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On Some Properties of the Permanent of Matrices of Small Orders

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Abstract—The permanent is a multilinear function that is a "symmetric" analog of the determinant. In the present paper, we consider several properties of the permanent of matrices of small orders.

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1. INTRODUCTION

As is known, the determinant of n-th order can be regarded as an n-ary multilinear antisymmetric function on the space of n-dimensional vectors, and the multilinearity and antisymmetry conditions define it uniquely up to a normalization coefficient, which is determined by the value of this function on the basis vectors. If we consider similarly a multilinear symmetric function, additionally assuming that the value of this function on an n-ary set of basis vectors is 0 if there are at least two identical vectors in this set, and is equal to 1 otherwise, we come to the concept of *permanent*. More precisely, let $\overline{e}_i$, $i = 1, \dots, n$, be basis vectors and let $\overline{v}_j = \sum_{i=1}^n a_{ij}\overline{e}_i$, $j = 1, \dots, n$, be arbitrary vectors. Then the permanent on the set of vectors $\overline{v}_j$ has the following value:

$$\mathrm{per}(\overline{v}_1, \overline{v}_2, \dots, \overline{v}_n) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

As in the case of the determinant, the coordinates of the vectors can be combined into a matrix $A = (a_{ij})$, and we can talk about the permanent as a function on the set of matrices [1, Sec. 1.1]. Despite the similarity of definitions, the permanent does not possess the rich set of properties of the determinant, which makes the latter one of the basic mathematical functions. For example, the analog of the property $\det(AB) = \det A \det B$ fails to hold for the permanent. As a consequence, the permanent, as compared with the determinant, is a much more "sensitive" function to linear matrix transformations. If we talk about linear transformations given on the entire set of matrices of n-th order, $n > 2$, then the permanent is preserved only when the rows and columns are permuted and possibly under some scaling of the rows and columns [2]–[4]. However, some properties of the determinant fully hold for the permanent as well. For example, it is easy to see that the permanent of a matrix is preserved under the transposition and, due to the multilinearity, the permanent has the decomposition along rows (columns). In practice, the permanent has found wide applications in graph theory and enumerative combinatorics [1], [5]–[7]. In this paper, we present several properties of the permanent of low-order matrices. Here we combine the multilinear vector and the matrix approaches to the permanent. In what follows, we consider the real case only.

The paper is organized as follows. In Sec. 2, we establish the geometric meaning of the permanent of real matrices of the second order. In Sec. 3 we introduce the concepts of a permanent vector and mixed product of three-dimensional vectors and consider some of their properties. In Sec. 4 we prove the properties of multilinear forms (including the permanent) that characterize the values of the forms on sets of radius vectors of vertices of regular polygons.

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2. GEOMETRIC MEANING OF THE PERMANENT OF A SECOND-ORDER MATRIX

In the simplest case of real (2×2)-matrices, it is easy to give a transparent geometric interpretation of the permanent.

Proposition 1. *Let $\overline{u}(a,b)$ and $\overline{v}(c,d)$ be two vectors on the plane. Denote by α the angle between the X-axis and the bisector of the angle between these vectors* (*Fig.* 1). *Then*

$$\operatorname{per}(\overline{u},\overline{v}) = |\overline{u}|\,|\overline{v}|\sin 2\alpha. \tag{2.1}$$

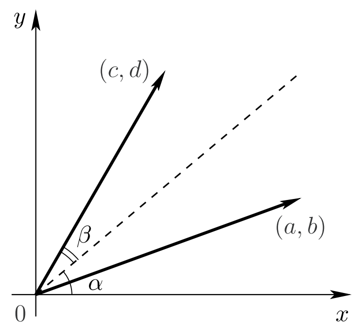

Fig. 1. The geometric meaning of the permanent of a 2×2 matrix.

Proof. Denote by β the angle between the vectors and the bisector of the angle between them. We can write

$$a = |\overline{u}|\cos(\alpha-\beta), \quad b = |\overline{u}|\sin(\alpha-\beta), \quad c = |\overline{v}|\cos(\alpha+\beta), \quad d = |\overline{v}|\sin(\alpha+\beta),$$

whence

$$\begin{aligned}\operatorname{per}(\overline{u},\overline{v}) = ad + bc &= |\overline{u}|\,|\overline{v}|\big[\cos(\alpha-\beta)\sin(\alpha+\beta) + \sin(\alpha-\beta)\cos(\alpha+\beta)\big]\\ &= |\overline{u}|\,|\overline{v}|\sin(\alpha+\beta+\alpha-\beta) = |\overline{u}|\,|\overline{v}|\sin 2\alpha. \quad \square\end{aligned}$$

