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The L_p-Mixed Geominimal Surface Areas*

C. J. Zhao[1]**

[1] *Department of Mathematics, China Jiliang University, Hangzhou, 310018, P. R. China*

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Abstract—In the paper, our main aim is to introduce a new concept and call it the *L_p-mixed geominimal surface area* $G_p(K_1, \ldots, K_n)$ of n convex bodies $K_1, \ldots, K_n$, which obeys the classical basic properties. The new affine geometric quantity in a special case yields Petty's geominimal surface area $G(K)$ of a convex body K, Lutwak's p-geominimal surface area $G_p(K)$ of K, and the newly established mixed geominimal surface area $G(K_1, \ldots, K_n)$ of n convex bodies $K_1, \ldots, K_n$. We establish some L_p-mixed geominimal surface area inequalities for the L_p-mixed geominimal surface area, whose some special cases are Petty's geominimal surface area inequality, Lutwak's p-geominimal surface area inequality, and some new mixed geominimal surface area inequalities.

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1. INTRODUCTION

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of the convex bodies (compact, convex sets with nonempty interiors) of $\mathbb{R}^n$. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call $V(K)$ the volume of K. Let B be the unit ball centered at the origin, and let its volume be ω_n. For the set of convex bodies containing the origin in their interiors, write $\mathcal{K}_o^n$, and let $\mathcal{K}_c^n$ denote the set of convex bodies whose centroids lie at the origin. The important concept of geominimal surface area was introduced by Petty [1]. The concept serves as a bridge connecting a number of areas of geometry: affine differential geometry, relative geometry, and Minkowski geometry. The geominimal surface area $G(K)$ of $K \in \mathcal{K}^n$ was defined by

$$\omega_n^{1/n} G(K) = \inf\{nV_1(K,Q)V(Q^*)^{1/n} : Q \in \mathcal{K}_o^n\}, \tag{1}$$

where $V_1(K,Q)$ denotes the usual mixed volume of K and Q, and Q^* is the polar of Q. Given $Q \in \mathcal{K}_o^n$, let Q^* denote the polar of the body Q, defined by (see, e.g. [2])

$$Q^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \qquad \text{for all} \quad y \in Q\},$$

where $\langle x, y\rangle$ denotes the usual inner product of x and y in $\mathbb{R}^n$.

The geominimal surface area of a body is invariant under the unimodular affine transformations of the body. Petty [1] showed that the geominimal surface area $G : \mathcal{K}^n \to (0, \infty)$ is continuous and also established the following fundamental inequality for the geominimal surface area:

Petty's geominimal surface area inequality. *If $K \in \mathcal{K}_o^n$, then*

$$G(K)^n \le n^n \omega_n V(K)^{n-1} \tag{2}$$

with equality if and only if K is an ellipsoid.

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**E-mail: chjzhao@163.com, chjzhao@cjlu.edu.cn

Some extension of Petty's geominimal surface area was presented by Lutwak [3]. The L_p-mixed geominimal surface area, $G_p(K)$ of $K \in \mathcal{K}_o^n$, with $p \geq 1$, was defined by

$$\omega_n^{p/n} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\}, \tag{3}$$

where $V_p(K,Q)$ denotes the usual L_p-mixed volume of K and Q. It was also shown that the L_p-mixed geominimal surface area of a body is invariant under the unimodular centro-affine transformations of the body. He showed also that $G_p : \mathcal{K}_o^n \to (0,\infty)$ is continuous. Some extension of Petty's geominimal surface area inequality was also obtained:

Lutwak's p-geominimal surface area inequality. *If $p \geq 1$ and $K \in \mathcal{K}_o^n$, then*

$$G_p(K)^n \leq n^n \omega_n^p V(K)^{n-p} \tag{4}$$

with equality if and only if K is an ellipsoid.

Recently, the geominimal surface area, p-geominimal surface area, Orlicz geominimal surface area and related inequalities have attracted extensive attention and research. The recent research on these matters can be found in the references [4]–[14].

To the convex bodies $K_1, \cdots, K_{n-1}$ in $\mathbb{R}^n$ there is a unique positive Borel measure on S^{n-1}, $S(K_1, \dots, K_{n-1}; \cdot)$, called the *mixed area measure* of $K_1, \dots, K_{n-1}$, such that for every convex body K_n we have the integral representation (see, e.g. [15], p. 280)

$$V(K_1, \dots, K_n) = \frac{1}{n}\int_{S^{n-1}} h(K_n, u)\, dS(K_1, \dots, K_{n-1}; u), \tag{5}$$

where $u \in S^{n-1}$, S^{n-1} stands for the unit sphere, and $h(K,x)$ is the support function of a convex body K (simply denoted by h_K).

The integration is with respect to the mixed area measure $S(K_1, \dots, K_{n-1}; \cdot)$ on S^{n-1}. The mixed area measure $S(K_1, \dots, K_{n-1}; \cdot)$ is symmetric in its first $n-1$ arguments. If $K_1 = \cdots = K_{n-i-1} = K$ and $K_{n-i} = \cdots = K_{n-1} = B$, the mixed area measure $S(K, \dots, K, B, \dots, B; \cdot)$ with i copies of B and $(n-i-1)$ copies of K will be written as $S_i(K, \cdot)$. If $K_1 = \cdots = K_{n-1} = K$, then $S(K_1, \dots, K_{n-1}; \cdot)$ becomes the surface area measure $S(K, \cdot)$.

We will present some natural extension of Lutwak's p-geominimal surface area in this paper. The L_p-mixed geominimal surface area $G_p(K_1, \dots, K_n)$ of $K_1, \dots, K_n \in \mathcal{K}_o^n$ with $p \geq 1$ was defined by

$$\omega_n^{p/n} G_p(K_1, \dots, K_n) = \inf\{nV_p(K_1, \dots, K_{n-1}, Q, K_n)V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\}, \tag{6}$$

where $V_p(K_1, \dots, K_{n-1}, Q, K_n)$ denotes the L_p-multiple mixed volume of $n+1$ convex bodies $K_1, \dots, K_n$ and Q and is defined by (see [16])

$$V_p(K_1, \dots, K_{n-1}, Q, K_n) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{h(Q,u)}{h(K_n,u)}\right)^p h(K_n, u)\, dS(K_1, \dots, K_{n-1}; u). \tag{7}$$

It will be shown that the L_p-mixed geominimal surface area is invariant under unimodular centro-affine transformations. It will also be shown that $G_p : \underbrace{\mathcal{K}_o^n \times \ldots \times \mathcal{K}_o^n}_{n} \to (0,\infty)$ is unique and continuous. In Sec. 4, we also establish an affine isoperimetric inequality for the L_p-mixed geominimal surface area.

