1}\bigg[R^0_{2r}\Big(x,t+\frac{h_1}{2h}-1\Big)-x^2R^0_{2r}\Big(x,t+1-\frac{h_1}{2h}\Big)\bigg]\,dt$$
$$+\int_{0.5}^{h_1/(2h)}|H_{2r}(t)|^{q-1}\bigg[xR^0_{2r}\Big(x,t+\frac{h_1}{2h}-1\Big)-x^3R^0_{2r}\Big(x,t+1-\frac{h_1}{2h}\Big)\bigg]\,dt$$

$$+\int_{h_1/(2h)}^{1}|H_{2r}(t)|^{q-1}\left[xR_{2r}^0\left(x,t+\frac{h_1}{2h}-1\right)-x^2R_{2r}^0\left(x,t-\frac{h_1}{2h}\right)\right]dt\Bigg\}. \tag{3.6}$$

Since $H_{2r}(t+1)=-H_{2r}(t)$, from (3.6) we derive that

$$\begin{aligned}&U_{2r+2}(x)\\&=\frac{1}{h(1-x)}\Bigg\{\int_0^{h_1/(2h)-0.5}R_{2r}^0(x,t)\left[\left|H_{2r}\left(t-\frac{h_1}{2h}\right)\right|^{q-1}-x^2\left|H_{2r}\left(t+\frac{h_1}{2h}\right)\right|^{q-1}\right]dt\\&\quad+\int_{h_1/(2h)-0.5}^{1.5-h_1/(2h)}R_{2r}^0(x,t)\left[x\left|H_{2r}\left(t-\frac{h_1}{2h}\right)\right|^{q-1}-x^2\left|H_{2r}\left(t+\frac{h_1}{2h}\right)\right|^{q-1}\right]dt\\&\quad+\int_{1.5-h_1/(2h)}^{1}R_{2r}^0(x,t)\left[x\left|H_{2r}\left(t-\frac{h_1}{2h}\right)\right|^{q-1}-x^3\left|H_{2r}\left(t+\frac{h_1}{2h}\right)\right|^{q-1}\right]dt\Bigg\}.\end{aligned} \tag{3.7}$$

Lemma 7. *One has the inequality* $\operatorname{sign}U_{2r+2}(-1)=(-1)^{r+1}$.

The proof of Lemma 7 follows by representation (3.7) for $x=-1$, Lemma 1, corollary 1 and the first statement of Lemma 4.

Lemma 8. *The solution of the system of equations in Lemma* 5 *satisfies the relations*

$$\operatorname{sign}U_{2r+2}(\eta_j(u_j))=(-1)^j,\qquad j=1,2,\dots,r.$$

Proof. Let us first note that from the proof of Lemma 5 it follows that for any $j=1,2,\dots,r$ the following inequalities hold: $\psi_{2r}(\eta_j(u_j),u)>0$ for $0\le u<u_j$ and $\psi_{2r}(\eta_j(u_j),u)<0$ for $u_j<u\le h_1/(2h)-1/2$. Therefore, from (3.7) and the second statement of Lemma 3, the validity of Lemma 8 follows. □

Lemma 9. *All* $2r+2$ *roots of the polynomial* $U_{2r+2}(x)$ *are negative and simple.*

Proof. By virtue of Lemma 8 we get that the polynomial $U_{2r+2}(x)$ has sign changes on every interval $(\eta_j(u_j);\eta_{j-1}(u_{j-1}))$, $j=1,2,\dots,r$. This means that the mentioned polynomial on the interval $(\eta_r(u_r);\eta_1(u_1))$ has at least $r-1$ negative roots. On the interval $(-1;\eta_r(u_r))$, the polynomial $U_{2r+2}(x)$ has at least one more root, since $\operatorname{sign}U_{2r+2}(\eta_r(u_r))=(-1)^r$, and due to Lemma 7, $\operatorname{sign}U_{2r+2}(-1)=(-1)^{r+1}$. The first statement of Lemma 3 implies that the free coefficient B_0 of the polynomial $U_{2r+2}(x)$ (i.e. the number $U_{2r+2}(0)$) is positive. However, on the other hand, $U_{2r+2}(\eta_1(u_1))<0$. This means that on the interval $(\eta_1(u_1);0)$ there is one more negative root. Thus, it is proved that on the interval $(-1;0)$ the polynomial $U_{2r+2}(x)$ has at least $r+1$ pairwise distinct negative roots. Recall that it is proved in Lemma 6 that this polynomial is self-reciprocal. Therefore, on the semiaxis $(-\infty;-1)$ this polynomial has $r+1$ more pairwise distinct negative roots, which completes the proof of Lemma 9. □

The previous auxiliary statements aimed to prove that the characteristic polynomial being studied satisfies the conditions of the following theorem.

