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Commuting Jordan Derivations on Triangular Rings Are Zero*

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Abstract—The main purpose of this article is to show that every commuting Jordan derivation on triangular rings (unital or not) is identically zero. Using this result, we prove that if $\mathcal{A}$ is a 2-torsion free ring that is either semiprime or satisfies Condition (P), then, under certain conditions, every commuting Jordan derivation of $\mathcal{A}$ into itself is identically zero.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{R}$ be an associative ring with center $Z(\mathcal{R})$. We write $[x,y]=xy-yx$ for all $x,y\in\mathcal{R}$. The ring $\mathcal{R}$ is said to be 2-*torsion free* if the relation $2a=0$ for $a\in\mathcal{R}$ implies that $a=0$. Further, $\mathcal{R}$ is called a *prime ring* if the relation $a\mathcal{R}b=\{0\}$ implies that $a=0$ or $b=0$ and a *semiprime ring* if the relation $a\mathcal{R}a=\{0\}$ implies that $a=0$. Finally, $\mathcal{R}$ satisfies Condition (P) if $xax=0$ for all $x\in\mathcal{R}$ implies that $a=0$. Clearly, every unital ring satisfies this condition. For nonunital rings that satisfy Condition (P), the reader is referred to [1].

Recall that an additive map $\Delta\colon\mathcal{R}\to\mathcal{R}$ is called a *derivation* if $\Delta(ab)=\Delta(a)b+a\Delta(b)$ for all $a,b\in\mathcal{R}$. Further, Δ is called a *Jordan derivation* if $\Delta(a^2)=\Delta(a)a+a\Delta(a)$ for all $a\in\mathcal{R}$. Also, Δ is called a *left* (respectively, *right*) *Jordan derivation* if $\Delta(a^2)=2a\Delta(a)$ (respectively, $\Delta(a^2)=2\Delta(a)a$) for any $a\in\mathcal{A}$. For more details about left (Jordan) derivations, e.g., see [2], [3].

The first result on a Jordan derivation to be a derivation is due to Herstein [4], who proved that every Jordan derivation on a 2-torsion free prime ring is a derivation. Cusack [5] generalized Herstein's result to 2-torsion free semiprime rings (see also [6] for an alternative proof). In 2008, Vukman [3] studied left Jordan derivations on semiprime rings. In that article he showed, that if $\mathcal{R}$ is a 2-torsion free semiprime ring and $\Delta\colon\mathcal{R}\to\mathcal{R}$ is a left Jordan derivation, then Δ is a derivation that maps $\mathcal{R}$ into $Z(\mathcal{R})$. In recent years, the characterizations of Jordan derivations on triangular rings have been studied. For example, it was proved in [1] that, under some conditions, every Jordan derivation on a triangular ring (without assuming unitality) is a derivation. For more studies concerning Jordan derivations, we refer the reader to [7], [8], and references therein.

A mapping $\mathfrak{F}\colon\mathcal{R}\to\mathcal{R}$ is said to be *centralizing* on a subset $\mathfrak{X}$ of $\mathcal{R}$ if $[\mathfrak{F}(x),x]\in Z(\mathcal{R})$ for all $x\in\mathfrak{X}$. In particular, if $[\mathfrak{F}(x),x]=0$ for all $x\in\mathfrak{X}$, then $\mathfrak{F}$ is said to be commuting on $\mathfrak{X}$. In the last few decades, commuting maps have been one most active topic in the study of mappings on rings. For commuting maps, we refer the readers to the very nice survey paper [9]. The history of commuting and centralizing mappings goes back to 1955, when Divinsky [10] proved that a simple Artinian ring is commutative if it has a nontrivial commuting automorphism. Two years later, Posner [11] achieved the first result on

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commuting derivations, which claims that if Δ is a commuting derivation on a prime ring $\mathcal{R}$, then either $\mathcal{R}$ is commutative or Δ is zero.

In the present paper, we aim at studying commuting Jordan derivations on triangular rings. Different from Ponser's result, we show that every commuting Jordan derivation on a triangular ring has to be zero. We also point out that our result in this paper does not require the triangular rings in question to be unital.

Throughout the paper, $\mathcal{A}$ and $\mathcal{B}$ are associative rings and $\mathcal{M}$ is an $(\mathcal{A},\mathcal{B})$-bimodule. Recall that a triangular ring $\mathfrak{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ is a ring of the form

$$\mathfrak{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B}) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a \in \mathcal{A},\ m \in \mathcal{M},\ b \in \mathcal{B} \right\}$$

under the usual matrix addition and multiplication (see [12]). Note that $\mathfrak{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ is unital if and only if both $\mathcal{A}$ and $\mathcal{B}$ are unital. Note that there exist many triangular rings without unity. For example, if $\mathcal{A}$ is a ring without unity, then every upper triangular matrix ring over $\mathcal{A}$ does not contain unity.

