ISSN 0001-4346, Mathematical Notes, 2022, Vol. 112, No. 5, pp. 697–708. © Pleiades Publishing, Ltd., 2022.
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Maslov Index on Symplectic Manifolds. With Supplement by A. T. Fomenko "Constructing the Generalized Maslov Class for the Total Space $W = \mathbb{T}^*(M)$ of the Cotangent Bundle"

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Received July 13, 2022; in final form, July 13, 2022; accepted July 15, 2022

Abstract—The geometric properties of the Maslov index on symplectic manifolds are discussed. The Maslov index is constructed as a homological invariant on a Lagrangian submanifold of a symplectic manifold. In the simplest case, a Lagrangian submanifold $\Lambda \subset \mathbb{R}^{2n} \approx \mathbb{R}^n \oplus \mathbb{R}^n$ is a submanifold in the symplectic space $\mathbb{R}^n \oplus \mathbb{R}^n$, in which the symplectic structure is given by the nondegenerate form $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ and $\Lambda \subset \mathbb{R}^{2n}$ is a submanifold, $\dim \Lambda = n$, on which the form ω is trivial. In the general case, a symplectic manifold (W, ω) and the bundle of Lagrangian Grassmannians $\mathcal{LG}(\mathbb{T}W)$ is considered. The question under study is as follows: when is the Maslov index, given on an individual Lagrangian manifold as a one-dimensional cohomology class, the image of a one-dimensional cohomology class of the total space $\mathcal{LG}(\mathbb{T}W)$ of bundles of Lagrangian Grassmannians? An answer is given for various classes of bundles of Lagrangian Grassmannians.

DOI: 10.1134/S0001434622110074

Keywords: *Maslov index, Maslov class, symplectic manifold, bundle of Lagrangian manifolds.*

Dedicated to Viktor Pavlovich Maslov, teacher, colleague, and friend, from a grateful disciple

1. STATEMENT OF THE PROBLEM

1.1. Consistent Structures

Let W be a symplectic manifold given by the symplectic structure ω, which is a (nondegenerate) symplectic form on the manifold W, $\dim W = 2n$, $d\omega = 0$. With a symplectic structure one can associate the following additional consistent structures:

- The Euclidean structure $E(u, v)$ on W, $u, v \in \Gamma(TW)$, $E(u, u) > 0$.
- The almost complex structure J: $J : TW \to TW$, $J^2 = -1$.
- The Hermitian structure $H(u, v) = E(u, v) + i\omega(u, v)$ on W,

$$H(Ju, v) = iH(u, v), \quad H(u, v) = \overline{H(v, u)}.$$

These structures have the following additional coordination between themselves:

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- $\omega(Ju, Jv) = \omega(u, v)$;
- $E(u, v) = \omega(u, Jv)$;
- $H(u, v) = E(u, v) + iE(u, Jv)$ is the Hermitian structure.

All these structures can be constructed starting from a given symplectic form ω. See, e.g., [1, Part V, Compatible Almost Complex Structures, p. 84].

1.2. Bundle of Lagrangian Grassmannians

By definition, the bundle of Lagrangian Grassmannians is constructed in the form of the total space $\mathcal{LG}(\mathbb{T}W)$ of the bundle

$$\begin{tikzcd} \mathcal{LG}(\mathbb{T}W) \arrow[d, "\pi_{LG}"] \\ W \end{tikzcd}$$

with fibers $\pi_{LG}^{-1}(x) = \mathcal{LG}(\mathbb{T}W)_x$ over points $x \in W$. The fiber $\mathcal{LG}(\mathbb{T}W)_x$ is the Lagrangian Grassmannian

$$\pi_{LG}^{-1}(x) = \mathcal{LG}(\mathbb{T}W)_x = \mathcal{LG}(\mathbb{T}_x W)$$

of the symplectic space $\mathbb{T}_x W$, which is a manifold consisting of all Lagrangian planes in the tangent space $\mathbb{T}_x W$:

$$\mathcal{LG}(\mathbb{T}_x W) = \{L \subset \mathbb{T}_x W : \dim_{\mathbb{R}} L = n,\, \omega_{|L} = 0\},$$

which are conveniently expressed as the diagram

$$\begin{tikzcd} \pi_{LG}^{-1}(x) \arrow[r, equal] \arrow[d] & \mathcal{LG}(\mathbb{T}_x W) \arrow[r, equal] \arrow[d] & \pi_{LG}^{-1}(x) \arrow[d] \\ \mathcal{LG}(\mathbb{T}W) \arrow[d, "\pi_{LG}"] & \mathcal{LG}(\mathbb{T}W)_x \arrow[l, hook'] \arrow[r, equal] \arrow[d] & \pi_{LG}^{-1}(x) \arrow[d] \\ W & x \arrow[l, hook'] \arrow[r, equal] & x. \end{tikzcd}$$

2. DEFINITION OF CHARACTERISTIC CLASSES OF LAGRANGIAN MANIFOLDS

The differential Dh generates the commutative diagram

$$\begin{tikzcd} \mathbb{T}\Lambda \arrow[r, "Dh"] \arrow[d, "\pi_{\mathbb{T}\Lambda}"'] & \mathbb{T}W \arrow[d, "\pi_{\mathbb{T}W}"] \\ \Lambda \arrow[r, hook, "h"] & W. \end{tikzcd}$$

A Lagrangian submanifold $h\colon \Lambda \to W$ is a submanifold for which $Dh(\mathbb{T}_x\Lambda) \subset \mathbb{T}_{h(x)}W$ is a Lagrangian plane. The differential Dh generates the fibered map Lh of the Lagrangian manifold into the total space $\mathcal{LG}(\mathbb{T}W)$ of the bundle of Lagrangian Grassmannians,

$$\begin{tikzcd} \Lambda \arrow[r, "Lh"] \arrow[dr, hook, bend right, "h"'] & \mathcal{LG}(\mathbb{T}W) \arrow[d, "\pi_{LG}"] \\ & W. \end{tikzcd}$$

We obtain the following map on cohomology:

$$H^*(\mathcal{LG}(\mathbb{T}W) \xrightarrow{(Lh)^*} H^*(\Lambda).$$

If $\alpha \in H^*(\mathcal{LG}(\mathbb{T}W)$, then the cohomology class

$$\alpha(\Lambda) = (Lh)^*(\alpha) \in H^*(\Lambda)$$

will be called the *characteristic class* of the Lagrangian submanifold

$$\Lambda \xhookrightarrow[h]{} W,$$

generated by the universal characteristic class $\alpha \in H^*(\mathcal{LG}(\mathbb{T}W)$.

2.1. Maslov Class for the Case of $W = \mathbb{T}^*(\mathbb{R}^n)$

There are at least three examples of characteristic classes of Lagrangian manifolds. One of them, the simplest, is the one-dimensional Maslov characteristic class whose value on a closed curve $\gamma \subset \Lambda$ coincides with the Maslov index of the curve γ (see the book by Trofimov and Fomenko [2, Sec. 63, Subsec. 3, p. 400]). The *Maslov class* is defined for Lagrangian manifolds in the symplectic space $W = \mathbb{R}^{2n}$,

$$\mathbb{R}^{2n} = \mathbb{T}^*(\mathbb{R}^n) = \mathbb{R}^n(p_k) \oplus \mathbb{R}^n(x^k) = \mathbb{C}^n(z^k), \qquad z^k = x^k + ip_k,$$

whose symplectic form ω is

$$\omega = \sum dp_k \wedge dx^k,$$

and let the complex coordinates (z^k) constitute the complex basis $(\tau_1, \dots, \tau_n)$ in the space $\mathbb{C}^n(z^k)$.