The L_p-mixed geominimal surface area inequality. *If $K_1, \dots, K_n \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$G_p(K_1, \dots, K_n)^n \leq n^n \omega_n^p V(K_1, \dots, K_n)^n / V(K_n)^p, \tag{8}$$

with equality if and only if $K_1, \dots, K_{n-1}$ and $K_n = E$ are mixed p-self-minimal, where E is an ellipsoid.

The equality conditions for this inequality involve "mixed p-self-minimal" bodies. These bodies are defined in Sec. 4.

Obviously, if $K_1 = \cdots = K_n = K$, then (8) becomes (4). If $p = 1$ and $K_1 = \cdots = K_n = K$, then (8) becomes (2).

It is worth mentioning here that other definitions of "L_p-mixed geominimal surface area" have recently been published. However, none of them is based on the newly established L_p-multiple mixed volume of $(n+1)$ convex bodies $K_1, \ldots, K_n$ and L_n, so neither of them is a natural extension. Because the L_p-multiple mixed volume was proposed which follows the spirit of introduction of Aleksandrov's mixed quermassintegrals and Lutwak's p-mixed quermassintegrals. Thus, the concept of L_p-mixed geominimal surface area based on the L_p-multiple mixed volume complies with the study line of Petty and Lutwak's geominimal surface area and the law of development.

2. NOTATION AND PRELIMINARIES

Let d denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, we have

$$d(K, L) = |h(K, u) - h(L, u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space $C(S^{n-1})$ of continuous functions. The Minkowski addition plays an important role in the Brunn–Minkowski theory. During the last few decades, the theory has been extended to L_p-Brunn–Minkowski theory. The well-known L_p addition is defined by (see Firey [17])

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p \tag{9}$$

for all $x \in \mathbb{R}^n$, $1 \le p \le \infty$, and compact convex sets K and L in $\mathbb{R}^n$ containing the origin.

2.1. Basics on Convex Bodies

Define the Santaló product of $K \in \mathcal{K}_o^n$ as $V(K)V(K^*)$. The Blaschke–Santaló inequality is one of the fundamental affine isoperimetric inequalities (see [18]–[21]). It states that if $K \in \mathcal{K}_c^n$, then

$$V(K)V(K^*) \le \omega_n^2, \tag{10}$$

with equality if and only if K is an ellipsoid.

The radial function $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, \infty)$ of a compact star-shaped $K \subset \mathbb{R}^n$ (about the origin) is defined for $x \in \mathbb{R}^n \backslash \{0\}$ as

$$\rho(K, x) = \max\{\lambda \ge 0 : \lambda x \in K\}.$$

If $\rho(K, u)$ (or simply ρ_K) is positive and continuous, then K is called a star body (about the origin). Write $\mathcal{S}_o^n$ for the set of star bodies in $\mathbb{R}^n$. For $K \in \mathcal{K}_o^n$, it is easily seen that

$$h(K^*, \cdot) = \frac{1}{\rho(K, \cdot)} \qquad \text{and} \qquad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \tag{11}$$

For $\phi \in \mathrm{GL}(n)$, we write ϕ^t for the transpose of ϕ and ϕ^{-t} for the inverse of the transpose of ϕ. It is easy that (see [22])

$$h(\phi K, x) = h(K, \phi^t x), \tag{12}$$

where $K \in \mathcal{K}_o^n$. Obviously,

$$(\phi Q)^* = \phi^{-t} Q^*. \tag{13}$$

where $Q \in \mathcal{K}_o^n$.

For $K \in \mathcal{K}_o^n$, define the inner radius $r(K)$ and outer radius $R(K)$ by

$$r(K) = \max\{\lambda > 0 : \lambda B \subset K\} \qquad \text{and} \qquad R(K) = \min\{\lambda > 0 : \lambda B \supset K\}. \tag{14}$$

Obviously,

$$r(K) = \min_{u \in S^{n-1}} h(K, u) \qquad \text{and} \qquad R(K) = \max_{u \in S^{n-1}} h(K, u).$$

2.2. L_p-Mixed Volumes

The Brunn–Minkowski inequality is the best-known inequality concerning volumes of compact convex sets. It states that if K and L are compact convex sets in $\mathbb{R}^n$, then

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{15}$$

with equality if and only if K and L are homothetic.

The mixed volume $V_1(K,L)$ of compact convex sets K and L is defined by

$$V_1(K,L) := \frac{1}{n}\lim_{\varepsilon\to 0^+}\frac{V(K+\varepsilon L)-V(K)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} h(L,u)\,dS(K,u). \tag{16}$$

The Minkowski inequality for K and L states that

$$V_1(K,L) \geq V(K)^{(n-1)/n}V(L)^{1/n}, \tag{17}$$

with equality if and only if K and L are homothetic.

The L_p Minkowski mixed volume inequality is as follows:

$$V_p(K,L) \geq V(K)^{(n-p)/n}V(L)^{p/n}, \tag{18}$$

with equality if and only if K and L are homothetic, where

$$V_p(K,L) := \frac{p}{n}\lim_{\varepsilon\to 0^+}\frac{V(K+_p\varepsilon\cdot L)-V(K)}{\varepsilon}, \tag{19}$$

and the L_p-mixed volume has the integral representation (see [23])

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p h(K,u)^{1-p}\,dS(K,u). \tag{20}$$

In particular, in the L_p-Brunn–Minkowski theory, $S(K,\cdot)$ is replaced by the p-surface area measure $S_p(K,\cdot)$ given by

$$dS_p(K,\cdot) = h(K,u)^{1-p}dS(K,u).$$

2.3. Mixed p-Quermassintegrals

The mixed quermassintegrals are, of course, the first variation of the ordinary quermassintegrals with respect to the Minkowski addition. The p-mixed quermassintegrals $W_{p,0}(K,L), W_{p,1}(K,L)\dots W_{p,n-1}(K,L)$, are the first variation of the ordinary quermassintegrals with respect to the Firey addition; i.e., for $K, L \in \mathcal{K}_o^n$ and real $p \geq 1$, we have (see e.g. [24])

$$W_{p,i}(K,L) = \frac{p}{n-i}\lim_{\varepsilon\to 0^+}\frac{W_i(K+_p\varepsilon\cdot L)-W_i(K)}{\varepsilon}. \tag{21}$$

For $p \geq 1$, $0 \leq i < n$, and each $K \in \mathcal{K}_o^n$, there exists a regular Borel measure $S_{p,i}(K;u)$ on S^{n-1} such that the p-mixed quermassintegral $W_{p,i}(K,L)$ has the integral representation

$$W_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p\,dS_{p,i}(K,u) \tag{22}$$

for all $L \in \mathcal{K}_o^n$. Obviously, for $p=1$, $W_{p,i}(K,L)$ becomes the well-known mixed quermassintegral $W_i(K,L)$ of K and L. The measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$ and has the Radon–Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)} = h(K,\cdot)^{1-p}. \tag{23}$$

The measure $S_{n-1}(K,\cdot)$ is independent of the body K, presenting just the ordinary Lebesgue measure $S(K,\cdot)$ on S^{n-1}. $S_i(B,\cdot)$ denotes the ith surface area measure of the unit ball in $\mathbb{R}^n$. In fact, $S_i(B,\cdot) = S$

for all i. The surface area measure $S_0(K,\cdot)$ will frequently be written simply as $S(K,\cdot)$. When $i=0$, $S_{p,i}(K,\cdot)$ is just the p-surface area measure $S_p(K,\cdot)$ (see [25]). Obviously, putting $i=0$ in (22), the mixed p-quermassintegral $W_{p,i}(K,L)$ becomes the L_p-mixed volume $V_p(K,L)$. The fundamental inequality for mixed p-quermassintegrals states that (see [24]) for $K,L\in\mathcal{K}_o^n$, $p\geq 1$, and $0\leq i<n$,

$$W_{p,i}(K,L)^{n-i}\geq W_i(K)^{n-i-p}W_i(L)^p, \tag{24}$$

with equality if and only if K and L are homothetic.