Theorem A. *If all zeros of the polynomial* $U_r(x)=\sum_{j=0}^{r}B_jx^j$, $B_j\in\mathbb{R}$, $B_r\ne0$, *are negative and simple*, $U_r(-1)\ne0$, *then the difference equation* $\sum_{j=0}^{r}B_jZ_{m+j}=K_m$, $m\in\mathbb{Z}$, *where* $K=\{K_m\}_{m=-\infty}^{\infty}\in\ell_p$, $1\le p\le\infty$, *has a unique solution* $Z^0=\{Z_m^0\}_{m=-\infty}^{\infty}\in l_p$ *given by the formula*

$$Z_m^0=\sum_{s=-\infty}^{\infty}a_{-s-m}K_s,$$

where $\sum_{s=-\infty}^{\infty}a_sx^s=1/U_r(x)$, *for which the bound*

$$\|Z^0\|_{l_p}\le\frac{\|K\|_{l_p}}{|U_r(-1)|}$$

holds.

The existence of a solution for the difference equation in Theorem A was proved by Krein [16], and an upper bound for the norm of this solution was obtained by Subbotin [5].

Let us return to the proof of Theorem 1 again and obtain a precise upper bound for the quantity $A_p(\mathcal{L}_{2r}, h, h_1)$. By virtue of what has been proved (see Lemmas 7 and 9), the polynomial $U_{2r+2}(x)$ (we replace the number r in Theorem A with $2r+2$) satisfies all the conditions of Theorem A and, therefore, difference equation (3.4) for $n = 2r$ has a unique solution $Z^0 = \{Z_m\}_{m=-\infty}^{\infty} \in \ell_p$, for which the bound

$$\|Z^0\|_{l_p} \le \frac{h_1 \|\Delta_h^{\mathcal{L}_{2r}} y\|_{\ell_p}}{h^2 |U_{2r+2}(-1)|}$$

is valid.

In particular, this statement implies that for an arbitrary sequence $y \in Y_{h,p}$ there exists a function $f \in F_{h,h_1,p}(y)$, for which, by virtue of (3.3), the inequality

$$\begin{aligned} \|\mathcal{L}_n(D) f\|_{L_p(\mathbb{R})} &= \biggl(\sum_{m=-\infty}^{\infty} |Z_m^0|^p \int_0^1 |H_{2r}(t)|^{(q-1)p}\,dt\biggr)^{1/p} = \|Z^0\|_{l_p} \biggl(\int_0^1 |H_{2r}(t)|^q\,dt\biggr)^{1/p} \\ &\le \biggl(\int_0^1 |H_{2r}(t)|^q\,dt\biggr)^{-1+1/p} = (\|H_{2r}\|_{L_q[0;1]})^{-1} \end{aligned}$$

holds. Hence, for $0 < h < h_0$, $h < h_1 < 2h$, $1 < p < \infty$, $1/p + 1/q = 1$ for the quantity $A_p(\mathcal{L}_{2r}, h, h_1)$ we obtain an upper bound

$$A_p(\mathcal{L}_{2r}, h, h_1) \le (\|H_{2r}\|_{L_q[0;1]})^{-1}, \tag{3.8}$$

which is valid for any linear formally self-adjoint differential operator of form (0.1) in the case $n = 2r$.

It stands to mention that we constructed the function $f \in F_{h,h_1,p}(y)$ (see (3.2)) assuming

$$\Delta_h^{\mathcal{L}_{2r}} y_m = \Delta_h^{\mathcal{L}_{2r}} \biggl(\frac{1}{h_1} \int_{-h_1/2}^{h_1/2} f(mh+t)\,dt\biggr), \qquad m \in \mathbb{Z}.$$

In this case, it is necessary to justify that this function satisfies the conditions for interpolation in the mean, i.e.

$$y_m = \frac{1}{h_1} \int_{-h_1/2}^{h_1/2} f(mh+t)\,dt, \qquad m \in \mathbb{Z}.$$

This fact for $p = \infty$ was proved in [10] for an arbitrary linear differential operator of form (0.1). In the case $1 < p < \infty$ the proof of the mentioned statement from [10] remains valid completely.