Set

$$\mathfrak{T}_{11} = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathcal{A} \right\},$$
$$\mathfrak{T}_{12} = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} : m \in \mathcal{M} \right\},$$
$$\mathfrak{T}_{22} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} : b \in \mathcal{B} \right\}.$$

Then $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$, and every element $A \in \mathfrak{T}$ can be written as $A = A_{11} + A_{12} + A_{22}$, where $A_{ij} \in \mathfrak{T}_{ij}$, $i, j \in \{1,2\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. A left (respectively, right) $\mathcal{A}$-module $\mathcal{M}$ is said to be *left* (respectively, *right*) *faithful* if $a = 0$ is the only element in $\mathcal{A}$ satisfying $a\mathcal{M} = 0$ (respectively, $\mathcal{M}a = 0$). An $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$ is said to be faithful if $\mathcal{M}$ is both a faithful left $\mathcal{A}$-module and a faithful right $\mathcal{B}$-module (see [12] for more details). A module $\mathcal{M}$ is said to be n-*torsion free*, where $n > 1$ is an integer, if, for any $x \in \mathcal{M}$, $nx = 0$ implies that $x = 0$.

Now let us state the main result of this paper. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-torsion free rings each of which is either semiprime or satisfies Condition (P), and let $\mathcal{M}$ be a 2-torsion free faithful $(\mathcal{A},\mathcal{B})$-bimodule such that if $\mathcal{A}m = \{0\}$ (respectively, $m\mathcal{B} = \{0\}$) for some $m \in \mathcal{M}$, then $m = 0$. If Δ is a commuting Jordan derivation on the triangular ring $\mathfrak{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$, then Δ is zero.

As stated above, Posner [11, Lemma 3] proved that if $\mathcal{R}$ is a prime ring and d is a commuting derivation of $\mathcal{R}$, then $\mathcal{R}$ is commutative or d is zero. As a corollary of the main theorem of this paper, we show that every commuting Jordan derivation on a 2-torsion free ring that is either semiprime or satisfies Condition (P)is identically zero under certain conditions. Some other related results are also presented.

2. RESULTS AND PROOFS

We begin our discussion with the following useful lemmas, which we will employ frequently to prove the main result of this paper.

Lemma 2.1. *Let $\mathcal{R}$ be a 2-torsion free ring, and let $\Delta\colon \mathcal{R} \to \mathcal{R}$ be a commuting Jordan derivation; i.e., $\Delta(x^2) = \Delta(x)x + x\Delta(x)$ and $\Delta(x)x = x\Delta(x)$ for all $x \in \mathcal{R}$. Then the following assertions hold for any $x, y \in \mathcal{R}$:*

(i) $\Delta(xy + yx) = 2\big(x\Delta(y) + y\Delta(x)\big)$.

(ii) $\Delta(xy + yx) = 2\big(\Delta(x)y + \Delta(y)x\big)$.

(iii) $\Delta(xyx) = 3\Delta(x)yx + \Delta(y)x^2 - \Delta(x)xy$.

(iv) $\Delta(xyx) = x^2\Delta(y) + 3xy\Delta(x) - yx\Delta(x)$.

Proof. It is clear that Δ is both a left and a right Jordan derivation. Relations (i) and (iv) were proved in [2, Proposition 1.1]. Likewise, relations (ii) and (iii) can be obtained for the right Jordan derivations, and to make the paper self-contained, we prove them here. Relation (ii) readily follows from $\Delta(x^2) = 2\Delta(x)x$ by linearization (i.e., substituting $x + y$ for x). Let us prove (iii). From (ii), we have

$$\begin{aligned}\Delta(x(xy + yx) + (xy + yx)x) &= 2\Delta(x)(xy + yx) + 2\Delta(xy + yx)x\\ &= 6\Delta(x)yx + 2\Delta(x)xy + 4\Delta(y)x^2\end{aligned}$$

for all $x, y \in \mathcal{R}$. On the other hand,

$$\begin{aligned}\Delta(x(xy + yx) + (xy + yx)x) &= \Delta(x^2y + yx^2) + 2\Delta(xyx)\\ &= 2\Delta(x^2)y + 2\Delta(y)x^2 + 2\Delta(xyx)\\ &= 4\Delta(x)xy + 2\Delta(y)x^2 + 2\Delta(xyx).\end{aligned}$$

Comparing these expressions and using the assumption that $\mathcal{R}$ is a 2-torsion free ring, we obtain

$$\Delta(xyx) = 3\Delta(x)yx + \Delta(y)x^2 - \Delta(x)xy$$

for all $x, y \in \mathcal{R}$, as desired. □

Lemma 2.2. *Let $\mathcal{R}$ be a 2-torsion free ring, and let $x^2a = 0$ for all $x \in \mathcal{R}$ and for some $a \in Z(\mathcal{R})$. The following assertions hold*:

(i) *If $\mathcal{R}$ is semiprime, then $a = 0$.*

(ii) *If $\mathcal{R}$ satisfies Condition* (P)*, then $a = 0$.*

Proof. (i) Replacing x by $x + y$ in the equation $x^2a = 0$ and then using this equation, we obtain $(xy + yx)a = 0$ for all $x, y \in \mathcal{R}$. Setting $y = a$ in the previous equation and using the assumption that $a \in Z(\mathcal{R})$, we obtain $2axa = 0$ for all $x \in \mathcal{R}$. Since $\mathcal{R}$ is 2-torsion free, we have $axa = 0$ for all $x \in \mathcal{R}$, and $\mathcal{R}$ being semiprime implies that $a = 0$, as desired.

