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Complexity of Recognizing Multidistance Graphs in $\mathbb{R}^d$

G. M. Sokolov[1*]

[1] *Moscow Institute of Physics and Technology, Moscow, 117303 Russia*

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Abstract—We study the complexity of recognizing A-distance graphs in $\mathbb{R}^d$ and prove that for all finite sets A such that any two elements of the set differ by a factor ≥ 2, the recognition problem for A-distance graphs is NP-hard for any $d \geq 3$.

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1. INTRODUCTION

1.1. History and Motivation of the Problem

One of the classical objects of study of combinatorial geometry is distance graphs. A graph is called a distance graph in $\mathbb{R}^d$ if its vertices can be mapped into points of $\mathbb{R}^d$ such that the distance between the images of any pair of vertices connected with an edge is equal to 1.

The distance graphs are closely related to the classical chromatic number problem of the space (see, e.g., book [1, Part 2]): the De Bruijn–Erdős theorem [2] implies that the chromatic number of $\mathbb{R}^d$ is equal to the maximum of chromatic numbers of finite distance graphs in $\mathbb{R}^d$. Various options of definitions of the distance graphs can be considered: namely, it can be required or not required that the mapping taking vertices of the graph to the points of $\mathbb{R}^d$ is injective, and also that nonadjacent vertices correspond to the points, distance between which is distinct from 1. Following work [3], we will call the mapping of the vertex set of the graph in $\mathbb{R}^d$ the embedding if the vertices joined with an edge are mapped into points at a distance of 1; the embedding is called injective if distinct vertices of the graph are mapped into distinct points, and it is called strict if distinct vertices, which are not joined with an edge, are mapped into points at a distance distinct from 1. A graph is called a (strictly) (injectively) embeddable into $\mathbb{R}^d$, if it admits an embedding with the respective properties.

One of the natural problems associated with distance graphs is to determine the complexity of recognizing graphs embeddable into $\mathbb{R}^d$, namely, determining for various types of embeddability and values of the parameter d whether this problem is polynomially solvable or NP-hard. It can be easily verified that for $d = 1$ the problem can be solved in polynomial time for any type of embeddability. In [4] it is proved that for $d = 2$ the problem is NP-hard for any type of embeddability. Finally, in [3] it is proved that for $d \geq 3$ the problem is NP-hard for any type of embeddability. Thus, the problem of determining the complexity of recognizing the graphs embeddable into $\mathbb{R}^d$ with unit distances is solved for all d and all types of embeddability. However, this problem can be generalized in a natural way: instead of ordinary distance graphs, the multidistance graphs can be considered. A graph is called multidistance in $\mathbb{R}^d$ with a distance set A if its vertices can be mapped into points of $\mathbb{R}^d$ such that the distance between the images of any pair of vertices joined with an edge belongs to the set A. Similarly to the ordinary distance graphs, strict, injective and noninjective A-embeddability into $\mathbb{R}^d$ can be considered. The complexity of recognition of graphs which are A-embeddable into $\mathbb{R}^d$ is studied only for some particular choices of A and d. For the case $d = 1$, for every type of embedding [5] describes the classification of all finite sets A into those for which the problem of recognition of graphs that are A-embeddable into $\mathbb{R}$ is polynomially

*E-mail: SokG2000@yandex.ru

solvable and those for which this problem is NP-hard. The case $A = (0, 1]$ was studied in [6], [7], and it is proved that the problem of recognition of graphs that are strictly injective $(0, 1]$-embeddable into $\mathbb{R}^d$ is polynomially solvable for $d = 1$ and NP-hard for $d = 2$.

We propose a method of reduction the problem of recognition of injectively embeddable graphs with a single allowed distance to the problem of recognition of graphs that are injectively A-embeddable, for some sets A. For $d \geq 3$, this method allows one to prove that for any finite set A any two elements of which differ at least twice, the problem of recognition of nonstrictly injective A-embeddable graphs is NP-hard and almost for every such A the problem of recognition of strictly A-embeddable graphs is NP-hard. It seems likely that in the future this method can be used to prove the NP-hardness of the problem of recognizing injectively A-embeddable graphs for other classes of A as well, since it allows one to reduce the proof of NP-hardness of this problem to a purely geometric (not related to computational complexity) problem.