Corollary 1. *Let $\overline{u}(a,b)$ and $\overline{v}(c,d)$ be two nonzero vectors. The sign of the permanent* $\operatorname{per}(\overline{u},\overline{v})$ *characterizes the direction of the bisector of the angle between these vectors as follows*:

- *If* $\operatorname{per}(\overline{u},\overline{v}) > 0$*, then the bisector belongs to the first or third coordinate angle.*
- *If* $\operatorname{per}(\overline{u},\overline{v}) < 0$*, then the bisector belongs to the second or fourth coordinate angle.*
- *If* $\operatorname{per}(\overline{u},\overline{v}) = 0$*, then the bisector coincides with one of the coordinate semiaxes.*

We mentioned in the introduction that, for $n > 2$, the permanent is preserved only under a very restricted set of linear transformations of matrices. It follows from Proposition 1 that, in the case of $n = 2$, the situation is somewhat different.

Corollary 2. *When changing the angle β between the vectors and the bisector of the angle between them, assuming that the lengths of the vectors and the angle α between the bisector and the X-axis are preserved, the permanent of the matrix composed of the coordinates of these vectors does not change.*

In other words, if we act on the first vector by the matrix M_1 of the rotation by an angle $-\gamma$ and on the other vector by the matrix M_2 of the rotation by an angle γ, then the permanent of the resulting matrix is equal to the permanent of the original matrix:

$$\operatorname{per}(\overline{u},\overline{v}) = \operatorname{per}(M_1\overline{u}, M_2\overline{v}),$$

or, in the matrix form,

$$\operatorname{per}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \operatorname{per}\begin{pmatrix} a\cos\gamma + b\sin\gamma & c\cos\gamma - d\sin\gamma \\ -a\sin\gamma + b\cos\gamma & c\sin\gamma + d\cos\gamma \end{pmatrix}.$$

This "pseudo-linear" transformation of the two-dimensional space can be represented as a linear transformation of the four-dimensional space. To do this, we are to assign, to a pair of vectors $\overline{u}(a,b)$ and $\overline{v}(c,d)$, the 4D vector $\overline{w}(a,b,c,d)$ and act on it by the block-diagonal orthogonal matrix

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\overline{w}.$$

This orthogonal transformation, in addition to preserving the lengths of the vectors, acts invariantly on the 3-manifolds of the form $ad + bc = \text{const}$.

3. PERMANENT OF REAL THIRD-ORDER MATRICES

In the case of a three-dimensional space, it is rather difficult to give a transparent geometric interpretation analogous to that in the plane case. We consider here the permanent only as an analog of the usual mixed product of vectors.

Let $\overline{i}, \overline{j}, \overline{k}$ be an orthonormal basis in $\mathbb{R}^3$. Consider two vectors,

$$\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}, \qquad \overline{b} = b_1\overline{i} + b_2\overline{j} + b_3\overline{k}.$$

Using them, define the third vector according to the following rule:

$$\overline{c} = \overline{i}\operatorname{per}\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} + \overline{j}\operatorname{per}\begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix} + \overline{k}\operatorname{per}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

We call this operation on the set of three-dimensional vectors the *permanent vector product* and denote it by $\langle\overline{a},\overline{b}\rangle$. It has the following properties:

1) $\langle\overline{a},\overline{b}\rangle = \langle\overline{b},\overline{a}\rangle$, the commutativity;
2) $\langle\alpha\overline{a} + \beta\overline{b},\overline{c}\rangle = \alpha\langle\overline{a},\overline{c}\rangle + \beta\langle\overline{b},\overline{c}\rangle$, the linearity;
3) $\langle\overline{i},\overline{i}\rangle = \langle\overline{j},\overline{j}\rangle = \langle\overline{k},\overline{k}\rangle = 0$, the nilpotency;
4) $\langle\overline{i},\overline{j}\rangle = \overline{k}$, $\langle\overline{i},\overline{k}\rangle = \overline{j}$, $\langle\overline{j},\overline{k}\rangle = \overline{i}$;
5) $\langle\alpha\overline{i} + \beta\overline{j}, \alpha\overline{i} - \beta\overline{j}\rangle = \langle\alpha\overline{i} + \beta\overline{k}, \alpha\overline{i} - \beta\overline{k}\rangle = \langle\alpha\overline{j} + \beta\overline{k}, \alpha\overline{j} - \beta\overline{k}\rangle = 0$.