Let $\Lambda \subset \mathbb{R}^{2n}$ be a Lagrangian submanifold, let $h\colon \Lambda \subset \mathbb{R}^{2n}$ be an embedding, and let $Lh\colon \Lambda \to \mathcal{LG}(\mathbb{R}^{2n})$ take each point $x \in \Lambda$ to the tangent (Lagrangian) plane $Lh(x) = \mathbb{T}_x(\Lambda) \in \mathcal{LG}(\mathbb{R}^{2n})$ as a point in the Lagrangian Grassmannian $\mathcal{LG}(\mathbb{R}^{2n})$ of all Lagrangian planes. We obtain the following map on cohomology:

$$H^*(\mathcal{LG}(\mathbb{R}^{2n}) \xrightarrow{(Lh)^*} H^*(\Lambda).$$

If $\alpha \in H^*(\Lambda(n))$, then the cohomology class

$$\alpha(\Lambda) = (Lh)^*(\alpha) \in H^*(\Lambda)$$

is the characteristic class of the Lagrangian submanifold $\Lambda \subset \mathbb{R}^{2n}$. An example of a universal characteristic class (the Maslov class) is the generating element $\mathbf{M}^a \in H^1(\mathcal{LG}(\mathbb{R}^{2n})) \approx \mathbb{Z}$; i.e., $\mathbf{M}^a(\Lambda) \in H^1(\Lambda)$.

The Maslov class $\mathbf{M}^a \in H^1(\mathcal{LG}(\mathbb{R}^{2n})) \approx \mathbb{Z}$ can be computed using a differential form on the manifold $\mathcal{LG}(\mathbb{R}^{2n})$. The manifold $\mathcal{LG}(\mathbb{R}^{2n})$ is diffeomorphic to the homogeneous space $\mathbb{U}(n)/\mathbb{O}(n)$ by means of the diffeomorphism

$$u\colon \mathcal{LG}(\mathbb{R}^{2n}) \to \mathbb{U}(n)/\mathbb{O}(n)$$

that takes each Lagrangian plane

$$L \subset \mathbb{R}^{2n} = \mathbb{C}^n, \qquad \dim_R L = n,$$

to the orthonormal real basis $(e_1, \dots, e_n) \subset L$. The same basis is a complex basis in the space $\mathbb{R}^{2n} = \mathbb{C}^n$, because

$$H(e_k, e_l) = E(e_k, e_l) + i\omega(e_k, e_l) = \delta_{k,l}.$$

Therefore,

$$(e_1, \dots, e_n) = (\tau_1, \dots, \tau_n)U, \qquad U \in \mathbb{U}(n),$$

for a unitary matrix $U \in \mathbb{U}(n)$. The basis $(e_1, \dots, e_n) \subset L$ can be replaced by another orthonormal real basis

$$(e'_1, \dots, e'_n) = (e_1, \dots, e_n)O \subset L, \qquad O \in \mathbb{O}(n);$$

i.e.,

$$(e'_1, \dots, e'_n) = (\tau_1, \dots, \tau_n)UO, \qquad U \in \mathbb{U}(n), \quad O \in \mathbb{O}(n),$$

and the coset class $u(L) \in \mathbb{U}(n)/\mathbb{O}(n)$ is well defined.

The composition f of the maps

$$f \colon \mathcal{LG}(\mathbb{R}^{2n}) \xrightarrow{u} \mathbb{U}(n)/\mathbb{O}(n) \xrightarrow{\det^2} \mathbb{S}^1$$

defines a one-dimensional cohomology class, the Maslov class

$$\mathbf{M}^a \in H^1(\mathcal{LG}(\mathbb{R}^{2n})), \qquad \mathbf{M}^a = f^*\left(\frac{dz}{2\pi iz}\right) \in H^1(\mathcal{LG}(\mathbb{R}^{2n})).$$

2.2. Generalized Maslov Class for the Case of $W = \mathbb{T}^*(M)$

The construction of a generalized Maslov class is described in the book of Trofimov and Fomenko (1995) [2].