In the rest of the introduction we will present the main definitions and formulate the results proved in [3] in the form, which will be convenient for us to use. The second section of the present paper describes a general method for reducing the recognition of injectively embeddable graphs with a single allowed distance to the recognition of injectively A-embeddable graphs. In the third section of the paper, this method is used to prove the NP-hardness of the problem of recognition injectively A-embeddable graphs for some classes of sets A.

1.2. Basic Definitions and Previously Known Results

Definition 1. Let $G = (V, E)$ be a graph with a vertex set V and an edge set E, and A be some set of positive numbers. An injective mapping $\varphi\colon V \to \mathbb{R}^d$ is called an *injective A-embedding* of G into $\mathbb{R}^d$ if it takes any pair of vertices connected with an edge to the points, the distance between which belongs to the set A. An embedding is called *strict* if it takes any pair of vertices not connected with an edge to the points, the distance between which does not belong to A. A graph is called *(strictly) injectively A-embeddable* into $\mathbb{R}^d$ if there exists a (strict) injective embedding of it into $\mathbb{R}^d$.

Notation 1. As in [3], we denote the problem of recognition of (strictly) injectively $\{1\}$-embeddable into $\mathbb{R}^d$ graphs by $\mathbb{R}^d$-UNIT-DIST-(STRICT)-INJ-EMB; the problem of recognition of (strictly) injective A-embeddable into $\mathbb{R}^d$ graphs is denoted by $\mathbb{R}^d$-A-(STRICT)-INJ-EMB.

We note that the problem $\mathbb{R}^d$-UNIT-DIST-(STRICT)-INJ-EMB is equivalent to the problem $\mathbb{R}^d$-$\{a\}$-(STRICT)-INJ-EMB for any positive a, because a $\{1\}$-embedding can be transformed into an $\{a\}$-embedding, and an $\{a\}$-embedding can be transformed into a $\{1\}$-embedding via homothety.

The problems $\mathbb{R}^d$-UNIT-DIST-INJ-EMB and $\mathbb{R}^d$-UNIT-DIST-STRICT-INJ-EMB are NP-hard for any $d > 2$ by [3, Theorem 1].

To work with the strict embeddings, we need not this theorem itself, but the construction used to prove it and its properties, stated in [3, Theorem 4]. We will introduce one more definition from [3] (generalizing it to A-embeddings) and give a closed statement of [3, Theorem 4] (its original formulation refers to the constructions previously described in the article).

Definition 2. A strict injective A-embedding is called *noncritical* if for any three vertices their images do not lie on the same line.

Theorem 1 [3, Theorem 4]. *For any $d > 2$ from any graph G it is possible to construct a graph H in polynomial time such that if G is properly colorable in 3 colors, then there exists a noncritical $\{1\}$-embedding for H into $\mathbb{R}^d$, and if there is no proper coloring for G in 3 colors, then there is no $\{1\}$-embedding of H into $\mathbb{R}^d$.*

1.3. Outline and Statement of the Main Results

In Sec. 2 we describe a general approach for proving the NP-hardness of recognition of injectively A-embeddable into $\mathbb{R}^d$ graphs using the rods construction. In Sec. 3 we prove the existence of the rods for some class of the sets A. The main results of the paper are the following theorems.

Theorem 2. *If $d \geq 3$ and A is a finite set of positive numbers, each pair of elements of which differ at least twice, then the problem of recognition of injectively A-embeddable into $\mathbb{R}^d$ graphs is* NP-*hard.*

Theorem 3. *There exists at most countable set M such that if $d \geq 3$ and A is a finite set of positive numbers, each pair of elements of which differ at least twice, with a maximal element a, and, moreover, $aM \cap A = \varnothing$, then the problem of recognition of strictly injectively A-embeddable into $\mathbb{R}^d$ graphs is* NP-*hard.*

2. (u, v, a)-RODS AND THEIR PROPERTIES

2.1. Nonstrict Embeddings

Definition 3. Let A be a set of positive numbers, a be a positive number. A graph H with labeled vertices u and v is called a *(strict) (u, v, a)-rod* in $\mathbb{R}^d$ with respect to the distance set A, if the following conditions hold

- there exists a (strict) injective A-embedding of H into $\mathbb{R}^d$ such that the images of any three vertices do not lie on the same line;
- for any nonstrict injective A-embedding of H into $\mathbb{R}^d$, the distance between vertices u and v is equal to a.