Obviously, when $p=1$, inequality (24) becomes the well-known Minkowski inequality for mixed quermassintegrals.

2.4. Mixed Projection Body

For $K\in\mathcal{K}^n$ and $u\in S^{n-1}$, let $v(K|u^\perp)$ denote the $(n-1)$-dimensional volume of $K|u^\perp$, the image of the orthogonal projection of K onto the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that is orthogonal to u. The projection body $\mathbf{\Pi}K\in\mathcal{K}_o^n$ of $K\in\mathcal{K}^n$ is the body whose support function is given by

$$h(\mathbf{\Pi}K,u)=v(K|u^\perp)=\frac{1}{2}\int_{S^{n-1}}|\langle u,u'\rangle|\,dS(K,u'), \tag{25}$$

where $u\in S^{n-1}$. Let $\mathbf{\Pi}(K_1,\,\dots\,,K_{n-1})$ be the mixed projection body of convex bodies $K_1,\,\dots\,,K_{n-1}$. It is defined by (see, e.g. [26])

$$h(\mathbf{\Pi}(K_1,\,\dots\,,K_{n-1}),u)=\frac{1}{2}\int_{S^{n-1}}|\langle u,u'\rangle|\,dS(K_1,\,\dots\,,K_{n-1};u'). \tag{26}$$

One of the fundamental inequalities for the mixed projection body is the following Aleksandrov–Fenchel inequality for mixed projection bodies: If $K_1,\,\dots\,,K_{n-1}$ are compact convex subsets and $1\leq r<n$, then (see [26])

$$V(\mathbf{\Pi}(K_1,\,\dots\,,K_{n-1}))\geq\prod_{j=1}^{r}V(\mathbf{\Pi}(K_j,\,\dots\,,K_j,K_{r+1},\,\dots\,,K_{n-1})^{1/r}. \tag{27}$$

If $K_1,\,\dots\,,K_{n-1}\in\mathcal{K}^n$, then the mixed projection body of convex bodies $K_1,\,\dots\,,K_{n-1}$ is denoted by $\mathbf{\Pi}(K_1,\,\dots\,,K_{n-1})$, and its support function is given for $u\in S^{n-1}$ as (see [26])

$$h(\mathbf{\Pi}(K_1,\,\dots\,,K_{n-1}),u)=v(K_1|u^\perp,\,\dots\,,K_{n-1}|u^\perp), \tag{28}$$

where $v(K_1|u^\perp,\,\dots\,,K_{n-1}|u^\perp)$ is the $(n-1)$-dimensional mixed volume of $K_1|u^\perp,\,\dots\,,K_{n-1}|u^\perp$.

3. L_p-Multiple Mixed Volume

Let us introduce the L_p-multiple mixed volume of $n+1$ convex bodies $K_1,\,\dots\,,K_n$ and L_n.

Definition 3.1. (see [16]) For $K_1,\,\dots\,,K_n,L_n\in\mathcal{K}_o^n$, and $p\geq 1$, the L_p-multiple mixed volume, denoted by $V_p(K_1,\cdots,K_n,L_n)$, is defined by

$$V_p(K_1,\cdots,K_n,L_n):=\frac{1}{n}\int_{S^{n-1}}\left(\frac{h(K_n,u)}{h(L_n,u)}\right)^p h(L_n,u)\,dS(K_1,\,\dots\,,K_{n-1};u). \tag{29}$$

Zhao [16] shown also that the L_p-multiple mixed volume has the limit representation; i.e.,

$$V_p(K_1,\cdots,K_n,L_n)=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}V(K_1,\cdots,K_{n-1},L_n+_p\varepsilon\cdot K_n). \tag{30}$$

Obviously, the classical mixed volume $V(K_1,\,\dots\,,K_n)$ of $K_1,\,\dots\,,K_n$, the L_p-mixed volume $V_p(K,L)$ of convex bodies K and L, and the p-mixed quermassintegral $W_{p,i}(K,L)$ of K and L are all special cases of the L_p-multiple mixed volume $V_p(K_1,\,\dots\,,K_{n-1},Q,K_n)$.

The fundamental inequality for L_p-multiple mixed volume of $K_1,\cdots,K_n,L_n$ is the following L_p-Aleksandrov–Fenchel inequality for L_p-multiple mixed volume: If $K_1,\cdots,K_n,L_n\in\mathcal{K}_o^n$, $1\le r\le n$, and $p\ge 1$, then (see [16])

$$V_p(K_1,\cdots,K_n,L_n)\ge\frac{\prod_{i=1}^{r}V(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{p/r}}{V(K_1,\cdots,K_{n-1},L_n)^{p-1}}. \tag{31}$$

The classical Aleksandrov–Fenchel inequality of $K_1,\cdots,K_n$ is an important special case of the L_p-Aleksandrov–Fenchel inequality. The Minkowski inequality (24) for mixed p-quermassintegrals is also a special case of the L_p-Aleksandrov–Fenchel inequality.

