ISSN 0001-4346, Mathematical Notes, 2024, Vol. 115, No. 6, pp. 998–1005. © Pleiades Publishing, Ltd., 2024.

New Proof of the Ostapenko–Tarasov Theorem*

R. Tapdigoglu[1,2] and **M. Garayev[3]***

[1] *Department of Mathematics, Azerbaijan State University of Economics (UNEC), Baku, 1001 Azerbaijan*

[2] *Khazar University, Baku, 1009 Azerbaijan*

[3] *Department of Mathematics, College of Science, King Saud University, Riyadh, 11451 Saudi Arabia*

Received January 1, 2024; in final form, April 4, 2024; accepted May 2, 2024

Abstract—We give a new proof of the Ostapenko–Tarasov unicellularity theorem for the classical Volterra integration operator on the space $C^{(n)}[0,1]$.

DOI: 10.1134/S0001434624050341

Keywords: *invariant subspace, Duhamel product, quasinilpotent operator, cyclic vector, invertible operator, unicellular operator.*

1. INTRODUCTION

Let X be a Banach space, and let $B(X)$ be the Banach algebra of bounded linear operators $A\colon X\to X$. A closed subspace $E\subset X$ is called an *invariant subspace* of A if $AE\subset E$, i.e., $Ax\in E$ for all $x\in E$. An operator $A\in B(X)$ is said to be *unicellular* if its lattice $\operatorname{Lat}(A)$ of invariant subspaces is linearly ordered, i.e., for any two A-invariant subspaces E and M one has either $E\subset M$ or $M\subset E$. A subspace $E\subset X$ is called a *hyperinvariant* subspace of A if $BE\subset E$ for each operator B such that $BA=AB$. The lattice of hyperinvariant subspaces of A is denoted by $\operatorname{Hyplat}(A)$.

It is well known [1], [2] that the classical indefinite integration operator V defined on the Lebesgue space $L^p[0,1]$ by

$$Vf(x)=\int_0^x f(t)\,dt$$

is unicellular for $p\in[1,\infty)$ and the lattice of invariant subspaces of V is anti-isomorphic to the interval $[0,1]$. The same is true (see [1], [3]) for the Riemann and Liouville fractional integration operators

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt,\qquad \operatorname{Re}\alpha>0,$$

which are complex powers of J. Namely,

$$\operatorname{Lat}(J^\alpha)=\operatorname{Hyplat}(J^\alpha)=\{E_a=\varkappa_{[0,a]}L^p[0,1]\colon 0\le a\le 1\}$$

(see [1], [4]). The results about the unicellularity of the operator V on $L^p[0,1]$ (see Donoghue [5]) was extended to the Sobolev spaces $W_2^{(k)}[0,1]$ (see Tsekanovskii [6]), $W_p^{(k)}[0,1]$ and $C^{(n)}[0,1]$ (see Ostapenko–Tarasov [7] and also [3], [8]). For the results on some double integration operators

$$Wf(x,y)=\int_0^x\int_0^y f(t,\tau)\,d\tau\,dt,$$

*The article was submitted by the authors for the English version of the journal.

**E-mail: tapdigoglu@gmail.com

***E-mail: mgarayev@ksu.edu.sa

we refer the reader to [9]. The Volterra integral operator

$$Vf(x)=\int_0^x f(t)\,dt \tag{1.1}$$

is well defined for functions f in the space $C^{(n)}[0,1]$ of n times continuously differentiable functions on the unit interval $[0,1]$ of the real axis $\mathbb{R}=(-\infty,\infty)$. A closed subspace $E\subset C^{(n)}[0,1]$ is said to be V*-invariant* if $VE\subset E$, i.e., $Vf\in E$ for every $f\in E$. The present paper is motivated by the papers of Ostapenko–Tarasov [7] and Tapdigoglu [8], where the unicellularity of the integration operator V on the space $C^{(n)}[0,1]$ is proved. Here we propose a new proof of the Ostapenko–Tarasov theorem by using some of the ideas in the papers [8], [10], and [11]. Namely, we prove that

$$\operatorname{Lat}(V)=\{E_\lambda, E^{(k)}:\, 0<\lambda<1,\quad k=1,\dots,n+1\},$$