The associativity property fails to hold. For example, $\langle\langle\overline{i},\overline{j}\rangle,\overline{j}\rangle \neq \langle\overline{i},\langle\overline{j},\overline{j}\rangle\rangle$. Thus, the set of three-dimensional vectors, with respect to this operation, forms a three-dimensional commutative and not associative algebra without unit. It can be defined as an algebra generated by three generators i, j, k and the relations

$$i^2 = j^2 = k^2 = 0, \qquad ij = ji = k, \qquad ik = ki = j, \qquad jk = kj = i.$$

It can readily be seen that part 5) exhausts all cases, up to multiplication by a scalar, in which the product $\langle\,\cdot\,,\cdot\,\rangle$ of nonzero vectors is zero.

Let three three-dimensional vectors $\overline{x}$, $\overline{a}$, and $\overline{b}$ be given. Consider the number $\overline{x}\cdot\langle\overline{a},\overline{b}\rangle$, where "$\cdot$" stands for the usual dot product of vectors. Let us call it the *permanent mixed product*. It is easy to see that

$$\overline{x}\cdot\langle\overline{a},\overline{b}\rangle = \operatorname{per}(\overline{x},\overline{a},\overline{b}). \tag{3.1}$$

It follows from (3.1) and the rule of expansion of a permanent along a column that

$$\overline{x}\cdot\langle\overline{a},\overline{b}\rangle = \langle\overline{x},\overline{a}\rangle\cdot\overline{b}. \tag{3.2}$$

Thus, as in the case of the ordinary mixed product, the signs of the dot product and permanent vector product in (3.2) can be omitted, and we can just write $\bar{x}\bar{a}\overline{b}$.

Consider the equality

$$\operatorname{per}(\overline{x}, \overline{a}, \overline{b}) = 0. \tag{3.3}$$

If the vectors $\overline{a}$ and $\overline{b}$ are assumed to be given and the vector $\overline{x}$ is unknown, then (3.3) defines an equation of the plane that is perpendicular to the nonzero vector $\langle \overline{a}, \overline{b} \rangle$. If the vector $\langle \overline{a}, \overline{b} \rangle$ is zero, then (3.3) is an identity with respect to $\overline{x}$. Thus, a necessary and sufficient condition for equality (3.3) is the perpendicularity of the vectors $\overline{x}$ and $\langle \overline{a}, \overline{b} \rangle$. In the next section, we consider properties that characterize the value of multilinear forms on sets of radius vectors of vertices of a regular polygon. As a consequence, we obtain an interesting special case in which equality (3.3) holds.

4. MULTILINEAR FORMS ON THE POSITION VECTORS OF VERTICES OF A REGULAR POLYGON

In this section, we consider properties that are valid not only for the permanent but also for a wider class of multilinear forms.

Let f be a bilinear form on $\mathbb{R}^2$, let g be the operator of rotation by the angle $2\pi/n$, $n \in \mathbb{N}$, $n \geq 3$, let $\overline{v}$ be an arbitrary vector in $\mathbb{R}^2$, and let $p \in \mathbb{Z}$. Introduce the following notation:

$$F_{n,p}(\overline{v}) = \sum_{k=0}^{n-1} f(g^k(\overline{v}), g^{k+p}(\overline{v})).$$

Theorem 1. *Choose an orthonormal basis $\overline{i}$, $\overline{j}$ in $\mathbb{R}^2$. Then, for the equality*

$$F_{n,p}(\overline{v}) = 0$$

to hold for any $n \in \mathbb{N}$, $n \geq 3$, $p \in \mathbb{Z}$, and $\overline{v} \in \mathbb{R}^2$, it is necessary and sufficient that f be symmetric and have the property

$$f(\overline{i}, \overline{i}) + f(\overline{j}, \overline{j}) = 0. \tag{4.1}$$

Proof. *Necessity.* Let g be the operator of rotation by the angle $\pi/2$ and let $\overline{v}(a, b)$ be an arbitrary vector. Consider the sum

$$\begin{aligned} F_{4,1}(\overline{v}) = \sum_{k=0}^{3} f(g^k(\overline{v}), g^{k+1}(\overline{v})) &= f(a\overline{i} + b\overline{j}, -b\overline{i} + a\overline{j}) + f(-b\overline{i} + a\overline{j}, -a\overline{i} - b\overline{j}) \\ &\quad + f(-a\overline{i} - b\overline{j}, b\overline{i} - a\overline{j}) + f(b\overline{i} - a\overline{j}, a\overline{i} + b\overline{j}) \\ &= 2(a^2 + b^2)(f(\overline{i}, \overline{j}) - f(\overline{j}, \overline{i})). \end{aligned}$$