Lemma 1. *If $K_1,\ldots,K_n\in\mathcal{K}_o^n$ and $p\ge 1$, then, for $A\in\mathrm{SL}(n)$, we have*

$$V_p(AK_1,\cdots AK_{n-1},K_n,AL_n)=V_p(K_1,\cdots,K_{n-1},A^{-1}K_n,L_n). \tag{32}$$

Proof. From (12) and (29), we obtain

$$\begin{aligned}V_p(AK_1,\cdots AK_{n-1},K_n,AL_n)&=\frac{1}{n}\int_{S^{n-1}}\left(\frac{h(K_n,u)}{h(AL_n,u)}\right)^p h(AL_n,u)\,dS(AK_1,\ldots,AK_{n-1};u)\\&=\frac{1}{n}\int_{S^{n-1}}\left(\frac{h(K_n,u)}{h(L_n,A^tu)}\right)^p h(L_n,A^tu)\,dS(K_1,\ldots,K_{n-1};A^tu)\\&=\frac{1}{n}\int_{S^{n-1}}\left(\frac{h(K_n,A^{-t}u)}{h(L_n,u)}\right)^p h(L_n,u)\,dS(K_1,\ldots,K_{n-1};u)\\&=\frac{1}{n}\int_{S^{n-1}}\left(\frac{h(A^{-1}K_n,u)}{h(L_n,u)}\right)^p h(L_n,u)\,dS(K_1,\ldots,K_{n-1};u)\\&=V_p(K_1,\cdots,K_{n-1},A^{-1}K_n,L_n).\end{aligned}$$

This completes the proof. □

Lemma 2. *Let $K_{i1},\ldots,K_{in},L_{in}\in\mathcal{K}_o^n$ and $p\ge 1$. If $K_{ij}\to K_{0j}$, $j=1,\ldots,n$, and $L_{in}\to L_{0n}$, then*

$$V_p(K_{i1},\cdots,K_{in},L_{in})\to V_p(K_{01},\cdots,K_{0n},L_{0n}) \tag{33}$$

as $i\to\infty$.

Proof. To see this, let $K_{ij}\in\mathcal{K}_o^n$, $i\in\mathbb{N}\cup\{0\}$, $j=1,\ldots,n$, be such that $K_{ij}\to K_{0j}$ as $i\to\infty$ and $L_{in}\to L_{0n}$ as $i\to\infty$. The mixed area measure is weakly continuous; i.e.,

$$dS(K_{i1},\ldots,K_{i(n-1)};u)\to dS(K_{01},\ldots,K_{0(n-1)};u)\qquad\text{weakly on } S^{n-1}.$$

Since $h(K_{in},u)\to h(K_{0n},u)$ and $h(L_{in},u)\to h(L_{0n},u)$ uniformly on S^{n-1}, it follows that, for $p\ge 1$,

$$\left(\frac{h(K_{in},u)}{h(L_{in},u)}\right)^p\to\left(\frac{h(K_{0n},u)}{h(L_{0n},u)}\right)^p.$$

Further,

$$\begin{aligned}&\int_{S^{n-1}}\left(\frac{h(L_{in},u)}{h(K_{in},u)}\right)^p h(K_{in},u)\,dS(K_{i1},\ldots,K_{i(n-1)};u)\\&\quad\to\int_{S^{n-1}}\left(\frac{h(L_{0n},u)}{h(K_{0n},u)}\right)^p h(K_{0n},u)\,dS(K_{01},\ldots,K_{0(n-1)};u).\end{aligned}$$

Hence

$$\lim_{i\to\infty}V_p(K_{i1},\cdots,K_{in},L_{in})=V_p(K_{01},\cdots,K_{0n},L_{0n}).$$

This shows that $V_p(K_1,\cdots,K_n,L_n)$ is continuous. □

Lemma 3. *Let $K_1, \ldots, K_n, L_n \in \mathcal{K}_o^n$ and $p_i, p \geq 1$. If $p_i \to p$, then*

$$V_{p_i}(K_1, \cdots, K_n, L_n) \to V_p(K_1, \cdots, K_n, L_n). \tag{34}$$

Proof. Note that $p_i \to p$ implies that

$$\left(\frac{h(L_n, u)}{h(K_n, u)}\right)^{p_i} \to \left(\frac{h(L_n, u)}{h(K_n, u)}\right)^{p}.$$

Further,

$$\int_{S^{n-1}} \left(\frac{h(L_n, u)}{h(K_n, u)}\right)^{p_i} h(K_n, u)\, dS(K_1, \ldots, K_{n-1}; u)$$

$$\to \int_{S^{n-1}} \left(\frac{h(L_n, u)}{h(K_n, u)}\right)^{p} h(K_n, u)\, dS(K_1, \ldots, K_{n-1}; u).$$

Hence

$$\lim_{i\to\infty} V_{p_i}(K_1, \cdots, K_n, L_n) = V_p(K_1, \cdots, K_n, L_n).$$

This completes the proof. □

Lemma 4 (see [16]). *If $K_1, \ldots, K_n, L_n \in \mathcal{K}_o^n$ and $p \geq 1$, then, for $A \in \mathrm{SL}(n)$, we have*

$$V_p(AK_1, \cdots, AK_n, AL_n) = V_p(K_1, \cdots, K_n, L_n). \tag{35}$$

4. L_p-Mixed Geominimal Surface Area

Definition 4.1. For $p \geq 1$ and $K_1, \cdots, K_n \in \mathcal{K}_o^n$, the L_p-mixed geominimal surface area of convex bodies $K_1, \cdots, K_n$, denoted by $G_p(K_1, \cdots, K_n)$, is defined by

$$\omega_n^{p/n} G_p(K_1, \cdots, K_n) := \inf\{nV_p(K_1, \ldots, K_{n-1}, Q, K_n) V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\}. \tag{36}$$

If $p = 1$, then $G_p(K_1, \ldots, K_n)$ will be written $G(K_1, \ldots, K_n)$ and called the mixed geominimal surface area of $K_1, \ldots, K_n$, and

$$\omega_n^{1/n} G(K_1, \ldots, K_n) = \inf\{nV_1(K_1, \ldots, K_{n-1}, Q, K_n) V(Q^*)^{1/n} : Q \in \mathcal{K}_o^n\}. \tag{37}$$

This is exactly another generalization of mixed angle of Petty's geominimal surface area.

If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$, and $K_n = K$, then $G_p(K_1, \ldots, K_n)$ will be written $G_{p,i}(K)$ and called the ith L_p-mixed geominimal surface area of K, and

$$\omega_n^{1/n} G_{p,i}(K) = \inf\{nW_{p,i}(K, Q) V(Q^*)^{1/n} : Q \in \mathcal{K}_o^n\}. \tag{38}$$

In case $i = 0$ in (38), $G_{p,i}(K)$ becomes the p-geominimal surface area $G_p(K)$. If $p = 1$, then $G_{p,i}(K)$ will be written $G_i(K)$ and called the ith geominimal surface area of K, and

$$\omega_n^{1/n} G_i(K) = \inf\{nW_i(K, Q) V(Q^*)^{1/n} : Q \in \mathcal{K}_o^n\}.$$

Lemma 5. *If $K_1, \ldots, K_n \in \mathcal{K}_o^n$, $p \geq 1$, and $A \in \mathrm{SL}(n)$, then*

$$G_p(AK_1, \cdots, AK_n) = G_p(K_1, \cdots, K_n). \tag{39}$$

Proof. From (13), (29) and Lemma 1, we have

$$\begin{aligned}
\omega_n^{p/n} G_p(AK_1, \cdots, AK_n) &= \inf\{nV_p(AK_1, \ldots, AK_{n-1}, Q, AK_n) \cdot V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\} \\
&= \inf\{nV_p(K_1, \ldots, K_{n-1}, A^{-1}Q, K_n) \cdot V((A^{-1}Q)^*)^{p/n} : A^{-1}Q \in \mathcal{K}_o^n\} \\
&= \omega_n^{p/n} G_p(K_1, \cdots, K_n).
\end{aligned}$$