where

$$E_\lambda=\{f\in C^{(n)}[0,1]:\, f(x)\equiv 0 \text{ on } [0,\lambda]\} \tag{1.2}$$

and

$$E^{(k)}=\{f\in C^{(n)}[0,1]:\, f(0)=f'(0)=\dots=f^{(k-1)}(0)=0\}. \tag{1.3}$$

Clearly,

$$\{0\}\subset E_\lambda\subset E_\mu\subset E^{(n+1)}\subset E^{(n)}\subset\dots\subset E^{(1)}\subset C^{(n)}[0,1]\quad(\lambda>\mu), \tag{1.4}$$

and hence V is a unicellular operator on $C^{(n)}[0,1]$.

2. LATTICE OF V-INVARIANT SUBSPACES

In this section, we describe the lattice of invariant subspaces of the operator V on the space $C^{(n)}[0,1]$ and prove its unicellularity. Our discussion is based on the Duhamel product of functions defined by

$$(f\circledast g)(x):=\frac{d}{dx}\int_0^x f(x-t)g(t)\,dt=\int_0^x f'(x-t)g(t)\,dt+f(0)g(x) \tag{2.1}$$

(see Wigley [12]).

Recall that the norm on $C^{(n)}[0,1]$ is defined by

$$\|f\|_n=\max_{0\le i\le n}\|f^{(i)}\|_\infty, \tag{2.2}$$

where

$$\|f^{(i)}\|_\infty:=\|f^{(i)}\|_{C^{(n)}[0,1]}=\max_{0\le x\le 1}|f^{(i)}(x)|.$$

Theorem 1. *Let V be the Volterra integration operator defined by* (1.1) *on the space $C^{(n)}[0,1]$. Then*

$$\operatorname{Lat}(V)=\{E_\lambda, E^{(k)}:\, \alpha<\lambda<1,\quad k=1,\dots,n+1\},$$

where E_λ and $E^{(k)}$ are the nontrivial V-invariant subspaces defined by formulas (1.2) *and* (1.3)*, and V is unicellular in $C^{(n)}[0,1]$.*

Proof. The proof is based on some of the arguments in the papers [3], [10], and [11]. Using (2.1), one can readily see that

$$(f\circledast g)^{(k)}(x)=\int_0^x f^{(k)}(x-t)g'(t)\,dt+\sum_{m=0}^{k-1}f^{(m)}(0)g^{(k-m)}(x)+g(0)f^{(k)}(x) \tag{2.3}$$

for all $f,g\in C^{(n)}[0,1]$ and $1\le k\le n$. Thus, it is an easy consequence of (2.3) that $f\circledast g$ belongs to $C^{(n)}[0,1]$ as well, and $(C^{(n)}[0,1],\circledast)$ becomes an algebra. One can use the results of operational calculus to show that $(C^n[0,1], \circledast)$ is a commutative and associative algebra with unit element $f(z) = \mathbf{1}$. Moreover, one can readily prove that $(C^{(n)}[0,1], \circledast)$ is a Banach algebra. Indeed, for all $f, g \in C^{(n)}[0,1]$, from (2.1)–(2.3) we have

$$|(f \circledast g)(x)| \le \max|f'| \max|g| + \max|f| \max|g|$$

and

$$\begin{aligned} |(f \circledast g)^{(k)}(x)| &\le \max|f^{(k)}| \max|g'| + \sum_{m=0}^{k-1} \max|f^{(m)}| \max|g^{(k-m)}| + \max|f^{(k)}| \max|g| \\ &\le \|f\|_n\|g\|_n + k\|f\|_n\|g\|_n + \|f\|_n\|g\|_n \\ &= (k+2)\|f\|_n\|g\|_n \le (n+2)\|f\|_n\|g\|_n \end{aligned}$$

for each $k \in \{1, 2, \dots, n\}$. Thus,

$$\|f \circledast g\|_n \le (n+2)\|f\|_n\|g\|_n, \tag{2.4}$$

and by setting $M = n + 2$ in (2.4), we obtain

$$\|f \circledast g\|_n \le M\|f\|_n\|g\|_n. \tag{2.5}$$

By passing to an equivalent norm on $C^{(n)}[0,1]$, from (2.5) we obtain the desired multiplicative inequality

$$\|f \circledast g\|_n \le \|f\|_n\|g\|_n,$$

which shows that $(C^{(n)}[0,1], \circledast)$ is a Banach algebra.