Since $F_{4,1}(\overline{v}) = 0$ by assumption for every $\overline{v}$, it follows that the form f is symmetric. It is easy to calculate similarly that, if g is the operator of rotation by the angle $2\pi/3$, then

$$F_{3,1}(\overline{v}) = -\frac{3}{4}(a^2 + b^2)\big(f(\overline{i}, \overline{i}) + f(\overline{j}, \overline{j}) - \sqrt{3} f(\overline{i}, \overline{j}) + \sqrt{3} f(\overline{j}, \overline{i})\big).$$

Since $F_{3,1}(\overline{v}) = 0$ for any vector $\overline{v} \in \mathbb{R}^2$ by assumption, it follows that the following equality holds:

$$f(\overline{i}, \overline{i}) + f(\overline{j}, \overline{j}) - \sqrt{3} f(\overline{i}, \overline{j}) + \sqrt{3} f(\overline{j}, \overline{i}) = 0.$$

Then equality (4.1) holds by the symmetry of the form f.

Sufficiency. Consider an arbitrary vector $\overline{v}$. Suppose that the angle between the X-axis and the vector $\overline{v}$ is equal to ϕ. Let g be the operator of rotation by the angle $2\pi/n$. Then

$$\begin{aligned} f(g^k(\overline{v}), g^{k+p}(\overline{v})) = |\overline{v}|^2 \bigg[& f(\overline{i}, \overline{i}) \cos\left(\phi + \frac{2\pi k}{n}\right) \cos\left(\phi + \frac{2\pi(k+p)}{n}\right) \\ & + f(\overline{j}, \overline{j}) \sin\left(\phi + \frac{2\pi k}{n} r\right) \sin\left(\phi + \frac{2\pi(k+p)}{n}\right) \end{aligned}$$

$$+ f(\bar{i},\bar{j})\cos\left(\phi+\frac{2\pi k}{n}\right)\sin\left(\phi+\frac{2\pi(k+p)}{n}\right)$$
$$+ f(\bar{j},\bar{i})\sin\left(\phi+\frac{2\pi k}{n}\right)\cos\left(\phi+\frac{2\pi(k+p)}{n}\right)\Bigg]$$
$$= |\overline{v}|^2\left[f(\bar{i},\bar{i})\cos\left(2\phi+\frac{2\pi p}{n}+\frac{4\pi k}{n}\right)+f(\bar{i},\bar{j})\sin\left(2\phi+\frac{2\pi p}{n}+\frac{4\pi k}{n}\right)\right].$$

Hence

$$F_{n,p}(\overline{v}) = |\overline{v}|^2\left[f(\bar{i},\bar{i})\sum_{k=0}^{n-1}\cos\left(2\phi+\frac{2\pi p}{n}+\frac{4\pi k}{n}\right)+f(\bar{i},\bar{j})\sum_{k=0}^{n-1}\sin\left(2\phi+\frac{2\pi p}{n}+\frac{4\pi k}{n}\right)\right].$$

It can readily be seen that

$$\sum_{k=0}^{n-1}\cos\left(2\phi+\frac{2\pi p}{n}+\frac{4\pi k}{n}\right),\qquad \sum_{k=0}^{n-1}\sin\left(2\phi+\frac{2\pi p}{n}+\frac{4\pi k}{n}\right)$$

are multiple sums of the first and second coordinates, respectively, of the vertices of some regular polygon centered at the origin. Therefore, these sums are zero, and $F_{n,p}(\overline{v}) = 0$. □

Corollary 3. *Choose an orthonormal basis in $\mathbb{R}^2$. Let g be the operator of rotation in $\mathbb{R}^2$ by the angle $2\pi/n$, let $\overline{v}\in\mathbb{R}^2$ be an arbitrary vector, and let f be a bilinear symmetric function on $\mathbb{R}^2$ satisfying condition* (4.1). *Denote by $D_n = \{(i,j),\ i,j=1,\dots,n,\ i<j\}$ the set of ordered pairs of integers from* 1 *to n. Then*

$$\sum_{(i,j)\in D_n} f(g^i(\overline{v}),g^j(\overline{v})) = 0,\qquad n\ge 3.$$