Notation 2. Let G be a graph, H be a graph with special vertices u and v. Denote by $f_H(G)$ a graph constructed in the following way: in the graph G each edge is replaced by the copy of H, identifying one head of edge with u and the second with v.

Notation 3. For a graph G, by $V(G)$ we denote its vertex set, and by $E(G)$ its edge set.

Lemma 1. *Let G be a graph, H be a (u, v, a)-rod in $\mathbb{R}^d$ with respect to the set of positive numbers A. Let $G' = f_H(G)$. Then*

- *if G' is injectively A-embeddable into $\mathbb{R}^d$, then G is injectively $\{a\}$-embeddable into $\mathbb{R}^d$;*
- *if G id injectively $\{a\}$-embeddable into $\mathbb{R}^d$, then G' is injectively A-embeddable into $\mathbb{R}^d$.*

Proof. 1. The second paragraph of the definition of the rod implies that the restriction to the vertices of G of the injective A-embedding of G' is an injective $\{a\}$-embedding of G.

2. Consider an injective $\{a\}$-embedding φ of a graph G into $\mathbb{R}^d$ construct from it an injective A-embedding of the graph G'. Let us arrange in an arbitrary way the edges of G; let $\{e_1, \dots, e_m\}$ be a resulting arrangement; denote by H_i a copy of the graph H, which replaces the edge e_i in construction of G'. Let us replace the edges G with copies of H one by one and complete the embedding φ for $H_i \setminus \{u, v\}$ in such a way that after every step the embedding remains injective. Let us construct for every $i \in \{0, \dots, m\}$ an embedding φ_i, which is an injective continuation of φ to $V(G) \cup V(H_1) \cup \dots \cup V(H_i)$ such that for any $j \leq i$ the restriction of φ_i into $V(H_j)$ is an A-embedding of H_j. Set $\varphi_0 = \varphi$. Construct φ_{i+1} as a continuation of φ. Let a, b be heads of the edge e_{i+1}. Consider an arbitrary injective embedding φ' of the rod H_{i+1} into $\mathbb{R}^d$, for which $\varphi'(u) = \varphi(a)$, $\varphi'(v) = \varphi(b)$. Let S be an infinite family of the rotations around the line $\varphi(a)\varphi(b)$ such that for any ψ_1, ψ_2 from S and for any point x not belonging to the line $\varphi(a)\varphi(b)$ unequality $\psi_1(x) \neq \psi_2(x)$ holds (it can be easily checked that for $d > 2$ such an infinite family exists). Let φ_ψ be a mapping $V(G) \cup V(H_1) \cup \dots \cup V(H_{i+1}) \to \mathbb{R}^d$ such that $\varphi_\psi(w) = \varphi_i(w)$ for $w \in V(G) \cup V(H_1) \cup \dots \cup V(H_i)$ and $\varphi_\psi(w) = \psi(\varphi'(w))$ for $w \in V(H_{i+1})$

(the definition is compatible in the intersection of $V(G) \cup V(H_1) \cup \cdots \cup V(H_i)$ and $V(H_{i+1})$ since $\varphi_i(a) = \varphi'(u) = \psi(\varphi'(u))$ and $\varphi_i(b) = \varphi'(v) = \psi(\varphi'(v))$). For any $\psi \in S$ the mapping φ_ψ satisfies all the requirements for φ_{i+1}, except for injectivity. From the injectivity of φ_i and φ' it follows that for any $\psi \in S$ and any $w_1, w_2 \in V(G) \cup V(H_1) \cup \cdots \cup V(H_i)$ unequality $\varphi_\psi(w_1) \neq \varphi_\psi(w_2)$ holds, and for any $w_1, w_2 \in V(H_{i+1})$ unequality $\varphi_\psi(w_1) \neq \varphi_\psi(w_2)$ holds, thus the injectivity follows from the fact that for any $w_1 \in V(G) \cup V(H_1) \cup \cdots \cup V(H_i) \setminus \{a, b\}$ and $w_2 \in V(H_{i+1}) \setminus \{u, v\}$ unequality $\varphi_\psi(w_1) \neq \varphi_\psi(w_2)$ holds. For fixed

$$w_1 \in V(G) \cup V(H_1) \cup \cdots \cup V(H_i) \setminus \{a, b\}, \qquad w_2 \in V(H_{i+1}) \setminus \{u, v\},$$