The set of V-cyclic vectors is denoted by $\mathrm{Cyc}(V)$; in other words,

$$\mathrm{Cyc}(V) = \{f \in C^{(n)}[0,1] \colon \ \mathrm{span}\{V^m f \colon m = 0, 1, \dots\} = C^{(n)}[0,1]\}.$$

Lemma 1. *One has $f \in \mathrm{Cyc}(V)$ if and only if $f(0) \ne 0$.*

Proof. Indeed, it follows from (2.1) that $x \circledast f(x) = Vf(x)$ for all $f \in C^{(n)}[0,1]$. More generally,

$$V^m f(x) = \frac{x^m}{m!} \circledast f, \qquad m \ge 0. \tag{2.6}$$

Then we have

$$\begin{aligned} \mathrm{span}\{V^m f \colon m \ge 0\} &= \mathrm{span}\left\{f \circledast \frac{x^m}{m!} \colon m \ge 0\right\} \\ &= \mathrm{span}\left\{D_f\left(\frac{x^m}{m!}\right) \colon m \ge 0\right\} \\ &= \mathrm{clos}\, D_f \,\mathrm{span}\{x^m \colon m \ge 0\} \\ &= \mathrm{clos}\, D_f[\mathrm{span}\{z\mathbf{1} \colon z \in \mathbb{C}\} \oplus \mathrm{span}\{x^m \colon m \ge 1\}] \\ &= \mathrm{clos}\, D_f C^{(n)}[0,1], \end{aligned}$$

where D_f and $\oplus$ stand for the Duhamel operator and the direct sum of subspaces, respectively. Thus,

$$\mathrm{span}\{V^m f \colon m \ge 0\} = \overline{D_f C^{(n)}[0,1]}, \tag{2.7}$$

which implies that $f \in \mathrm{Cyc}(V)$ if and only if the range of the Duhamel operator D_f is dense, i.e.,

$$\overline{D_f C^{(n)}[0,1]} = C^{(n)}[0,1]. \tag{2.8}$$

Now if $f \in \mathrm{Cyc}(V)$, then it follows from (2.8) that there exists a sequence

$$\{g_m\}_{m \ge 1} \subset C^{(n)}[0,1]$$

such that

$$\lim_{m \to \infty} f \circledast g_m = \mathbf{1} \qquad \text{in} \quad C^{(n)}[0,1].$$

Therefore, $(f \circledast g_m)(0) \to 1$ as $m \to \infty$, or, equivalently, $f(0)g_m(0) \to 1$ as $m \to \infty$, which shows that $f(0) \neq 0$. Now it remains to prove that if $f \in C^{(n)}[0,1]$ and $f(0) \neq 0$, then $f \in \mathrm{Cyc}(V)$. To this end, in view of (2.8), we must show that the range of the operator D_f is dense under the condition $f(0) \neq 0$. However, in the following Statement we prove even more than we actually need.

Statement. *If $f(0) \neq 0$, then D_f is invertible on $C^{(n)}[0,1]$.*

Indeed, for $F = f - f(0)$ we have $D_f = f(0)I_n + D_F$, where I_n is the identity operator on $C^{(n)}[0,1]$. To prove that D_f is invertible, it suffices to show that D_F is a quasinilpotent operator on $C^{(n)}[0,1]$, i.e., $\sigma(D_F) = \{0\}$, or, equivalently, the spectral radius $r(D_F)$ of the operator D_F is zero. In fact, by the well-known Gelfand formula (e.g., see Dunford–Schwartz [13]),

$$r(D_F) = \lim_{m\to\infty} \|D_F^m\|^{\frac{1}{m}},$$

and so we will estimate $\|D_F^m\|$. For every $g \in C^{(n)}[0,1]$, we have

$$\begin{aligned} D_F g(x) &= \frac{d}{dx}\int_0^x F(x-t)g(t)\,dt = \int_0^x F'(x-t)g(t)\,dt \\ &\overset{\text{def}}{=} (F' * g)(x) \overset{\text{def}}{=} (K_{F'}g)(x). \end{aligned}$$