Proof. The vectors $g(\overline{v}),g^2(\overline{v}),\dots,g^n(\overline{v})$ are radius vectors of the vertices of some regular n-gon centered at the origin; therefore, their sum is zero. Hence, taking into account the bilinearity and symmetry of the function f, we obtain

$$0 = f\left(\sum_{k=1}^{n} g^k(\overline{v}),\sum_{k=1}^{n} g^k(\overline{v})\right) = 2\sum_{(i,j)\in D_n} f(g^i(\overline{v}),g^j(\overline{v})) + \sum_{k=1}^{n} f(g^k(\overline{v}),g^k(\overline{v})). \tag{4.2}$$

Since $g^n(\overline{v}) = \overline{v}$, it follows that the second sum on the right-hand side (4.2) is equal to $F_{n,0}(\overline{v})$ and, therefore, by Theorem 1, is equal to zero. Thus, the first sum on the right-hand side (4.2) is equal to zero, as was to be proved. □

Corollary 4. *Choose an orthonormal basis in $\mathbb{R}^3$. Let $\overline{v}$ be an arbitrary vector, and let g be the operator of rotation around the Z-axis by the angle $2\pi/3$. Then*

$$\operatorname{per}(\overline{v},g(\overline{v}),g^2(\overline{v})) = 0.$$

Proof. Since g is an operator of rotation around the Z-axis, we see that all three vectors $\overline{v}$, $g(\overline{v})$, and $g^2(\overline{v})$ have the same coordinate z, say, equal to c. Expanding the permanent along the third row, we obtain

$$\operatorname{per}(\overline{v},g(\overline{v}),g^2(\overline{v})) = c\big(\operatorname{per}(\overline{u},\widetilde{g}(\overline{u}))+\operatorname{per}(\widetilde{g}(\overline{u}),\widetilde{g}^2(\overline{u}))+\operatorname{per}(\widetilde{g}^2(\overline{u}),\overline{u})\big),$$

where $\overline{u}$ is the projection of the vector $\overline{v}$ to the plane XOY and $\widetilde{g}$ is the operator of rotation in the plane by the angle $2\pi/3$. By Theorem 1, the last expression is zero. □

Corollary 5. *Choose an orthonormal basis in $\mathbb{R}^2$ and consider a regular n-gon centered at the origin with the vertices (a_i,b_i), where i ranges from* 1 *to n. Then the sum of the products of the coordinates of the vertices of the given n-gon is zero*:

$$\sum_{i=1}^{n} a_i b_i = 0.$$

Proof. Let $\overline{v}$ be the radius vector of the first vertex of the polygon and let g be the operator of rotation by the angle $2\pi/n$. Then the vectors $g^k(\overline{v})$, $k = 0, \dots, n-1$, correspond to the radius vectors of all the vertices of the polygon. Considering the permanent as a function f and assuming that $p = 0$, we obtain, by Theorem 1,

$$\sum_{k=0}^{n-1} \operatorname{per}(g^k(\overline{v}), g^k(\overline{v})) = 0.$$

However, it can readily be seen that $\operatorname{per}(g^k(\overline{v}), g^k(\overline{v}))$ is equal to the doubled product of the coordinates of the vector $g^k(\overline{v})$. This implies the assertion of the corollary. □

Now let us pass to the three-dimensional space. In this case, an analog of the previous property is proved even simpler and has more general form.

Let h be a trilinear form on $\mathbb{R}^3$, let $\overline{v}$ be an arbitrary vector, let g be the operator of rotation by angle $2\pi/n$, $n \in \mathbb{N}$, $n \geq 4$, around an arbitrary axis passing through the origin perpendicularly to the vector $\overline{v}$, and let $p \in \mathbb{Z}$. Introduce the notation

$$H_{n,p}(\overline{v}) = \sum_{k=0}^{n-1} h\big(g^k(\overline{v}), g^{k+p}(\overline{v}), g^{k+2p}(\overline{v})\big).$$