This completes the proof. □

Lemma 6 ([3]). *Let $K_i \in \mathcal{K}_o^n$ and $K_i \to L \in \mathcal{S}^n$. If the sequence $V(K_i^*)$ is bounded, then $L \in \mathcal{K}_o^n$.*

Lutwak [3] introduced the L_p-compact convex set P_pK whose support function for $x \in \mathbb{R}^n$ is given by

$$h(P_pK,x)^p = \frac{1}{n}\int_{S^{n-1}} \frac{1}{2^p}(|\langle x,u\rangle| + \langle x,u\rangle)^p\, dS_p(K,u), \tag{40}$$

where $K \in \mathcal{K}_o^n$ and $p \geq 1$. Since $h(P_pK,\cdot) : \mathbb{R}^n \to [0,\infty)$ is convex, it is the support function of a compact convex set. Lutwak was the first to give a lower bound estimate of $h(P_pK,\cdot)$ and prove the following important result for L_p-mixed geominimal surface area $G_p(K)$: If $K \in \mathcal{K}_o^n$, $p \geq 1$, then there exists a unique body $\bar{K} \in \mathcal{K}_o^n$ such that

$$G_p(K) = nV_p(K,\bar{K}) \qquad \text{and} \qquad V(\bar{K}^*) = \omega_n.$$

For the L_p-mixed geominimal surface area $G_p(K_1, \ldots, K_n)$, the introduction of such a compact convex set is difficult but can also be done. However, here we will prove the following result on the L_p-mixed geominimal surface area not introducing the compact convex set but using a new technique.

Theorem 1. *If $K_1, \ldots, K_n \in \mathcal{K}_o^n$, $p \geq 1$, then there exists a unique body $\bar{K} \in \mathcal{K}_o^n$ such that*

$$G_p(K_1,\cdots,K_n) = nV_p(K_1,\cdots,K_{n-1},\bar{K},K_n) \qquad \textit{and} \qquad V(\bar{K}^*) = \omega_n. \tag{41}$$

Proof. From (36), there exists a sequence $M_i \in \mathcal{K}_o^n$ such that $V(M_i^*) = \omega_n$ with

$$V_p(K_1,\cdots,K_{n-1},B,K_n) \geq V_p(K_1,\cdots,K_{n-1},M_i,K_n)$$

for all i, and

$$nV_p(K_1,\cdots,K_{n-1},M_i,K_n) \to G_p(K_1,\cdots,K_n).$$

Suppose that $R_i = R(M_i) = \rho(M_i,u_i) = \max\{\rho(M_i,u) : u \in S^{n-1}\}$. The convex set

$$e_i = \{\lambda u_i : 0 \leq \lambda \leq R_i\} \subseteq M_i, \qquad \text{where} \quad u_i \in S^{n-1},$$

is such that $\rho(M_i,u_i) = R_i$, whence

$$h(e_i,u) = \frac{1}{2}R_i(|\langle u_i,u\rangle| + \langle u_i,u\rangle).$$

From the well-known Jensen's inequality, and choosing c such that $h(\mathbf{\Pi}(K_1,\cdots,K_{n-1}),u) \geq c > 0$ on S^{n-1}, we obtain

$$\begin{aligned} V_p(K_1,\cdots,K_{n-1},B,K_n) &\geq V_p(K_1,\cdots,K_{n-1},M_i,K_n) \\ &= \frac{1}{n}\int_{S^{n-1}} \left(\frac{h(M_i,u)}{h(K_n,u)}\right)^p h(K_n,u)\, dS(K_1,\ldots,K_{n-1};u) \\ &\geq V(K_1,\cdots,K_n)^{1-p}\left(\frac{1}{n}\int_{S^{n-1}} h(M_i,u)\, dS(K_1,\cdots,K_{n-1};u)\right)^p \\ &\geq V(K_1,\cdots,K_n)^{1-p}\left(\frac{1}{n}\int_{S^{n-1}} h(e_i,u)\, dS(K_1,\cdots,K_{n-1};u)\right)^p \\ &= V(K_1,\cdots,K_n)^{1-p} \\ &\quad\times \left(\frac{R_i}{2n}\int_{S^{n-1}} (|\langle u_i,u\rangle| + \langle u_i,u\rangle)\, dS(K_1,\cdots,K_{n-1};u)\right)^p. \end{aligned}$$

Note that the mixed area measure, when considered as defining a mass distribution on the sphere, always has centroid at the origin (see [15, p. 281]),

$$\int_{S^{n-1}} u\, dS(K_1,\cdots,K_{n-1};u) = o.$$

Hence

$$V_p(K_1, \cdots, K_{n-1}, B, K_n) \geq V(K_1, \cdots, K_n)^{1-p} \cdot \left(\frac{R_i \cdot v(K_1|u^\perp, \cdots, K_{n-1}|u^\perp)}{n}\right)^p$$
$$\geq n^{-p} c^p V(K_1, \cdots, K_n)^{1-p} R_i^p.$$

Noting that M_i are uniformly bounded, the Blaschke selection theorem guarantees the existence of a subsequence of the M_i, which will also denote by M_i, and a compact convex $L \in \mathcal{S}^n$ such that $M_i \to L$. In view of $V(M_i^*) = \omega_n$, Lemma 6 gives $L \in \mathcal{K}_o^n$. Hence $M_i \to L$ implies that $M_i^* \to L^*$, and since $V(M_i^*) = \omega_n$, it follows that $V(L^*) = \omega_n$. Lemma 2 can now be used to conclude that L will serve as the desired body $\bar{K}$.

The uniqueness of the minimizing body is proved as follows: let $L_1, L_2 \in \mathcal{K}_o^n$ satisfy $V(L_1^*) = V(L_2^*) = \omega_n$, and let

$$V_p(K_1, \cdots, K_{n-1}, L_1, K_n) V(L_1^*)^{p/n} = \inf\{V_p(K_1, \cdots, K_{n-1}, Q, K_n) V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\}$$
$$= V_p(K_1, \cdots, K_{n-1}, L_2, K_n) V(L_2^*)^{p/n}.$$