Hence we obtain

$$\begin{aligned} |(K_{F'}g)(x)| &= \left|\int_0^x F'(x-t)g(t)\,dt\right| \le \int_0^x |F'(x-t)||g(t)|\,dt \le \int_0^x \|F'\|_\infty \|g\|_\infty\,dt \\ &\le \|F\|_n \|g\|_n (x), \end{aligned}$$

$$\begin{aligned} |(K_{F'}^2 g)(x)| &= \left(\int_0^x F'(x-t)(K_g)(t)\,dt\right) = \left|\int_0^x F'(x-t)\left(\int_0^t F'(t-\tau)g(\tau)\,d\tau\right)dt\right| \\ &\le \int_0^x |F'(x-t)|\left(\left|\int_0^t F'(t-\tau)g(\tau)\,d\tau\right|\right)dt \\ &\le \|F\|_n^2 \|g\|_n \frac{x^2}{2!}. \end{aligned}$$

Thus, by induction,

$$|(K_{F'}^m g)(x)| \le \|F\|_n^m \|g\|_n \frac{x^m}{m!}$$

for each integer $m \ge 0$. Further, we can prove that

$$|(K_{F'}^m g)'(x)| \le \|F\|_n^m \|g\|_n \frac{(x+1)^m}{m!}$$

for all $x \in [0,1]$ and $m \ge 0$. The proof is similar to that in the paper [11] and hence is omitted. In a similar way, one can also prove by induction (we omit the proof) that

$$|(K_{F'}^m g)^{(j)}(x)| \le \|F\|_n^m \|g\|_n \frac{(x+j)^m}{m!}, \qquad j = 0,1,\dots,n. \tag{2.9}$$

It follows from (2.9) that

$$\|K_{F'}^m g\|_n \le \|F\|_n^m \|g\|_n \frac{(n+1)^m}{m!},$$

and hence

$$\|K_{F'}^m\|^{\frac{1}{m}} \le \|F\|_n \frac{n+1}{(m!)^{\frac{1}{m}}} \to 0 \qquad \text{as} \quad m \to \infty.$$

Consequently, $r(K_{\alpha,F'}) = 0$, i.e., $K_{\alpha,F'}$ is quasinilpotent, and hence $D_{\alpha,f}$ is an invertible operator on $C^{(n)}[0,1]$, which proves the Statement. The proof of the lemma is complete. ☐

Lemma 2. *Let $f \in C^{(n)}[0,1]$.*

(i) *If $f \in E^{(k)}$, $1 \le k \le n$, then $f \in \operatorname{Cyc}(V \mid E^{(k)})$ if and only if $f \in E^k \backslash E^{k+1}$, i.e., if*

$$f(0) = f'(0) = \cdots = f^{k-1}(0) = 0, \qquad f^{(k)}(0) \neq 0.$$

(ii) *If $f \in E^{(n+1)}$, then $f \in \operatorname{Cyc}(V \mid E^{(n+1)})$ if only if $f \in E^{(n+1)} \backslash E_\lambda$ for every $\lambda \in (0,1)$.*

(iii) *If $f \in E_\lambda$, then $f \in \operatorname{Cyc}(V \mid E_\lambda)$ if and only if $f \in X_\lambda \backslash X_\mu$ for all $\mu > \lambda$.*

Proof. Let us define the following convolution products on the subspaces $E^{(k)}$, $1 \le k \le n+1$:

$$(f \overset{k}{\circledast} g)(x) = \frac{d}{dx} \int_0^x \frac{f(x-t)}{(x-t)^k} g(t)\, dt, \qquad f, g \in E^{(k)}. \tag{2.10}$$