Theorem 2. *For every $n \in \mathbb{N}$, where $n \geq 4$, $p \in \mathbb{Z}$, and $\overline{v} \in \mathbb{R}^3$, we have*

$$H_{n,p}(\overline{v}) = 0.$$

Proof. Let $\overline{v}$ be an arbitrary vector in $\mathbb{R}^3$, let g be the operator of rotation by the angle $2\pi/n$, $n \in \mathbb{N}$, $n \geq 4$, around an arbitrary axis passing through the origin perpendicularly to $\overline{v}$. Since all vectors $g^k(\overline{v})$, $k \in \mathbb{Z}$, lie in the same plane, it follows that each of them can be expressed linearly in terms of the vectors $\overline{v}$ and $g(\overline{v})$. In order to obtain this dependence explicitly, let us temporarily pass to the auxiliary coordinate system in which the vectors $g^k(\overline{v})$ lie in a plane XOY and the direction of the vector $\overline{v}$ coincides with the positive direction of the X-axis. In this coordinate system, the vector $g^k(\overline{v})$ has the coordinates $(\cos(2\pi k/n), \sin(2\pi k/n))$. Consider the equality $g^k(\overline{v}) = \alpha\overline{v} + \beta g(\overline{v})$. Writing out this equation using the coordinates and solving the resulting system of two linear equations for α and β, we see that the vectors $g^k(\overline{v})$, $k \in \mathbb{Z}$, are expressed in terms of the vectors $\overline{v}$ and $g(\overline{v})$ as follows:

$$g^k(\overline{v}) = -\frac{\sin(2(k-1)\pi/n)}{\sin(2\pi/n)}\overline{v} + \frac{\sin(2k\pi/n)}{\sin(2\pi/n)} g(\overline{v}). \tag{4.3}$$

By the trilinearity property, we can write

$$\begin{aligned} H_{n,p}(\overline{v}) = {} & \alpha h(\overline{v}, \overline{v}, \overline{v}) + \beta h\big(g(\overline{v}), g(\overline{v}), g(\overline{v})\big) + \gamma h\big(\overline{v}, \overline{v}, g(\overline{v})\big) + \delta h\big(\overline{v}, g(\overline{v}), \overline{v}\big) \\ & + \mu h\big(g(\overline{v}), \overline{v}, \overline{v}\big) + \nu h\big(g(\overline{v}), g(\overline{v}), \overline{v}\big) + \rho h\big(g(\overline{v}), \overline{v}, g(\overline{v})\big) + \sigma h\big(g(\overline{v}), g(\overline{v}), \overline{v}\big), \end{aligned}$$

where α, β, γ, δ, μ, ν, ρ, and σ are some coefficients. Consider each of these coefficients separately. By (4.3), we obtain

$$\begin{aligned} \alpha &= -\frac{1}{(\sin(2\pi/n))^3} \sum_{k=0}^{n-1} \sin\frac{2(k-1)\pi}{n} \sin\frac{2(k+p-1)\pi}{n} \sin\frac{2(k+2p-1)\pi}{n} \\ &= -\frac{1}{2(\sin(2\pi/n))^3} \sum_{k=0}^{n-1} \left(\cos\frac{4p\pi}{n} - \cos\frac{4(k+p-1)\pi}{n}\right) \sin\frac{2(k+p-1)\pi}{n} \\ &= -\frac{1}{2(\sin(2\pi/n))^3} \Bigg[\cos\frac{4p\pi}{n} \sum_{k=0}^{n-1} \sin\frac{2(k+p-1)\pi}{n} - \frac{1}{2}\sum_{k=0}^{n-1} \sin\frac{6(k+p-1)\pi}{n} \end{aligned}$$

$$+ \frac{1}{2}\sum_{k=0}^{n-1} \sin\frac{2(k+p-1)\pi}{n}\bigg].$$

Consider the sums in square brackets. The first and third of them are exactly equal to the sum of the imaginary parts of the n-th roots of unity and, therefore, are equal to zero. It can readily be seen that the second sum is proportional to the sum of the imaginary parts of the n-th roots of unity or of the roots of unity of some lesser order and, therefore, this sum is also equal to zero. Thus, the coefficient α is equal to zero. It can be proved in exactly the same way that all other coefficients are also equal to zero. This implies the assertion of the theorem. □

Similarly to the plane case, using Theorem 2 and considering the permanent as a trilinear form h, one can prove the following assertion.

Corollary 6. *Consider a regular n-gon in $\mathbb{R}^3$, $n \geq 4$, centered at the origin and with the vertices (a_i, b_i, c_i), where i ranges from 1 to n. Then the sum of the products of the coordinates of the vertices of the given n-gon is equal to zero*:

$$\sum_{i=1}^{n} a_i b_i c_i = 0.$$

In conclusion, we conjecture that analogs of Theorems 1 and 2 hold also for spaces of higher dimensions.

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