(ii) Since $x^2a = 0$ for all $x \in \mathcal{R}$ and some $a \in Z(\mathcal{R})$, we conclude that $xax = 0$ for all $x \in \mathcal{R}$. Condition (P) for $\mathcal{R}$ implies that $a = 0$. □

Lemma 2.3 [13, Lemma 3]**.** *Let $\mathcal{R}$ be a semiprime ring, and let $f : \mathcal{R} \to \mathcal{R}$ be an additive mapping. If either $f(x)x = 0$ or $xf(x) = 0$ holds for all $x \in \mathcal{R}$, then $f = 0$.*

We are now in a position to prove our main result.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-torsion free rings each of which is either semiprime or satisfies Condition* (P)*, and let $\mathcal{M}$ be a 2-torsion free faithful $(\mathcal{A}, \mathcal{B})$-bimodule such that if $m \in \mathcal{M}$ and $\mathcal{A}m = \{0\}$* (*respectively, $m\mathcal{B} = \{0\}$*)*, then $m = 0$. If Δ is a commuting Jordan derivation on the triangular ring $\mathfrak{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$, then Δ is zero.*

Proof. Without loss of generality, we assume that $\mathcal{A}$ is a semiprime ring and $\mathcal{B}$ satisfies Condition (P). The proof is divided into the following six steps.

Step 1. For any

$$A_{11} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{T}_{11}, \qquad A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in \mathfrak{T}_{22},$$

we have

(i) $[\Delta(A_{11})]_{22} = 0$.

(ii) $[\Delta(A_{11})]_{12} = 0$.

(iii) $[\Delta(A_{22})]_{11} = 0$.

Note that $A_{11}A_{22} = A_{22}A_{11} = 0$. It follows from Lemma 2.1 (i) that

$$\begin{aligned} 0 &= \Delta(A_{11}A_{22} + A_{22}A_{11}) = 2\big(\Delta(A_{11})A_{22} + \Delta(A_{22})A_{11}\big) \\ &= 2\begin{bmatrix} [\Delta(A_{11})]_{11} & [\Delta(A_{11})]_{12} \\ 0 & [\Delta(A_{11})]_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} + 2\begin{bmatrix} [\Delta(A_{22})]_{11} & [\Delta(A_{22})]_{12} \\ 0 & [\Delta(A_{22})]_{22} \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2[\Delta(A_{11})]_{12}b \\ 0 & 2[\Delta(A_{11})]_{22}b \end{bmatrix} + \begin{bmatrix} 2[\Delta(A_{22})]_{11}a & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2[\Delta(A_{22})]_{11}a & 2[\Delta(A_{11})]_{12}b \\ 0 & 2[\Delta(A_{11})]_{22}b \end{bmatrix}, \end{aligned}$$

which implies that

$$\begin{bmatrix} 2[\Delta(A_{22})]_{11}a & 2[\Delta(A_{11})]_{12}b \\ 0 & 2[\Delta(A_{11})]_{22}b \end{bmatrix} = 0.$$

It follows that $[\Delta(A_{22})]_{11}a = 0$ for any $a \in \mathcal{A}$ and that $[\Delta(A_{11})]_{12}b = 0$ and $[\Delta(A_{11})]_{22}b = 0$ for any $b \in \mathcal{B}$. Since the ring $\mathcal{A}$ is semiprime and the ring $\mathcal{B}$ satisfies Condition (P), we have $[\Delta(A_{22})]_{11} = 0$ and $[\Delta(A_{11})]_{22} = 0$. Using the fact that $m = 0$ is the only element in $\mathcal{M}$ such that $m\mathcal{B} = \{0\}$, we see that $[\Delta(A_{11})]_{12} = 0$.

Step 2. For any

$$A_{11} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{T}_{11}, \qquad A_{12} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in \mathfrak{T}_{12},$$

we have

(i) $[\Delta(A_{12})]_{11} = 0$.

(ii) $[\Delta(A_{11})]_{11} = 0$.

To show that $[\Delta(A_{12})]_{11} = 0$, first, we show that $[\Delta(A_{12})]_{11} \in Z(\mathcal{A})$. According to Step 1, $[\Delta(A_{11})]_{22} = 0$, which implies that $A_{12}\Delta(A_{11}) = 0$. Hence we have

$$\begin{aligned} \Delta(A_{11}A_{12}) &= \Delta(A_{11}A_{12} + A_{12}A_{11}) \\ &= 2\big(A_{11}\Delta(A_{12}) + A_{12}\Delta(A_{11})\big) \\ &= 2A_{11}\Delta(A_{12}) \\ &= 2\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} [\Delta(A_{12})]_{11} & [\Delta(A_{12})]_{12} \\ 0 & [\Delta(A_{12})]_{22} \end{bmatrix} \\ &= \begin{bmatrix} 2a[\Delta(A_{12})]_{11} & 2a[\Delta(A_{12})]_{12} \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which means that

$$\Delta(A_{11}A_{12}) = \begin{bmatrix} 2a[\Delta(A_{12})]_{11} & 2a[\Delta(A_{12})]_{12} \\ 0 & 0 \end{bmatrix} \tag{2.1}$$

On the other hand, by Step 1, we have

$$\begin{aligned} \Delta(A_{11}A_{12}) &= \Delta(A_{11}A_{12} + A_{12}A_{11}) \\ &= 2\big(\Delta(A_{11})A_{12} + \Delta(A_{12})A_{11}\big) \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} 2[\Delta(A_{11})]_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2[\Delta(A_{12})]_{11} & 2[\Delta(A_{12})]_{12} \\ 0 & 2[\Delta(A_{12})]_{22} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2[\Delta(A_{12})]_{11}a & 2[\Delta(A_{11})]_{11}m \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

which means that

$$
\Delta(A_{11}A_{12}) = \begin{bmatrix} 2[\Delta(A_{12})]_{11}a & 2[\Delta(A_{11})]_{11}m \\ 0 & 0 \end{bmatrix}. \tag{2.2}
$$

Comparing Eqs. (2.1) and (2.2) and using the assumption that $\mathcal{A}$ is a 2-torsion free ring, we obtain $[\Delta(A_{12})]_{11} \in Z(\mathcal{A})$.