Let $L \in \mathcal{K}_o^n$ be defined by

$$L = \frac{1}{2} \cdot L_1 +_p \frac{1}{2} \cdot L_2.$$

Using (9) and (29) and noticing that φ is convex and strictly increasing on $[0, \infty)$, we have

$$\begin{aligned}
V_p(K_1, \cdots, K_{n-1}, L, K_n) &= \frac{1}{n} \int_{S^{n-1}} \left(\frac{h(L,u)}{h(K_n,u)}\right)^p h(K_n,u)\, dS(K_1, \cdots, K_{n-1}; u) \\
&= \frac{1}{2n} \int_{S^{n-1}} \left(\frac{h(L_1,u)}{h(K_n,u)}\right)^p h(K_n,u)\, dS(K_1, \cdots, K_{n-1}; u) \\
&\quad + \frac{1}{2n} \int_{S^{n-1}} \left(\frac{h(L_2,u)}{h(K_n,u)}\right)^p h(K_n,u)\, dS(K_1, \cdots, K_{n-1}; u) \\
&= \frac{1}{2} V_p(K_1, \cdots, K_{n-1}, L_1, K_n) + \frac{1}{2} V_p(K_1, \cdots, K_{n-1}, L_2, K_n) \\
&= V_p(K_1, \cdots, K_{n-1}, L_1, K_n) \\
&= V_p(K_1, \cdots, K_{n-1}, L_2, K_n).
\end{aligned}$$

If $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \geq 0$ (not both zero), then, for $p \geq 1$, the harmonic p-combination $\lambda \circ K \widehat{+}_p \mu \circ L \in \mathcal{S}_o^n$ is defined as (see [10]–[11])

$$\rho(\lambda \circ K \widehat{+}_p \mu \circ L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}.$$

Hence

$$L^* = \frac{1}{2} \circ L_1^* \widehat{+}_p \frac{1}{2} \circ L_2^*,$$

and $V(L_1^*) = \omega_n = V(L_2^*)$.

Suppose that $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \geq 0$. If $p \geq 1$, then (see [3])

$$V(\lambda \circ K \widehat{+}_p \mu \circ L)^{-p/n} \geq \lambda V(K)^{-p/n} + \mu V(L)^{-p/n},$$

with equality if and only if K and L are dilates. This yields that

$$V(L^*) \leq \omega_n,$$

with equality if and only if $L_1 = L_2$. Thus,

$$V_p(K_1, \cdots, K_{n-1}, L, K_n) V(L^*)^{p/n} < V_p(K_1, \cdots, K_{n-1}, L_1, K_n) V(L_1^*)^{p/n}.$$

This is a contradiction if $L_1 \neq L_2$. ☐

The unique body whose existence is guaranteed by Theorem 1 will be denoted by $T_p(K_1, \dots, K_n)$ and called the L_p-mixed Petty body of $K_1, \dots, K_n$. Thus, for $K_1, \dots, K_n \in \mathcal{K}_o^n$ and $p \geq 1$, the body $T_p(K_1, \dots, K_n)$ is defined so as

$$G_p(K_1, \dots, K_n) = nV_p(K_1, \cdots, K_{n-1}, T_p(K_1, \dots, K_n), K_n), V(T_p^*(K_1, \dots, K_n)) = \omega_n,$$

where $T_p^*(K_1, \dots, K_n)$ denotes the polar body of $T_p(K_1, \dots, K_n)$. That is,

$$T_p(K_1, \dots, K_n) := \{\bar{K} \in \mathcal{K}_o^n : G_p(K_1, \dots, K_n) = nV_p(K_1, \dots, K_{n-1}, \bar{K}, K_n), V(\bar{K}^*) = \omega_n\}. \tag{42}$$

Lemma 7. *If $K_1, \dots, K_n \in \mathcal{K}_o^n$, $p \geq 1$, then, for $A \in \mathrm{SL}(n)$,*

$$T_p(AK_1, \cdots, AK_n) = T_p(K_1, \cdots, K_n). \tag{43}$$

Proof. Let $\bar{K} \in T_p(K_1, \cdots, K_n)$; from the definition of $T_p(K_1, \dots, K_n)$, Lemma 4, Lemma 5 and Theorem 1, we have

$$\begin{aligned} G_p(AK_1, \cdots, AK_n) &= G_p(K_1, \cdots, K_n) \\ &= nV_p(K_1, \dots, K_{n-1}, \bar{K}, K_n) \\ &= nV_p(AK_1, \dots, AK_{n-1}, A\bar{K}, AK_n). \end{aligned}$$

Observe that

$$V((A\bar{K})^*) = V(\bar{K}^*) = \omega_n.$$

So

$$A\bar{K} \in T_p(AK_1, \cdots, AK_n).$$

On the other hand, let $\bar{K} \in T_p(AK_1, \cdots, AK_n)$; from Lemma 1, Lemma 5, and Theorem 1, we obtain

$$\begin{aligned} G_p(K_1, \cdots, K_n) &= G_p(AK_1, \cdots, AK_n) \\ &= nV_p(AK_1, \dots, AK_{n-1}, \bar{K}, AK_n) \\ &= nV_p(K_1, \dots, K_{n-1}, A^{-1}\bar{K}, K_n). \end{aligned}$$

Note that

$$V((A^{-1}\bar{K})^*) = V(\bar{K}^*) = \omega_n.$$

Hence

$$A^{-1}\bar{K} \in T_p(K_1, \cdots, K_n).$$

This completes the proof. □

In much the same way as before, here we do not introduce the compact convex set P_pK and do not estimate the relevant lower boundary either. By using a new technique, we will prove the following bound on the size of L_p-mixed geominimal surface area $T_p(K_1, \dots, K_n)$.

Lemma 8. *Suppose that $p \geq 1$ and $K_1, \dots, K_n \in \mathcal{K}_o^n$. If $r, R > 0$ are such that*

$$rB \subset K_i \subset RB, \qquad i = 1, 2, \dots, n,$$

then, for all $u \in S^{n-1}$,

$$h(T_p(K_1, \dots, K_n), u) \leq \frac{n\omega_n}{\omega_{n-1}} (R/r)^n. \tag{44}$$

Proof. It follows from (29) that

$$V_p(K_1, \cdots, K_{n-1}, B, K_n) \le r^{-p} V(K_1, \cdots, K_n) \le r^{-p}\omega_n R^n. \tag{45}$$

Let $\bar{K} = T_p(K_1, \ldots, K_n)$. Then

$$V_p(K_1, \cdots, K_{n-1}, \bar{K}, K_n) \le V_p(K_1, \cdots, K_{n-1}, B, K_n). \tag{46}$$

Let u_0 be a point in S^{n-1} such that

$$\rho(\bar{K}, u_0) = \max\{\rho(\bar{K}, u) : u \in S^{n-1}\} = R(\bar{K}).$$

Since the support function of $\bar{K}$ majorizes that of the convex set $e_0 = \{\lambda u_0 : 0 \le \lambda \le R(\bar{K})\} \subset \bar{K}$, from the proof of Theorem 1, it follows that

$$V_p(K_1, \cdots, K_{n-1}, \bar{K}, K_n) \ge V(K_1, \cdots, K_n)^{1-p} \cdot \left(\frac{R(\bar{K}) \cdot v(K_1|u_0^{\perp}, \cdots, K_{n-1}|u_0^{\perp})}{n} \right)^p.$$

Hence

$$r^n \omega_n \le V(K_1, \cdots, K_n) \le R^n \omega_n \qquad \text{and} \qquad v(K_1|u_0^{\perp}, \cdots, K_{n-1}|u_0^{\perp}) \ge r^{n-1}\omega_{n-1}.$$

So

$$V_p(K_1, \cdots, K_{n-1}, \bar{K}, K_n) \ge \frac{R(\bar{K})^p (R^n \omega_n)^{1-p} (r^{n-1}\omega_{n-1})^p}{n^p}. \tag{47}$$

From (45), (46) and (47), (44), the desired result easily follows.