We now obtain a lower bound for the quantity $A_p(\mathcal{L}_{2r}, h, h_1)$. Let N be a positive integer number, $N > n+1$. Consider an arbitrary sequence $y^* = \{y_m^*\}_{m=-\infty}^{\infty}$ satisfying the condition

$$\Delta_h^{\mathcal{L}_{2r}} y_m^* = \begin{cases} (-1)^m (2N+1)^{-1/p}, & |m| \le N, \\ 0, & |m| > N. \end{cases}$$

It can be easily verified that $y^* \in Y_{h,p}$. For any function $f \in F_{h,h_1,p}(y^*)$ in [10] and for $0 < h_1 < h < h_0$, in particular, for $n = 2r$, the inequality

$$\|\mathcal{L}_n(D) f\|_{L_p(\mathbb{R})} \ge (\|H_n\|_{L_q[0;1]})^{-1}$$

was proved. For $h < h_1 < 2h$ the proof from [10] remains valid. Therefore, for $h < h_1 < 2h$ for the quantity $A_p(\mathcal{L}_{2r}, h, h_1)$ a lower bound

$$A_p(\mathcal{L}_{2r}, h, h_1) \ge (\|H_{2r}\|_{L_q[0;1]})^{-1}$$

holds, and it coincides with upper bound (3.8), which completes the proof of Theorem 1. □

Corollary 2. *Let* $0 < h < h_0$*,* $1 < p < \infty$ *and* $\mathcal{L}_n(D)$ *be an arbitrary formally self-adjoint linear differential operator of form* (0.1) *for* $n = 2r$*. Then*

$$A_p(\mathcal{L}_n, h, 2h) = \infty.$$

Proof. The proof follows from the fact that for $h_1 = 2h$ the function $H_{2r}(t) \equiv 0$ (see (0.5)), and the passage to the limit as $h_1 \to 2h$ in Theorem 1. □

4. CONCLUSIONS

In the present article, it was not possible to solve the problem of extremal interpolation, that is, the problem of accurately computing the value of $A_p(\mathcal{L}_n, h, h_1)$ for $h < h_1 \leq 2h$ for an arbitrary linear differential operator $\mathcal{L}_n$ of form (0.1). The properties of the functions H_n and S_n in the general case require more detailed studies. Note that for $h_1 > 2h$ it is not even clear whether the quantity $A_p(\mathcal{L}_n, h, h_1)$ is finite (in particular, if the operator $\mathcal{L}_n$ is formally self-adjoint).

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

REFERENCES

1. A. O. Gelfond, *The Calculus of Finite Differences* (Gosudarstv. Izdat. Tekhn.-Teor. Lit., Moscow–Leningrad, 1952) [in Russian].
2. J. Favard, "Sur l'interpolation," J. Math. Pures Appl. (9) **19**, 281–306 (1940).
3. Yu. N. Subbotin, "On the connection between finite differences and corresponding derivatives," in *Trudy Mat. Inst. Steklova,* Vol. 78: *Extremal Properties of Polynomials* (Nauka, Moscow, 1965), pp. 24–42.
4. Yu. N. Subbotin, "Functional interpolation in the mean with smallest n derivative," Proc. Steklov Inst. Math. **88**, 31–63 (1967).
5. Yu. N. Subbotin, "Extremal problems of functional interpolation, and mean interpolation splines," Proc. Steklov Inst. Math. **138**, 127–185 (1977).
6. Yu. N. Subbotin, S. I. Novikov, and V. T. Shevaldin, "Extremal functional interpolation and splines," Trudy Inst. Mat. Mekh. UrO RAN **24** (3), 200–225 (2018).
7. Yu. N. Subbotin, "Extremal functional interpolation in the mean with least value of the n-th derivative for large averaging intervals," Math. Notes **59** (1), 83–96 (1996).
8. Yu. N. Subbotin, "Some extremal problems of interpolation and interpolation in the mean," East J. Approx. **2** (2), 155–167 (1996).
9. Yu. N. Subbotin, "Extremal L_p interpolation in the mean with intersecting averaging intervals," Izv. Math. **61** (1), 183–205 (1997).
10. V. T. Shevaldin, "Some problems of extremal interpolation in the mean for linear differential operators," Proc. Steklov Inst. Math. **164**, 233–273 (1985).
11. V. T. Shevaldin, "Extremal interpolation in the mean with overlapping averaging intervals and L-splines," Izv. Math. **62** (4), 833–856 (1998).
12. V. T. Shevaldin, "Extremal interpolation in the mean with overlapping averaging intervals and the smallest norm of a linear differential operator," Trudy Inst. Mat. i Mekh. UrO RAN **29** (1), 219–232 (2023).
13. V. T. Shevaldin, "Extremal interpolation in the mean in the space $L_1(\mathbb{R})$ with overlapping averaging intervals," Math. Notes **115** (1), 102–113 (2024).
14. V. T. Shevaldin, "Extremal interpolation with least norm of linear differential operator," Math. Notes **27** (5), 344–354 (1980).
15. V. T. Shevaldin, "A problem of extremal interpolation," Math. Notes **29** (4), 310–320 (1981).
16. M. G. Krein, "Integral equations on the half-line with a kernel depending on the difference of the arguments," Uspekhi Mat. Nauk **13** (5(83)), 3–120 (1958).