Now let us prove that $[\Delta(A_{12})]_{11} = 0 = [\Delta(A_{11})]_{11}$. Using Lemma 2.1 (iii), we have

$$
\begin{aligned}
0 = \Delta(A_{11}A_{12}A_{11}) &= 3\Delta(A_{11})A_{12}A_{11} + \Delta(A_{12})A_{11}^2 - \Delta(A_{11})A_{11}A_{12} \\
&= \begin{bmatrix} [\Delta(A_{12})]_{11} & [\Delta(A_{12})]_{12} \\ 0 & [\Delta(A_{12})]_{22} \end{bmatrix} \begin{bmatrix} a^2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} [\Delta(A_{11})]_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & am \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} [\Delta(A_{12})]_{11}a^2 & -[\Delta(A_{11})]_{11}am \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

which means that

$$
\begin{bmatrix} [\Delta(A_{12})]_{11}a^2 & -[\Delta(A_{11})]_{11}am \\ 0 & 0 \end{bmatrix} = 0.
$$

Therefore,

$$
[\Delta(A_{12})]_{11}a^2 = 0, \tag{2.3}
$$
$$
[\Delta(A_{11})]_{11}am = 0, \tag{2.4}
$$

for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. Since $[\Delta(A_{12})]_{11} \in Z(\mathcal{A})$, the relation $[\Delta(A_{12})]_{11}a^2 = 0$ for all $a \in \mathcal{A}$ implies that $[\Delta(A_{12})]_{11} = 0$ by Lemma 2.2 (i).

It follows from Eq. (2.4) that $[\Delta(A_{11})]_{11}a = 0$ for each $a \in \mathcal{A}$, because $\mathcal{M}$ is a faithful left $\mathcal{A}$-module. Now we define a mapping $F_{11}\colon \mathcal{A} \to \mathcal{A}$ by setting $F_{11}(a) = [\Delta(A_{11})]_{11}$, where

$$
A_{11} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.
$$

Obviously, the mapping F_{11} is additive. Thus, we have $F_{11}(a)a = 0$ for all $a \in \mathcal{A}$. It follows from Lemma 2.3 that $F_{11}(a) = [\Delta(A_{11})]_{11} = 0$ for all $A_{11} \in \mathfrak{T}_{11}$, as desired.

Step3. $[\Delta(A_{22})]_{12} = 0$ for all $A_{22} \in \mathfrak{T}_{22}$.

Obviously, $A_{11}A_{22} = 0 = A_{22}A_{11}$ for all $A_{11} \in \mathfrak{T}_{11}$ and $A_{22} \in \mathfrak{T}_{22}$. Applying Lemma 2.1 (i) and Step 1 (i) and (iii), we have

$$
\begin{aligned}
0 &= \Delta(A_{11}A_{22} + A_{22}A_{11}) \\
&= 2\big(A_{11}\Delta(A_{22}) + A_{22}\Delta(A_{11})\big) \\
&= 2\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & [\Delta(A_{22})]_{12} \\ 0 & [\Delta(A_{22})]_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2a[\Delta(A_{22})]_{12} \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

which implies that $2a[\Delta(A_{22})]_{12} = 0$ for all $a \in \mathcal{A}$. Since $\mathcal{M}$ is a 2-torsion free ring, we have $a[\Delta(A_{22})]_{12} = 0$ for all $a \in \mathcal{A}$, and so $[\Delta(A_{22})]_{12} = 0$ for all $A_{22} \in \mathfrak{T}_{22}$, as desired.

Step 4. For any

$$A_{12}=\begin{bmatrix}0 & m\\ 0 & 0\end{bmatrix}\in\mathfrak{T}_{12},\qquad A_{22}=\begin{bmatrix}0 & 0\\ 0 & b\end{bmatrix}\in\mathfrak{T}_{22},$$

we have

(i) $[\Delta(A_{12})]_{22}=0$.

(ii) $[\Delta(A_{22})]_{22}=0$.

First, let us show that $[\Delta(A_{12})]_{22}\in Z(\mathcal{B})$. On the one hand, using Step 1 (iii), Step 2 (i), and Step 3, we obtain the following expressions:

$$\begin{aligned}\Delta(A_{12}A_{22})&=\Delta(A_{22}A_{12}+A_{12}A_{22})\\&=2\big(\Delta(A_{22})A_{12}+\Delta(A_{12})A_{22}\big)\\&=2\begin{bmatrix}0 & 0\\ 0 & [\Delta(A_{22})]_{22}\end{bmatrix}\begin{bmatrix}0 & m\\ 0 & 0\end{bmatrix}+2\begin{bmatrix}0 & [\Delta(A_{12})]_{12}\\ 0 & [\Delta(A_{12})]_{22}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & b\end{bmatrix}\\&=\begin{bmatrix}0 & 2[\Delta(A_{12})]_{12}b\\ 0 & 2[\Delta(A_{12})]_{22}b\end{bmatrix};\end{aligned}$$