the value of $\varphi_\psi(w_1)$ does not depend on ψ, and the values of $\varphi_\psi(w_2)$ are distinct for all ψ, thus the equality $\varphi_\psi(w_1) = \varphi_\psi(w_2)$ holds at most for a single ψ. The set of such pairs w_1, w_2 is finite, and S is infinite, thus there exists $\psi \in S$ such that φ_ψ is injective, for this ψ we set $\varphi_{i+1} = \varphi_\psi$. Executing this procedure for all edges, we get an injective mapping $\varphi_m : G' \to \mathbb{R}^d$, the restriction of which to H_i is an A-embedding. Any edge G' is an edge of some H_i, so φ_m is an injective A-embeding of G' into $\mathbb{R}^d$. □

Theorem 4. *Let A be some set of positive numbers, $d > 2$. If for some a there exists an (u, v, a)-rod in $\mathbb{R}^d$ with respect to A, then the problem of recognition of injective A-embeddable into $\mathbb{R}^d$ graphs is* NP-*hard.*

Proof. We can construct $f_H(G)$ from G in a polynomial (with respect to the size of G) time, and, according to Lemma 1, this construction is a reduction of the problem $\mathbb{R}^d$-$\{a\}$-INJ-EMB (and, therefore, $\mathbb{R}^d$-UNIT-DIST-INJ-EMB) to the problem $\mathbb{R}^d$-A-INJ-EMB. According to [3, Theorem 1], the problem $\mathbb{R}^d$-UNIT-DIST-INJ-EMB is NP-hard and, therefore, the problem $\mathbb{R}^d$-A-INJ-EMB is NP-hard as well. □

2.2. Strict Embeddings

The analog of Theorem 4 for a strict embedding turns out to be weaker, while the proof turns out to be more complicated. Formally, we will reduce the problem of coloring a graph in three colors to the recognition of strictly injective A-embeddable graphs, using the construction from [3] as one of the reduction steps. This is due to the fact that in order for the construction of an embedding of G' from an embedding of G, similar to that used in Theorem 4, to work, the original embedding of G must be noncritical and no vertices not connected by an edge must lie at a distance of A from each other.

Lemma 2. *Let G be a graph, H be a strict (u, v, a)-rod in $\mathbb{R}^d$ with respect to the set of positive numbers A. Let $G' = f_H(G)$. Then*

- *if G' is injective A-embeddable into $\mathbb{R}^d$, then G is injective $\{a\}$-embeddable into $\mathbb{R}^d$;*
- *if there exists a noncritical embedding of G into $\mathbb{R}^d$ such that the images of any pair of vertices do not lie at the distance from $A \setminus \{a\}$ from each other, then there exists a noncritical A-embedding of G' into $\mathbb{R}^d$.*

Proof. 1. From the second paragraph of the definition of the rod it follows that the restriction to the vertices of injective A-embedding G' is an injective $\{a\}$-embedding G.

2. In the proof of the second point of Theorem 4 we require that the embedding φ_i be noncritical. The rest of the proof repeats the proof of Theorem 4, except for the fact that for the embedding φ_0 to be strict it is necessary to use that the images of any pair of vertices in the original embedding do not lie at the distance from $A \setminus \{a\}$ from each other, and also that the checking of existence of such ψ that the embedding φ_ψ is strict and injective, requires more technical details. We omit this checking since it completely coincides with the analog checking in the proof of [3, Lemma 1]. □

Lemma 3. *If there exists some (possibly nonstrict) (u, v, a)-rod in $\mathbb{R}^d$ with respect to A, then there exists a strict (u, v, a)-rod in $\mathbb{R}^d$ with respect to A.*

Proof. Let H be a (u, v, a)-rod in $\mathbb{R}^d$ with respect to A. Let φ be an embedding from the first paragraph of the definition of the rod. Let H' be a graph obtained from H by addition of edges (x, y) for all $x, y \in V(H)$ such that the distance between $\varphi(x)$ and $\varphi(y)$ belongs to A. Then H' is a strict (u, v, a)-rod in $\mathbb{R}^d$ with respect to A. Indeed, the first paragraph of the definition holds for the same embedding φ, since any two vertices having images at the distance from A are now joined with an edge, and any two vertices at the distance not from A remain not joined with an edge; the second paragraph of the definition holds, since it remains true after adding edges. □