This completes the proof. □

Lemma 9. *Let $p \ge 1$. If $K_{ij} \in \mathcal{K}_o^n$ $(j = 1, 2, \ldots, n)$ is a family of bodies for which there exist $r, R > 0$ such that*

$$rB \subset K_{i1}, \cdots, K_{in} \subset RB \qquad \textit{for all} \quad i,$$

then there exist $r', R' > 0$ such that

$$r'B \subset T_p(K_{i1}, \cdots, K_{in}) \subset R'B \qquad \textit{for all} \quad i. \tag{48}$$

Proof. Let $\bar{K}_i = T_p(K_{i1}, \cdots, K_{in})$. The existence of $R' > 0$, implying that the $\bar{K}_i$ are uniformly bounded, is contained in Lemma 8. Let $r_i = r(\bar{K}_i)$ denote the inner radius of $\bar{K}_i$. Thus,

$$r_i = \min_{u \in S^{n-1}} h(\bar{K}_i, u) = h(\bar{K}_i, u_i),$$

where $u_i \in S^{n-1}$ is any point where this minimum is attained. Suppose that $\inf\{r_i\} = 0$. Thus, there exists a subsequence of the $\bar{K}_i$, which will not be relabeled, such that

$$h(\bar{K}_i, u_i) \to 0.$$

The Blaschke selection theorem, in conjunction with Lemma 6, demonstrates the existence of $M \in \mathcal{K}_o^n$ such that, for a subsequence of the $\bar{K}_i$, which will also not be relabeled,

$$\bar{K}_i \to M.$$

But $h(\bar{K}_i, u_i) \to 0$, and $\max |h_{K_i} - h_M| \to 0$, so that $h(M, u_i) \to 0$, which is impossible, because the continuous function h_M is positive.

This completes the proof. □

Theorem 2. *If $p \ge 1$, then $G_p : \underbrace{\mathcal{K}_o^n \times \cdots \times \mathcal{K}_o^n}_{n} \to (0, \infty)$ is continuous.*

Proof. First, we show that G_p is upper semicontinuous. If $p \geq 1$ and $K_{ij} \in \mathcal{K}_o^n$, $j = 1, 2, \ldots, n$, such that $K_{ij} \to K_{0j} \in \mathcal{K}_o^n$, then $S(K_{i1}, \ldots, K_{in}; \cdot) \to S(K_{01}, \ldots, K_{0n}; \cdot)$, weakly on S^{n-1}. Hence for $L \in \mathcal{S}_o^n$ we have

$$V_p(\cdot, \ldots, \cdot, L^*, \cdot) : \underbrace{\mathcal{K}_o^n \times \cdots \times \mathcal{K}_o^n}_{n} \to (0, \infty) \quad \text{is continuous.}$$

Therefore, the L_p-mixed geominimal surface area $\omega_n^{p/n} G_p : \underbrace{\mathcal{K}_o^n \times \cdots \times \mathcal{K}_o^n}_{n} \to (0, \infty)$ is defined by the infimum of the continuous functions $nV_p(K_1, \ldots, K_{n-1}, Q^*, \cdot)V(Q)^{p/n} : \underbrace{\mathcal{K}_o^n \times \cdots \times \mathcal{K}_o^n}_{n} \to (0, \infty)$ as Q ranges over $\mathcal{K}_o^n$.

To see that G_p is lower semicontinuous at $K_{01}, \ldots, K_{0n} \in \mathcal{K}_o^n$, let $K_{ij} \in \mathcal{K}_o^n$ be a sequence of bodies such that $K_{ij} \to K_{0j}$ $(j = 1, 2, \ldots, n)$ with $G_p(K_{i1}, \ldots, K_{in}) \to l \in \mathbb{R}$. We will show that $l \geq G_p(K_{01}, \ldots, K_{0n})$, and thus

$$\lim_{k\to\infty} \inf_{i>k} G_p(K_{i1}, \ldots, K_{in}) \geq G_p(K_{01}, \ldots, K_{0n}).$$

By Lemma 9, the $\bar{K}_i = T_p(K_{i1}, \ldots, K_{in})$ are uniformly bounded. The Blaschke selection theorem, in conjunction with Lemma 6, yields the existence of a body $M \in K_o^n$ and a subsequence of the $\bar{K}_i$, which will not be relabeled, such that $\bar{K}_i \to M$ and $V(M^*) = \omega_n$. By Lemma 2 and the facts that $K_{ij} \to K_{0j}$ and $\bar{K}_i \to M$, we have

$$G_p(K_{i1}, \ldots, K_{in}) = nV_p(K_{i1}, \ldots, K_{i,n-1}, \bar{K}_i, K_{in}) \to nV_p(K_{01}, \ldots, K_{0,n-1}, M, K_{0n}).$$

Since $G_p(K_{i1}, \ldots, K_{in}) \to l$, we have $nV_p(K_{01}, \ldots, K_{0,n-1}, M, K_{0n}) = l$. But it follows from the definition of $G_p(K_{01}, \ldots, K_{0n})$ that

$$\omega_n^{p/n} l = nV_p(K_{01}, \ldots, K_{0,n-1}, M, K_{0n})V(M^*)^{p/n} \geq \omega_n^{p/n} G_p(K_{01}, \ldots, K_{0n}).$$

This completes the proof. □

Lemma 10. *If $p \geq 1$, then $T_p : \underbrace{\mathcal{K}_o^n \times \cdots \times \mathcal{K}_o^n}_{n} \to (0, \infty)$ is continuous.*

Proof. Suppose that $K_{ij} \in \mathcal{K}_o^n$ $(j = 1, 2, \ldots, n)$ such that $K_{ij} \to K_{0j}$. Let $\bar{K}_i = T_p(K_{i1}, \ldots, K_{in})$ denote a subsequence of $\bar{K}_i$. Lemma 9 shows that the $\bar{K}_i$ are uniformly bounded. The Blaschke selection theorem, in conjunction with Lemma 6, yields the existence of a body $M \in \mathcal{K}_o^n$ and a subsequence of the $\bar{K}_i$, which will not be relabeled, such that $\bar{K}_i \to M$ and $V(M^*) = \omega_n$. Lemma 2 and the fact that $K_{ij} \to K_{0j}$ and $\bar{K}_i \to M$ may be used to conclude that

$$G_p(K_{i1}, \ldots, K_{in}) = nV_p(K_{i1}, \ldots, K_{i,n-1}, \bar{K}_i, K_{in}) \to nV_p(K_{01}, \ldots, K_{0,n-1}, M, K_{0n}).$$

But, by Theorem 2,

$$G_p(K_{i1}, \ldots, K_{in}) \to G_p(K_{01}, \ldots, K_{0n}).$$

Hence

$$G_p(K_{01}, \ldots, K_{0n}) = nV_p(K_{01}, \ldots, K_{0,n-1}, M, K_{0n}),$$

and the uniqueness part of Theorem 1 shows that $\bar{K}_0 = M$.