(i) Let $f \in E^{(k)}$ and $f \notin E^{(k+1)}$, where $1 \le k \le n$. It is easily seen from (2.10) that

$$\frac{x^{k+m}}{m!} \overset{k}{\circledast} g = V^m g, \qquad g \in E^{(k)}, \tag{2.11}$$

for all $m \ge 0$. The Maclaurin series expansion of f gives

$$f(x) = \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(k+1)}(0)}{(k+1)!} x^{k+1} + \cdots + \frac{f^{(n)}(0)}{n!} x^n + q(x),$$

whence it follows that

$$f(x) = \frac{f^{(k)}(0)}{k!} x^k + \widetilde{q}(x), \tag{2.12}$$

where $f^{(k)}(0) \neq 0$ and

$$\widetilde{q}(x) = \frac{f^{(k+1)}(0)}{(k+1)!} x^{k+1} + \cdots + \frac{f^{(n)}(0)}{n!} x^n + q(x) \in E^{(k)}.$$

We define the *k-Duhamel operator* $D_{k,f}$ on the subspace $E^{(k)}$ by the formula

$$D_{k,f} g = f \overset{k}{\circledast} g, \qquad g \in E^{(k)}.$$

It is obvious from (2.11) and (2.12) that

$$D_{k,f} = f^{(k)}(0) I_{E^{(k)}} + D_{k,\widetilde{q}}.$$

Since $f^{(k)}(0) \neq 0$, it can be proved by the same argument as in the papers [3, Lemma 2] and [10, Lemma 1] that $D_{k,f}$ is invertible on $E^{(k)}$ (we omit the proof). On the other hand,

$$\operatorname{span}\{x^{k+m} \colon m \ge 0\} = E^{(k)}.$$

In view of the representation (2.11), we have

$$\begin{aligned} E_f(V \mid E_n^{(k)}) &:= \operatorname{span}\{V^m f \colon m \ge 0\} = \operatorname{span}\left\{ \frac{x^{k+m}}{m!} \overset{k}{\circledast} f \colon m \ge 0 \right\} \\ &= \operatorname{span}\left\{ D_{k,f} \frac{x^{k+m}}{m!} \colon m \ge 0 \right\} \\ &= \operatorname{clos} D_{k,f} \operatorname{span}\{x^{k+m} \colon m \ge 0\} \\ &= \operatorname{clos} D_{k,f} E^{(k)} = E^{(k)}; \end{aligned}$$

i.e., if $f \in E^{(k)} \backslash E^{(k+1)}$, then $f \in \operatorname{Cyc}(V \mid E^{(k)})$.

Conversely, the equality $E_f(V) = E^{(k)}$ readily implies that $f^{(k)}(0) \neq 0$, whence $f \notin E^{(k+1)}$. Thus, if $f \in E^{(k)}$ and $f \in \operatorname{Cyc}(V \mid E^{(k)})$, then $f^{(k)}(0) \neq 0$, which proves (i).

The proof of (ii) is very similar to that of (i).

Indeed, the Maclaurin formula for the function $f \in E^{(n+1)}$ has the integral representation

$$\begin{aligned} f(x) &= \frac{1}{(n-1)!}\int_0^x f^{(n)}(t)(x-t)^{n-1}\,dt \\ &= \frac{x^n}{n!} + \left(\frac{1}{(n-1)!}\int_0^x f^{(n)}(t)(x-t)^{n-1}\,dt - \frac{x^n}{n!}\right) \\ &= \frac{x^n}{n!} + q_1(x). \end{aligned}$$

Clearly, $q_1 \in E^{(n)}$ and $f \overset{n}{\circledast} g \in E^{(n+1)}$ for every $g \in E^{(n)}$. Since $f \notin E_\lambda$ for all $0 < \lambda < 1$, we have $\ker(D_{n,f}) = \{0\}$ by the Titchmarsh convolution theorem. Further, the condition $q_1 \in E^{(n)}$ implies that D_{n,q_1} is a compact operator on the space $E^{(n+1)}$ (see [3, Lemma 2]). Since

$$D_{n,f} = D_{n,\frac{x^n}{n!}+q_1(x)} = D_{n,\frac{x^n}{n!}} + D_{n,q_1(x)} = \frac{1}{n!}I_{E^{(n+1)}} + D_{n,q_1(x)}$$