Theorem 5. *There exists at most countable set M such that for any d and for any set of positive numbers A such that for some $a \in A$ there exists (possibly nonstrict) (u, v, a)-rod in $\mathbb{R}^d$ with respect to A, and $aM \cap A = \varnothing$, the problem of recognition of the injectively A-embeddable into $\mathbb{R}^d$ graphs is* NP*-hard. (aM stands for the set $\{ax \mid x \in M\}$).*

Proof. Consider all graphs, which can be colored in 3 colors. For every such graph G and for every $d > 3$ denote by $h_d(G)$ a graph constructed from it from Theorem 4, and fix one its noncritical $\{1\}$-embedding of $\varphi_{G,d}$ into $\mathbb{R}^d$ (at least one such embedding exists according to Theorem 4). Let M be a set of all pairwise distances between the images of the vertices not joined with an edge in all fixed embeddings (obviously, this set is at most countable). Let A be some set of positive numbers, $a \in A$, and the set A does not intersect with aM and there exists a (u, v, a)-rod H. According to Lemma 3, from this it follows that there exists a strict (u, v, a)-rod H'. Let us demonstrate that the composition of the construction from Theorem 4 and Lemma 2 is a reduction of the problem of 3-coloring of the graph to the problem of the recognition of the strictly injectively A-embeddable graphs and, therefore, this problem is NP-hard.

If a graph G is 3-colorable, then, according to the construction of M, for a graph $h_d(G)$ there exists a noncritical $\{1\}$-embedding $\varphi_{G,d}$ such that the images of any two nonadjacent vertices lie at the distance from M. Using the homothety, we get from this embedding such a noncritical $\{a\}$-embedding, so that images of any two nonadjacent vertices lie at the distance from aM. From condition $aM \cap A = \varnothing$, we get that the images of any two nonadjacent vertices do not lie at the distance from A, i.e.. the images of any two vertices do not lie at the distance from $A \setminus \{a\}$. According to Lemma 2, it implies that $f_{H'}(h_d(G))$ is strictly injectively A-embeddable.

If $f_{H'}(h_d(G))$ is strictly injectively A-embeddable into $\mathbb{R}^d$, then, according to Lemma 2, the graph $h_d(G)$ is injectively $\{a\}$-embeddable into $\mathbb{R}^d$, and, according to the Theorem 4, the graph G is 3-colorable. □

Theorems 4 and 5 reduce the proof of the NP-hardness of the recognition of A-embeddable graphs to the construction of the rod with respect to the distance set A. Note that this problem is purely geometric, i.e. it does not involve the computational complexity. In the rest of the paper, the construction of rods for some sets A is described.

3. CONSTRUCTION OF RODS

In this section we construct a rod in $\mathbb{R}^d$ with respect to the set A for any $d \geq 2$ and any set A such that any two elements from A differ at least twice.

Describe the graph, for which it will be proved that it is a rod. Let $G(d, n) = (V, E)$, where $V = \{1, \ldots, d + 2(1 + d(n - 1))\}$, and $(u, v) \in E$ if and only if $u \leq d$ or $v \leq d$. Let $B = \{1, \ldots, d\}$, $C = V \setminus B$. Then $V = B \sqcup C$, moreover, $|B| = d$, $|C| = 2(1 + d(n - 1))$ and between any two vertices from B there is an edge, between any two vertices from C there is no edge, and between any vertex from B and any vertex from C there is an edge. In other words, $G(d, n)$ divides into clique of d vertices and an independent set of $2(1 + d(n - 1))$ vertices, moreover, any vertex from clique is joined with any vertex from the independent set. We will prove that for any $d \geq 2$ and any set A such that any two elements of A differ at least twice, the graph $G(d, |A|)$ is a (u, v, a)-rod in $\mathbb{R}^d$ with respect to A for $a = \max A$, and any u and v from B (for, e.g., $u = 1$, $v = 2$).

Let us verify that the first paragraph of the definition of the rod holds.