Hence every subsequence of $\bar{K}_i$ has a subsequence converging to $\bar{K}_0$.

This completes the proof. □

Petty [1] called a body $K \in \mathcal{K}^n$ *self-minimal* if TK and K are homothetic. Lutwak [3] called a body $K \in \mathcal{K}_o^n$ *p-self-minimal* if T_pK and K are dilates of each other, and showed that the class of p-self-minimal bodies is a centro-affine invariant class of bodies. In this article, for some bodies $K_1, \ldots, K_n \in \mathcal{K}_o^n$, we will say that $K_1, \ldots, K_{n-1}$ and K_n are mixed p-self-minimal if $T_p(K_1, \ldots, K_n)$ and K_n are dilates of each other.

Lemma 11. *If* $p \geq 1$, *and* $K_1, \dots, K_n \in \mathcal{K}_o^n$, *then*

$$\omega_n \left(\frac{V(K_n)^p G_p(K_1, \dots, K_n)^n}{n^n V(K_1, \dots, K_n)^n} \right)^{1/p} \leq V(K_n)V(K_n^*), \tag{49}$$

with equality if and only if $K_1, \dots, K_{n-1}$ *and* K_n *are mixed* p*-self-minimal.*

Proof. Putting $Q = K_n$ in the definition of L_p-mixed geominimal surface area, we have

$$\begin{aligned} \omega_n^{p/n} G_p(K_1, \cdots, K_n) &= \inf\{nV_p(K_1, \dots, K_{n-1}, Q, K_n)V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\} \\ &\leq nV_p(K_1, \dots, K_{n-1}, K_n, K_n)V(K_n^*)^{p/n} \\ &= nV(K_1, \dots, K_n)V(K_n^*)^{p/n}. \end{aligned}$$

To obtain the equality conditions, let $\bar{K} = T_p(K_1, \dots, K_n)$ and assume first that $K_1, \dots, K_n$ are mixed p-self-minimal. From $\bar{K} = T_p(K_1, \dots, K_n) = \delta K_n$, $\delta > 0$, we have

$$\begin{aligned} G_p(K_1, \cdots, K_n) &= nV_p(K_1, \dots, K_{n-1}, \bar{K}, K_n) \\ &= nV_p(K_1, \dots, K_{n-1}, \delta K_n, K_n) \\ &= n\delta^p V(K_1, \cdots, K_n). \end{aligned}$$

Moreover, $\bar{K} = \delta K_n$ implies that $\delta \bar{K}^* = K_n^*$. Since $V(\bar{K}^*) = \omega_n$, this yields $\delta = (V(K_n^*)/\omega_n)^{1/n}$. This shows that there is equality in the inequality.

Conversely, suppose that there is equality in the inequality

$$G_p(K_1, \cdots, K_n)^n = n^n \left(\frac{V(K_n^*)}{\omega_n} \right)^p V(K_1, \cdots, K_n)^n.$$

But

$$G_p(K_1, \cdots, K_n) = nV_p(K_1, \dots, K_{n-1}, \bar{K}, K_n).$$

Hence

$$\begin{aligned} V_p(K_1, \dots, K_{n-1}, \bar{K}, K_n)^n &= \left(\frac{V(K_n^*)}{\omega_n} \right)^p V(K_1, \cdots, K_n)^n \\ &= V_p\Big(K_1, \dots, K_{n-1}, [(V(K_n^*)/\omega_n)^{1/n} K_n], K_n\Big)^n. \end{aligned}$$

Since

$$V([(V(K_n^*)/\omega_n)^{1/n} K_n]^*) = \omega_n,$$

it follows from the uniqueness of $\bar{K}$ that

$$T_p(K_1, \dots, K_n) = (V(K_n^*)/\omega_n)^{1/n} K_n.$$

Thus, $K_1, \dots, K_{n-1}$ and K_n are mixed p-self-minimal.

This completes the proof. □

Theorem 3. *If* $p \geq 1$ *and* $K_1, \dots, K_n \in \mathcal{K}_o^n$, *then*

$$G_p(K_1, \dots, K_n)^n \leq n^n \omega_n^p V(K_1, \dots, K_n)^n / V(K_n)^p, \tag{50}$$

with equality if and only if $K_1, \dots, K_{n-1}$ *and* $K_n = E$ *are mixed* p*-self-minimal.*

Proof. The inequality is immediate from Lemma 11 and the Blaschke–Santaló inequality.

From the equalities of Lemma 11 and the Blaschke–Santaló inequality, it follows that equality in (50) holds if and only if $K_1, \dots, K_{n-1}$ and K_n are mixed p-self-minimal and K_n is an ellipsoid. This yields that the equality in (50) holds if and only if $K_1, \dots, K_{n-1}$ and an ellipsoid E are mixed p-self-minimal.

This completes the proof. □

Remark 1. When $K_1 = \cdots = K_n = K$, (50) becomes Lutwak's p-geominimal surface area inequality: If $p \geq 1$, and $K \in \mathcal{K}_o^n$, then

$$G_p(K)^n \leq n^n \omega_n^p V(K)^{n-p},$$

with equality if and only if K is an ellipsoid.

The following mixed geominimal surface area inequality of $K_1, \ldots, K_n$ is also derived.

The mixed geominimal surface area inequality. *If $K_1, \ldots, K_n \in \mathcal{K}_o^n$, then*

$$G(K_1, \ldots, K_n)^n \leq n^n \omega_n V(K_1, \ldots, K_n)^n / V(K_n), \tag{51}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ and $K_n = E$ are mixed p-self-minimal.

The following ith L_p-mixed geominimal surface area inequality is a special case of (50). If $p \geq 1$, $0 \leq i < n$, and $K \in \mathcal{K}_o^n$, then

$$G_{p,i}(K)^n \leq n^n \omega_n^p W_i(K)^n / V(K)^p, \tag{52}$$

with equality if and only if K is an ellipsoid.

The following ith mixed geominimal surface area inequality is also a special case of (52). If $0 \leq i < n$ and $K \in \mathcal{K}_o^n$, then

$$G_i(K)^n \leq n^n \omega_n W_i(K)^n / V(K), \tag{53}$$

with equality if and only if K is an ellipsoid.

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