and $D_{n,f}$ is invertible on $E^{(n+1)}$, we have

$$D_{n,f}E^{(n+1)} = E^{(n+1)}.$$

Now

$$\begin{aligned} E_f(V \mid E^{(n)}) &= \operatorname{span}\{V^m f \colon m \ge 0\} = \operatorname{span}\left\{\frac{x^{n+m}}{m!} \overset{n}{\circledast} f \colon m \ge 0\right\} \\ &= \operatorname{span}\left\{D_{n,f}\frac{x^{n+m}}{m!} \colon m \ge 0\right\} \supset \operatorname{span}\left\{D_{n,f}\frac{x^{n+1+m}}{(m+1)!} \colon m \ge 0\right\} \\ &= \operatorname{clos} D_{n,f}\operatorname{span}\left\{\frac{x^{n+1+m}}{(m+1)!} \colon m \ge 0\right\} = \operatorname{clos} D_{n,f}E^{(n+1)} = E^{(n+1)}. \end{aligned}$$

Hence $E_f(V \mid E^{(n)}) \supset E^{(n+1)}$. On the other hand, $VE^{(n+1)} \subset E^{(n+1)}$, and so $E_f(V \mid E^{(n)}) \subset E^{(n+1)}$. Thus, $E_f(V \mid E^{(n)}) = E^{(n+1)}$, whence $f \in \operatorname{Cyc}(V \mid E^{(n+1)})$. Conversely, if $f \in \operatorname{Cyc}(V \mid E^{(n+1)})$, then $f \in E^{(n+1)} \backslash E_\lambda$ for all $\lambda \in (0,1)$, which proves (ii).

The proof of (iii) can be obtained from Lemma 1 by a standard argument based on a simple change of variables (e.g., see Ostapenko–Tarasov [7], Kalisch [14], and Gohberg–Krein [1]) and hence it is omitted. The proof of Lemma 2 is complete. □

Now let us return to the proof of Theorem 1. We will show that there exist no V-invariant subspaces other than those in the chain (1.4) and hence

$$\operatorname{Lat}(V) = \{E_\lambda, E^{(k)} \colon 0 < \lambda < 1; \quad k = 1, 2, \dots, n+1\}.$$

Indeed, assume the contrary: there exists a nontrivial V-invariant subspace $E \subset C^{(n)}[0,1]$ different from the invariant subspaces in (1.4). It is clear that

$$E = \bigcup_{g \in E} E_g(V \mid E),$$

where, as before,

$$E_g(V \mid E) = \operatorname{span}\{V^m g \colon m = 0, 1, \dots\}.$$

Then it is clear by Lemma 1 that there exists a function $f \in E$ such that $f(0) \neq 0$. Consequently, by Lemma 1, we conclude that $E = C^{(n)}[0,1]$, which contradicts our assumption that E is a nontrivial subspace. Since the set of subspaces in (1.4) is linearly ordered, it follows that the operator V is unicellular. The proof of the theorem is complete. □

Recall that the commutant of an operator A on a Banach space X is the set

$$\{A\}' = \{B \in B(X) \colon BA = AB\},$$

where $B(X)$ is the Banach algebra of all bounded linear operators on X. Set

$$\mathrm{Hyplat}(B) := \{E \subset X \colon AE \subset E \text{ for each } A \in \{B\}'\}.$$

Corollary 1. $\mathrm{Hyplat}(V) = \{E_\lambda, E^{(k)} \colon 0 < \lambda < 1; \quad k = 1, \dots, n+1\}$.

The proof is immediate by the well-known general theorem stating that the lattice of hyperinvariant subspaces of any unicellular operator A coincides with the lattice of nontrivial A-invariant subspaces (e.g., see Radjavi–Rosenthal [2]).

The following corollary describes the commutant of the operator V.

Corollary 2. $\{V\}' = \{D_f \colon f \in C^{(n)}[0,1]\}$.