that is,

$$\Delta(A_{12}A_{22})=\begin{bmatrix}0 & 2[\Delta(A_{12})]_{12}b\\ 0 & 2[\Delta(A_{12})]_{22}b\end{bmatrix}\tag{2.5}$$

for all $b\in\mathcal{B}$. On the other hand, using Lemma 2.1 (i), Step 1 (iii), Step 2 (i), and Step 3, we obtain

$$\begin{aligned}\Delta(A_{12}A_{22})&=\Delta(A_{22}A_{12}+A_{12}A_{22})\\&=2\big(A_{22}\Delta(A_{12})+A_{12}\Delta(A_{22})\big)\\&=2\begin{bmatrix}0 & 0\\ 0 & b\end{bmatrix}\begin{bmatrix}0 & [\Delta(A_{12})]_{12}\\ 0 & [\Delta(A_{12})]_{22}\end{bmatrix}+2\begin{bmatrix}0 & m\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & [\Delta(A_{22})]_{22}\end{bmatrix}\\&=\begin{bmatrix}0 & 2m[\Delta(A_{22})]_{22}\\ 0 & 2b[\Delta(A_{12})]_{22}\end{bmatrix};\end{aligned}$$

that is,

$$\Delta(A_{12}A_{22})=\begin{bmatrix}0 & 2m[\Delta(A_{22})]_{22}\\ 0 & 2b[\Delta(A_{12})]_{22}\end{bmatrix}\tag{2.6}$$

for all $b\in\mathcal{B}$. Comparing Eqs. (2.5) and (2.6), we see that $[\Delta(A_{12})]_{22}\in Z(\mathcal{B})$ for all $A_{12}\in\mathfrak{T}_{12}$. Using Lemma 2.1 (iv), Step 1 (iii), Step 2 (i), and Step 3, we obtain

$$\begin{aligned}0=\Delta(A_{22}A_{12}A_{22})&=A_{22}^2\Delta(A_{12})+3A_{22}A_{12}\Delta(A_{22})-A_{12}A_{22}\Delta(A_{22})\\=\begin{bmatrix}0 & 0\\ 0 & b^2\end{bmatrix}&\begin{bmatrix}0 & [\Delta(A_{12})]_{12}\\ 0 & [\Delta(A_{12})]_{22}\end{bmatrix}-\begin{bmatrix}0 & m\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & b\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & [\Delta(A_{22})]_{22}\end{bmatrix}\\=\begin{bmatrix}0 & -mb[\Delta(A_{22})]_{22}\\ 0 & b^2[\Delta(A_{12})]_{22}\end{bmatrix}&,\end{aligned}$$

which implies that

$$b^2[\Delta(A_{12})]_{22}=0\qquad(b\in\mathcal{B}),\tag{2.7}$$

$$mb[\Delta(A_{22})]_{22}=0\qquad(b\in\mathcal{B},\quad m\in\mathcal{M}).\tag{2.8}$$

Note that $[\Delta(A_{12})]_{22} \in Z(\mathcal{B})$ for all $A_{12} \in \mathfrak{T}_{12}$. This, along with Eq. (2.7), implies $b[\Delta(A_{12})]_{22}b = 0$ for all $b \in \mathcal{B}$. Since $\mathcal{B}$ satisfies Condition (P), we obtain $[\Delta(A_{12})]_{22} = 0$ for all $A_{12} \in \mathfrak{T}_{12}$. Further, it follows from Eq. (2.8) that $\mathcal{M}b[\Delta(A_{22})]_{22} = \{0\}$ for all

$$A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in \mathfrak{T}_{22}.$$

Since $\mathcal{M}$ is a faithful right $\mathcal{B}$-module, we obtain

$$b[\Delta(A_{22})]_{22} = 0 \tag{2.9}$$

for all

$$A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in \mathfrak{T}_{22}.$$

It is clear that the mapping $F_{22}\colon \mathcal{B} \to \mathcal{B}$ defined by $F_{22}(b) = [\Delta(A_{22})]_{22}$, where

$$A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix},$$

is additive. It follows from Eq. (2.9) that

$$bF_{22}(b) = 0 \qquad (b \in \mathcal{B}). \tag{2.10}$$

Since Δ is a commuting map, it can be verified that $F_{22}(b)b = bF_{22}(b)$ for any $b \in \mathcal{B}$. Therefore,

$$F_{22}(b)b = bF_{22}(b) = 0 \qquad (b \in \mathcal{B}). \tag{2.11}$$

Replacing b by $b_1 + b_2$ in Eq. (2.10) and then using Eq. (2.10), we have

$$b_1F_{22}(b_2) + b_2F_{22}(b_1) = 0 \qquad (b_1, b_2 \in \mathcal{B}).$$

Multiplying the previous equality on the right by b_1 and then using identity (2.11), we obtain

$$b_1F_{22}(b_2)b_1 = 0, \qquad (b_1, b_2 \in \mathcal{B}). \tag{2.12}$$

Equation (2.12) and the assumption that the ring $\mathcal{B}$ satisfies Condition (P) imply that $F_{22}(b) = 0$ for all $b \in \mathcal{B}$. This means that $[\Delta(A_{22})]_{22} = 0$ for all $A_{22} \in \mathfrak{T}_{22}$.

Step 5. $[\Delta(A_{12})]_{12} = 0$ for every $A_{12} \in \mathfrak{T}_{12}$.