Lemma 4. *If $|A| = n$, $d \geq 2$, then there exists an injective A-embedding of $G(d, n)$ into $\mathbb{R}^d$ such that the images of any three vertices do not lie on the same line.*

Proof. Let

$$A = \{a_1, \dots, a_n\}, \qquad a_1 < a_2 < \dots < a_n.$$

Let $V(G(d, n)) = B \sqcup C$ be a partition of the graph $G(d, n)$ into clique of d vertices and the independent set of $2(1 + d(n - 1))$ vertices. To simplify the description of the embedding construction, we will identify the vertices of the graph with their images under the embedding. Let us place the vertices from B at the vertices of a regular $(d - 1)$-dimensional simplex with side a_n. It is necessary to check that in $\mathbb{R}^d$ there are $2(1 + d(n - 1))$ points such that the distance between them and any vertices from B belong to A, then it is possible to allocate at these points the vertices from C and obtain the required embedding. Let us find $1 + d(n - 1)$ points at each of open semispaces, into which $\mathbb{R}^d$ is divided by $(d - 1)$-dimensional subspace containing the vertices from B. Let us fix one of semispaces. For every point x of d points in B, consider a set of points in that semispace, that are at the distance of a_n from all other points from B. This set is a semicircle with a diameter $2a_n$, and x belongs to one of its endpoints, therefore, for any $i \leq n$ there is a point on it, which lies at the distance of a_i from x. All these points are at some distance from A from each of points from B. The points, which are at the distance of a_n from x, coincide for different x, and all other points are distinct (since ordered sets of distances from them to the points from B are distinct). Thus, at every open semispace there are $1 + d(n - 1)$ distinct points, and at the entire space there are $2(1 + d(n - 1))$ distinct points, the distance from each of which to each point from B belongs to A. Thus, by mapping into these points the vertices of the graph from C, we get an injective A-embedding of $G(d, n)$ in $\mathbb{R}^d$. The images of all vertices lie at the boundary of the intersection of d balls with centers at the vertices from B and radius a_n, thus none of three points lie on the same line. $\square$

To prove that the second paragraph of the definition of the rod holds, it is necessary to demonstrate that for any embedding of the clique B, for which not all edges have length a_n, there are no $2(1 + d(n - 1))$ distinct points lying at the distance from A from the images of all vertices from B. For a fixed ordered distance set between the images of vertices from B, there exist at most two points with this distance set, thus, it is sufficient to prove that if not every edge from B has length a_n, then the number of distance sets, for which there exists at least one such point, is less than $1 + d(n - 1)$. We will show that in this case the number of distance sets such that for a triple of vertices the triangle inequality holds is less than $1 + d(n - 1)$. In order to do so, we will prove the following lemma.

Definition 4. Edge coloring of a graph in n colors $\{1, \dots, n\}$ satisfies the triangle inequality if in any triangle two colors with maximal color number coincide.

Lemma 5. *Let G be a complete graph with $d + 1$ vertices, H be its induced subgraph with d vertices. Then any edge coloring of H in n colors can be continued in at most $1 + d(n - 1)$ ways to the edge coloring of G, which satisfies the triangle inequality, and, moreover, the equality is achieved only if all edges of H are colored with the color n.*

Proof. If the coloring of H does not satisfy the triangle inequality, then it can not be continued to the coloring satisfying the triangle inequality, so in what follows we assume that the coloring of H satisfies the triangle inequality.

Let us prove the statement of Lemma by induction on the number of colors n. For $n = 1$ the statement is trivial: all the edges of H are colored with the color n, and there exists the only way to continue the coloring to the rest edges with one color.

Now prove the transition from n to $n + 1$. Fix the edge coloring of H. Denote by x a vertex of G not belonging to H. Consider a subgraph H, which is obtained from H by removing all the edges of color n, and denote it H'. Note that for any three distinct vertices $u, v, w \in V(H)$ if $(u, v) \in E(H')$ and $(v, w) \in E(H')$, then $(u, w) \in E(H')$ as well, i.e. all connected components of H' are cliques. Indeed, if it is not so, then in the graph H the edge (u, w) is colored with color n, and edges (u, v) and (v, w) are colored with colors of less number, which contradicts to the fact that the edge coloring of H satisfies the triangle inequality. Moreover, if the continuation satisfying the triangle inequality some vertex is joined by an edge of color n with the vertex x, then all the other vertices from this connected component of H' must be connected with x by an edge of color n. Indeed, if the vertex u is joined with x by the edge of number n, and the vertex v of the same connected component is joined by an edge of different color, then the edge (u, x) is a unique edge of color n in the triangle with vertices x, u, v, and the coloring does not satisfy the triangle inequality. Finally, if two vertices lie at different connected components of H', then for any continuation of the coloring satisfying the triangle inequality at least one of them is joined with the vertex x by an edge of color n, since otherwise in a triangle with vertices at these two vertices and x, there is exactly one edge is colored with color n. So, in any continuation of the coloring satisfying the triangle inequality either all vertices are joined with the vertex x by an edge of color n or there is only one connected component of H' all the vertices of which are joined with x by edges of colors less that n by the number, and all vertices from the other connected components are joined with x by an edge of color n. Let H' be decomposed into k connected components $H'_1, \dots, H'_k$, consisting of $d_1, \dots, d_k$ vertices respectively. Then, due to the assumption of the induction, the edges joining H_i and x, can be colored in at most $1 + d_i(n-2)$ ways with the colors $\{1, \dots, n-1\}$ such that the triangle inequality holds on them, therefore, the number of continuations of the coloring of H to the coloring of G does not exceed