Proof. Indeed, it is clear from the formula

$$Vf(x) = x \circledast f \qquad \forall f \in C^{(n)}[0,1]$$

that $D_f V = V D_f$ for all $f \in C^{(n)}[0,1]$ and hence

$$\{D_f \colon f \in C^{(n)}[0,1]\} \subset \{V\}'. \tag{2.13}$$

Let $T \in \{V\}'$. Then $TV = VT$, and hence $TV^m = V^m T$ for all $m \geq 0$. Applying formula (2.6) to the identity function $f = \mathbf{1}$, we obtain

$$TV^m\mathbf{1} = \frac{x^m}{m!} \circledast T\mathbf{1}, \qquad m \geq 0,$$

i.e.,

$$T\left(\frac{x}{m!} \circledast \mathbf{1}\right) = D_{T\mathbf{1}}\left(\frac{x^m}{m!}\right), \qquad m \geq 0.$$

Since

$$\frac{x^m}{m!} \circledast \mathbf{1} = \frac{x^m}{m!}, \qquad m \geq 0,$$

we have

$$T\left(\frac{x^m}{m!}\right) = D_{T\mathbf{1}}\left(\frac{x^m}{m!}\right),$$

or, equivalently,

$$T(x^m) = D_{T\mathbf{1}}(x^m), \qquad m \geq 0.$$

Since

$$\mathrm{span}\{x^m \colon m \geq 0\} = \mathrm{span}\{z\mathbf{1} \colon z \in \mathbb{C}\} \oplus \mathrm{span}\{x^m \colon m \geq 1\} = C^{(n)}[0,1],$$

it follows from the last equalities that $Tf(x) = D_{T\mathbf{1}}f(x)$ for all $f \in C^{(n)}[0,1]$ and hence $T = D_{T\mathbf{1}}$. Thus, $\{V\}' \subset \{D_f \colon f \in C^{(n)}[0,1]\}$, which, together with (2.13), proves the corollary. $\square$

FUNDING

The work of the first author was supported by ongoing institutional funding. The work of second author was supported by the Research Support Project no. RSPD2024R1056, King Saud University, Riyadh, Saudi Arabia.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. I. C. Gohberg and M. G. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, in *Transl. Math. Monogr.* (Amer. Math. Soc., Providence, R. I., 1970), Vol. 24.
2. H. Radjavi and P. Rosenthal, *Invariant Subspaces* (Dover, Mineola, N. Y., 2003).
3. M. T. Karaev, "Invariant subspaces, cyclic vectors, commutant and extended eigenvectors of some convolution operators," Methods Funct. Anal. Topol. **11** (1), 48–59 (2005).
4. M. S. Brodskij, *Triangular and Jordan Representations of Linear Operators*, in *Transl. Math. Monogr.* (Amer. Math. Soc., Providence, R. I., 1971), Vol. 32.
5. W. F. Donoghue, "The lattice of invariant subspaces of a completely continious quasinilpotent transformation," Pac. J. Math. **7**, 1031–1035 (1957).
6. E. R. Tsekanovskii, "On description of invariant subspaces and unicellularity of the integration operator in the space $W_2^{(p)}$," Usp. Mat. Nauk **20** (6(126)), 169–172 (1965).
7. P. V. Ostapenko and V. G. Tarasov, "Unicellularity of the integration operator in certain function spaces," Teor. Funkts. Funkts. Anal. Prilozh. **27**, 121–128 (1977).
8. R. Tapdigoglu, "On the description of invariant subspaces in the space $C^{(n)}[0,1]$," Houston J. Math. **39** (1), 169–176 (2013).
9. R. Tapdigoglu, "On the Banach algebra structure for $C^{(n)}$ of the bidisc and related topics," Ill. J. Math. **64** (2), 185–197 (2020).
10. R. Tapdigoglu, "Invariant subspaces of Volterra integration operator: Axiomatical approach," Bull. Sci. Math. **136**, 574–578 (2012).
11. M. T. Garayev, H. Guediri, and H. Sadraoui, "On some problems in the space $C^{(n)}[0,1]$," Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar. **78** (1), 147–156 (2016).
12. N. M. Wigley, "The Duhamel product of analytic functions," Duke Math. J. **41**, 211–217 (1974).
13. N. Dunford and J. T. Schwartz, *Linear operators. Part I: General Theory* (Interscience, New York–London, 1958).
14. G. K. Kalisch, "On similarity, reducing manifolds, and unitary equivalence of certain Volterra operators," Ann. Math. (2) **66**, 481–494 (1957).