On the one hand, using Lemma 2.1 (i), Step 1 (i), Step 2 (i) and Step 4 (i), for any

$$A_{11} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{T}_{11}, \qquad A_{12} \in \mathfrak{T}_{12}$$

we have

$$\begin{aligned} \Delta(A_{11}A_{12}) &= \Delta(A_{11}A_{12} + A_{12}A_{11}) \\ &= 2\big(A_{11}\Delta(A_{12}) + A_{12}\Delta(A_{11})\big) \\ &= \begin{bmatrix} 2a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & [\Delta(A_{12})]_{12} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2a[\Delta(A_{12})]_{12} \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which means that

$$\Delta(A_{11}A_{12}) = \begin{bmatrix} 0 & 2a[\Delta(A_{12})]_{12} \\ 0 & 0 \end{bmatrix} \tag{2.13}$$

for all

$$A_{12} \in \mathfrak{T}_{12}, \qquad A_{11} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{T}_{11}.$$

On the other hand, using Lemma 2.1 (ii) and Step 2, we have

$$\begin{aligned}\Delta(A_{11}A_{12}) &= \Delta(A_{11}A_{12} + A_{12}A_{11}) \\ &= 2\big(\Delta(A_{11})A_{12} + \Delta(A_{12})A_{11}\big) \\ &= 0,\end{aligned}$$

which means that

$$\Delta(A_{11}A_{12}) = 0 \qquad (A_{11} \in \mathfrak{T}_{11}, \quad A_{12} \in \mathfrak{T}_{12}). \tag{2.14}$$

Comparing Eqs. (2.13) and (2.14) and using the assumption that $\mathcal{M}$ is 2-torsion free, we conclude that $a[\Delta(A_{12})]_{12} = 0$ for all $a \in \mathcal{A}$ and all $A_{12} \in \mathfrak{T}_{12}$. Using the fact that $m = 0$ is the only element in $\mathcal{M}$ satisfying $\mathcal{A}m = \{0\}$, we see that $[\Delta(A_{12})]_{12} = 0$ for all $A_{12} \in \mathfrak{T}_{12}$.

Step 6. $\Delta(A) = 0$ for all $A \in \mathfrak{T}$.

It follows from Steps 1–5 that $\Delta(A_{11}) = 0$, $\Delta(A_{12}) = 0$, and $\Delta(A_{22}) = 0$ for all $A_{11} \in \mathfrak{T}_{11}$, $A_{12} \in \mathfrak{T}_{12}$, and $A_{22} \in \mathfrak{T}_{22}$. Therefore, for any $A \in \mathfrak{T}$ we have

$$\Delta(A) = \Delta(A_{11} + A_{12} + A_{22}) = 0.$$

Thereby, our proof is complete. The proof for the cases in which both rings $\mathcal{A}$ and $\mathcal{B}$ are semiprime, or both satisfy Condition (P), or $\mathcal{B}$ is semiprime and $\mathcal{A}$ satisfies Condition (P) is exactly the same as above, and we leave it to the interested reader. □

In the following, we provide an example showing that the assumptions of Theorem 2.4 are not superfluous. In this example, we use zero square rings, so we define these rings first.

Definition 2.5. (i) A ring $\mathcal{R}$ is called a *zero square ring of type* 1 if $x^2 = 0$ for all $x \in \mathcal{R}$ and there exist two elements $y, z \in \mathcal{R}$ such that $yz \neq 0$.

(ii) A ring $\mathcal{R}$ is called a *zero square ring of type* 2 if $x^2 = 0$ for all $x \in \mathcal{R}$.

Examples of the above-mentioned concepts can be found in [14, Example 2.2]; to make our exposition self-contained, let us present an example of a zero square ring of type 1.

Example 2.6 [14, Example 2.2 (iv)]**.** Let $\mathcal{Z}$ be a nonnull ring (that is, $\mathcal{Z}^2 \neq \{0\}$). Write $\mathcal{R} = \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z}$. Define addition on $\mathcal{R}$ componentwise, and define a multiplication $\bullet$ on $\mathcal{R}$ by setting

$$(x_1, y_1, z_1) \bullet (x_2, y_2, z_2) = (0, 0, x_1y_2 - x_2y_1).$$

Stanley [15] mentioned that $\mathcal{R}^2 \neq \{0\}$ and $r^2 = 0$ for all $r \in \mathcal{R}$. Hence $\mathcal{R}$ is a zero square ring of type 1.

For more examples and details concerning zero square rings, e.g., see [14]–[17] and references therein.

Example 2.7. Let $\mathcal{R}$ be a zero square ring of type 1; that is, $\mathcal{R}$ is a ring such that the square of each element in $\mathcal{R}$ is zero but the product of some nonzero elements in $\mathcal{R}$ is nonzero. Let

$$\mathfrak{A} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} : a, b \in \mathcal{R} \right\}.$$

Clearly, $\mathfrak{A}$ is a ring with the annihilator

$$\operatorname{ann}(\mathfrak{A}) = \left\{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix} : x \in \operatorname{ann}(\mathcal{R}), y \in \mathcal{R} \right\},$$

where $\operatorname{ann}(\mathcal{R})$ denotes the annihilator of $\mathcal{R}$. We know that the annihilator of any ring is an ideal of that ring. Thus, $\operatorname{ann}(\mathfrak{A})$ is an ideal of $\mathfrak{A}$, and we can view it as an $\mathfrak{A}$-bimodule. Define a mapping $\delta\colon \mathfrak{A} \to \mathfrak{A}$ by

$$\delta\left(\begin{bmatrix}0 & a & b\\ 0 & 0 & a\\ 0 & 0 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & a & 0\\ 0 & 0 & a\\ 0 & 0 & 0\end{bmatrix}.$$