$$1 + \sum_{i=1}^{k}(1 + d_i(n-2)) = 1 + k + d(n-2) \le 1 + d(n-1),$$

moreover, in the latter inequality the equality is achieved only for $d = k$, i.e. only if H' does not contain edges, i.e. only if all the edges of H are colored with the color n. □

Let us complete the proof of the fact that the graph $G(d, n)$ is a rod in $\mathbb{R}^d$ with respect to n-element sets, any two elements of which differ at least twice.

Theorem 6. *Let $A = \{a_1, \dots, a_n\}$ and for any i the inequality holds $2a_i < a_{i+1}$. Then the graph $G(d, n)$ is a $(1, 2, a_n)$-rod in $\mathbb{R}^d$ with respect to A.*

Proof. Let $V(G(d, n)) = B \sqcup C$ be a partition of $G(d, n)$ into a clique of d vertices and an independent set of $2(1 + d(n-1))$ vertices. The first paragraph of the definition of the rod holds true due to Lemma 4. Let us verify the second paragraph of the definition. Consider an arbitrary injective A-embedding of $G(d, n)$ into $\mathbb{R}^d$. Let us construct from this embedding an edge coloring of $G(d, n)$ into n colors: let us paint the edge in color i if the distance between the images of its heads is equal to a_i. From the triangle inequality in $\mathbb{R}^d$ and the fact that any two elements from A differ at least twice it follows that this coloring satisfies the triangle inequality. Therefore, for each vertex i from C the coloring of the complete graph on the vertices $B \cup \{i\}$ defined by it is a continuation of the coloring of the complete graph with d vertices to the coloring of the complete graph with $d + 1$ vertices, which satisfies the triangle inequality. If for some vertices from C these continuations coincide, then for their images the ordered sets of distances from their images to the vertices from B coincide and, therefore, each continuation of the coloring occur at most two times. Thus, the number of distinct continuations is at least $1 + d(n-1)$. According to Lemma 5, this implies that all edges between the vertices from B are colored with the same color n, i.e. the distance between the images of any two vertices from B is equal to a_n, in particular, the distance between images of vertices 1 and 2 is equal to a_n. □

Proof of Theorems 2, 3. According to Theorem 6, for a set A from Theorems 2 and 3 there exist (u, v, a)-rods. According to Theorem 4, this implies the statement of Theorem 2, and due to Theorem 5, the statement of Theorem 3 is true. □

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

REFERENCES

1. A. Soifer, *The Mathematical Coloring Book. Mathematics of Coloring and the Colorful Life of its Creators* (Springer, New York, 2009).
2. N. G. de Bruijn and P. Erdos, "A colour problem for infinite graphs and a problem in the theory of relations," Indag. Math. **13**, 369–373 (1951).
3. M. Tikhomirov, "On computational complexity of length embeddability of graphs," Discrete Math. **339** (11), 2605–2612 (2016).
4. B. Horvat, J. Kratochvil, and T. Pisanski, "On the computational complexity of degenerate unit distance representations of graphs," in *Combinatorial Algorithms*, *Lecture Notes in Comput. Sci.* (Springer, Heidelberg, 2011), Vol. 6460, pp. 274–285.
5. M. Tikhomirov, "On complexity of multidistance graph recognition in $\mathbb{R}^1$," Elect. Notes Discr. Math. **61**, 1039–1045 (2017).
6. K. S. Booth and G. S. Lueker, "Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms," J. Comput. System Sci. **13** (3), 335–379 (1976).
7. H. Breu and D. G. Kirkpatrick, "Unit disk graph recognition is NP-hard," Comput. Geom. **9** (1–2), 30–24 (1998).