It is easily seen that δ is a commuting Jordan derivation (see [18, Example 2.8]). Now let

$$\mathfrak{T} = Tri(\mathfrak{A}, \operatorname{ann}(\mathfrak{A}), \mathfrak{A}) = \left\{ \begin{bmatrix}A & X\\ 0 & B\end{bmatrix} : A, B \in \mathfrak{A}, X \in \operatorname{ann}(\mathfrak{A}) \right\}.$$

Define a mapping $\Delta\colon \mathfrak{T} \to \mathfrak{T}$ by

$$\Delta\left(\begin{bmatrix}A & X\\ 0 & B\end{bmatrix}\right) = \begin{bmatrix}\delta(A) & 0\\ 0 & 0\end{bmatrix}.$$

A straightforward verification shows that $\Delta(\mathfrak{X}^2) = 2\Delta(\mathfrak{X})\mathfrak{X} = 2\mathfrak{X}\Delta(\mathfrak{X})$ for all $\mathfrak{X} \in \mathfrak{T}$. As can be seen, Δ is a nonzero commuting Jordan derivation of $\mathfrak{T}$, and this shows that the conditions outlined in Theorem 2.4 are not superfluous.

As mentioned in Sec. 1, the following result is quite different from that in [11].

Corollary 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-torsion free rings each of which is either semiprime or satisfies Condition* (P)*, and let $\mathcal{M}$ be as in Theorem* 2.4*. If the triangular ring $\mathfrak{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is commutative, then every Jordan derivation on $\mathfrak{T}$ is identically zero.*

Corollary 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-torsion free rings each of which is either semiprime or satisfies Condition* (P)*, let $\mathcal{M}$ be as in Theorem* 2.4*, and let $\{d_n\}_{n=0}^{\infty}$ be a commuting Jordan higher derivation on the triangular ring $\mathfrak{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$; i.e., $d_n(A^2) = \sum_{k=0}^{n} d_{n-k}(A)d_k(A)$, $d_0(A) = A$ and $[d_n(A), A] = 0$ for all nonnegative integers n and any $A \in \mathfrak{T}$. Then $d_n = 0$ for all $n \in \mathbb{N}$.*

Proof. We prove this corollary by induction on n. According to Theorem 2.4, the result holds trivially for $n = 1$. Thus, we observe that

$$d_2(A^2) = d_2(A)A + (d_1(A))^2 + Ad_2(A) = d_2(A)A + Ad_2(A)$$

for all $A \in \mathfrak{T}$, which means that d_2 is a Jordan derivation on $\mathfrak{T}$. By the induction assumption, $[d_2(A), A] = 0$ for all $A \in \mathfrak{T}$, and so d_2 is a commuting Jordan derivation on the triangular ring $\mathfrak{T}$. It follows from Theorem 2.4 that d_2 is zero. Let n be an arbitrary positive integer, and assume that the result holds for any $k < n$. Let us prove the result for n. In view of our assumption, $[d_n(A), A] = 0$ for all $A \in \mathfrak{T}$. Hence we have

$$d_n(A^2) = \sum_{k=0}^{n} d_{n-k}(A)d_k(A) = d_n(A)A + Ad_n(A) = 2d_n(A)A = 2Ad_n(A)$$

for all $A \in \mathfrak{T}$, which means that d_n is a commuting Jordan derivation on $\mathfrak{T}$. Reusing Theorem 2.4 gives the result. □

Corollary 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-torsion free rings each of which is either semiprime or satisfies Condition* (P)*, let $\mathcal{M}$ and $\mathfrak{T}$ be as in Theorem* 2.4*, and let $B_0 \in \mathfrak{T}$. If $[[A, B_0], A] = 0$ for all $A \in \mathfrak{T}$, then $B_0 \in Z(\mathfrak{T})$, the center of $\mathfrak{T}$.*

Proof. It is clear that $\Delta_{B_0}\colon \mathfrak{T} \to \mathfrak{T}$ defined by $\Delta_{B_0}(A) = [A, B_0]$ is a derivation. Since $[[A, B_0], A] = 0$ for all $A \in \mathfrak{T}$, it follows that Δ_{B_0} is a commuting derivation on $\mathfrak{T}$. By Theorem 2.4, Δ_{B_0} is zero, which implies that $B_0 \in Z(\mathfrak{T})$, as desired. □

Posner [11, Lemma 3] proved that if $\mathcal{R}$ is a prime ring and d is a commuting derivation of $\mathcal{R}$, then either $\mathcal{R}$ is commutative or d is zero. In the following corollary, we show that every commuting Jordan derivation on a 2-torsion free ring that is either semiprime or satisfies Condition (P) is identically zero under certain conditions.

Corollary 2.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-torsion free rings each of which is either semiprime or satisfies Condition* (P)*, and let $\mathcal{M}$ and $\mathfrak{T}$ be as in Theorem 2.4. Let $d\colon \mathcal{A}\to\mathcal{A}$ and $D\colon \mathcal{B}\to\mathcal{B}$ be two commuting Jordan derivations, and let $\mathfrak{G}\colon \mathcal{M}\to\mathcal{M}$ be a mapping satisfying*

$$\mathfrak{G}(am+mb) = 2d(a)m + 2\mathfrak{G}(m)b = 2a\mathfrak{G}(m) + 2mD(b) \tag{2.15}$$

for all $a\in\mathcal{A}$, $b\in\mathcal{B}$, and $m\in\mathcal{M}$. Then d, D, and $\mathfrak{G}$ are identically zero.

Proof. Define $\Delta\colon \mathfrak{T}\to\mathfrak{T}$ by

$$\Delta\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} d(a) & \mathfrak{G}(m) \\ 0 & D(b) \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$$

be an arbitrary element of $\mathfrak{T}$. We have

$$\begin{aligned}\Delta(A^2) &= \Delta\left(\begin{bmatrix} a^2 & am+mb \\ 0 & b^2 \end{bmatrix}\right) = \Delta\left(\begin{bmatrix} d(a^2) & \mathfrak{G}(am+mb) \\ 0 & D(b^2) \end{bmatrix}\right) \\ &= \begin{bmatrix} 2ad(a) & 2(a\mathfrak{G}(m)+mD(b)) \\ 0 & 2bD(b) \end{bmatrix} \\ &= 2A\Delta(A).\end{aligned}$$

One can readily show that $\Delta(A^2) = 2\Delta(A)A$ for all $A\in\mathfrak{T}$, and this means that Δ is a commuting Jordan derivation on $\mathfrak{T}$. It follows from Theorem 2.4 that Δ is identically zero, and this implies that d, D, and $\mathfrak{G}$ are all zero, as desired. □

At the end of this article, we present an example of rings $\mathcal{A}$ and $\mathcal{B}$ and a module $\mathcal{M}$ satisfying the assumptions of Theorem 2.4.

Example 2.12. Let $\mathbb{Z}$ be the set of all integers. Set

$$\mathcal{A} = \mathcal{B} = \left\{\begin{bmatrix} 2n & 0 \\ 0 & 2n \end{bmatrix} \;:\; n\in\mathbb{Z}\right\}.$$

It is obvious that $\mathcal{A}$ and $\mathcal{B}$ do not contain identity. Let

$$\mathcal{M} = \left\{\begin{bmatrix} i & j \\ 0 & k \end{bmatrix} \;:\; i,j,k\in\mathbb{Z}\right\}.$$

A straightforward verification shows that $\mathcal{A}$ is a semiprime ring that also satisfies Condition (P), and further, $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule. Moreover, $m=0$ is the only element of $\mathcal{M}$ satisfying $\mathcal{A}m=\{0\}$ (respectively, $m\mathcal{B}=\{0\}$). Therefore, the module $\mathcal{M}$ satisfies all the assumptions in Theorem 2.4.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. A. Fošner and W. Jing, "A note on Jordan derivations of triangulr rings," Aequationes Math. **94**, 277–285 (2020).
2. M. Brešar and J. Vukman, "On left drivations and related mappings," Proc. Amer. Math. Soc. **110** (1), 7–16 (1990).
3. J. Vukman, "On left Jordan derivations of rings and Banach algebras," Aequationes Math. **75** (3), 260–266 (2008).
4. I. N. Herstein, "Jordan derivations of prime rings," Proc. Amer. Math. Soc. **8**, 1104–1110 (1957).
5. J. Cusack, "Jordan derivations on rings," Proc. Amer. Math. Soc. **53**, 1104–1110 (1975).
6. M. Brešar, "Jordan derivations on semiprime rings," Proc. Amer. Math. Soc. **104** (4), 1003–1006 (1988).
7. H. Ghahramani, "Characterizing Jordan derivations of matrix rings through zero products," Math. Slovaca **65**, 1277–1290 (2015).
8. J. Zhang and W. Yu, "Jordan derivation of triangular algebras," Linear Algebra Appl. **419** (1), 251–255 (2006).
9. M. Brešar, "Commuting maps: a survey," Taiwanese J. Math. **8** (3), 361–397 (2004).
10. N. Divinsky, "On commuting automorphisms of rings," Trans. Roy. Soc. Canada, Sec. III **3** (49), 19–22 (1955).
11. E. Posner, "Derivations in prime rings," Proc. Amer. Soc. **8**, 1093–1100 (1957).
12. W. S. Cheung, "Commuting maps on triangular algebras," J. Lond. Math. Soc. **63**, 117–127 (2002).
13. J. Vukman, "Identities with derivations and automorphisms on semiprime rings," Int. J. Math. Math. Sci. **7**, 1031–1038 (2005).
14. S. Bhavanari, G. Lungisile, and N. Dasari, "Ideals and direct product of zero square rings," East Asian Math. J. **24** (4), 377–387 (2008).
15. R. P. Stanley, "Zero square rings," Pac. J. Math. **30** (3), 811–824 (1969).
16. M. Dugas and S. Feigelstock, "Additive groups of zero squre rings," Acta Math. Hungar. **107** (1–2), 55–64 (2005).
17. E. A. Hussein and S. O. Alsalihi, "Some new results on zero near rings," Int. J. Nonlinear Anal. Appl. **14** (1), 979–985 (2023).
18. A. Hosseini and A. Fošner, "The image of Jordan left derivations on algebras," Bol. Soc. Paran. Mat. **38** (6), 53–61 (2020).

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