Let M be a smooth manifold, and let ω be a symplectic structure on the total space $\mathbb{T}^*M$. Each tangent space $\mathbb{T}_z(\mathbb{T}^*M)$, $z \in \mathbb{T}^*M$, is a symplectic vector space and in it we can take the Lagrangian subspace V_z tangent to the vertical, i.e., consisting of tangent vectors ξ such that $d\pi_z\xi = 0$, where the letter π denotes the standard projection $\pi\colon \mathbb{T}^*M \to M$. The choice of a Riemannian metric on M induces the positive definite inner product on V_z, which allows identifying $\mathcal{L}G(\mathbb{T}_z(\mathbb{T}^*M))$ with $\mathbb{U}(n)/\mathbb{O}(n)$. This identification is ambiguous but allows us to find the well-defined differential form $(\det^2)^*(dz/2\pi iz)$ on the total space of the bundle $\mathcal{L}G(\mathbb{T}^*M)$ over $\mathbb{T}^*M$, whose fiber above, the point $z \in \mathbb{T}^*M$, consists of all Lagrangian subspaces in $\mathbb{T}_z(\mathbb{T}^*M)$.

If N is a Lagrangian submanifold in $\mathbb{T}^*M$, then, for any curve γ on N, the curve on $\mathcal{L}G(\mathbb{T}^*M)$ is defined naturally. In this case, using the formula

$$l = \oint_\gamma (\det^2)^* \frac{dz}{2\pi iz},$$

we define an integer; thus, we obtain an element in $H^1(N;\mathbb{Z})$, which is called a *generalized Maslov class* of submanifolds N. This class does not depend on the choice of the Riemannian metric.

For an accurate proof of the well-posedness of the definition of the generalized Maslov class for the total space of the cotangent bundle of an arbitrary manifold, see the Supplement kindly provided by A. T. Fomenko.

2.3. Maslov–Trofimov Class. The Case of an Arbitrary Symplectic Manifold (W, ω)

Consider an arbitrary symplectic manifold (W, ω). Let ∇ be a connection consistent with the symplectic form ω (or an almost symplectic connection), i.e., a connection such that $\nabla\omega = 0$. It was shown in the book [2, Sec. 63, Sec. 9, p. 402] that, on a symplectic manifold (W, ω), $d\omega = 0$, there exists an almost symplectic connection ∇ with zero torsion. In this case, the almost symplectic connection ∇ is called a *symplectic connection*.

Theorem 1. *The connection ∇ can be chosen to be complex linear and preserving the Hermitian structure H. The connection ∇ is given by the Christoffel symbols*

$$2\Gamma_{j;ik} = \frac{\partial H_{ij}}{\partial e_k} + \frac{\partial H_{ki}}{\partial e_j} - \frac{\partial H_{jk}}{\partial e_i},$$

where $e_j \in \Gamma(\mathbb{T}(W))$ is the set of basic sections in the space $\Gamma(\mathbb{T}(W))$.

The parallel translation operation specifies a unitary linear transformation $ptr(\gamma)\colon \mathbb{T}_{x_0}(W) \to \mathbb{T}_{x_1}(W)$, $\gamma \in \Pi(x_0, x_1, W)$,

$$ptr\colon \Pi(x_0, x_1, W) \to \mathbf{U}(\mathbb{T}_{x_0}(W), \mathbb{T}_{x_1}(W)).$$

Let $\Pi(x_0, W)$ be the set of all closed paths with origin and end at a point $x_0 \in W$, $\Pi(x_0, W) = \Pi(x_0, x_0, W)$. The parallel translation operation along paths $\gamma \in \Pi(x_0, W)$ generates the group of unitary transformations of the tangent space $\mathbb{T}_{x_0}W$ of the manifold W at the point $x_0 \in W$:

$$ptr\colon \Pi(x_0, W) \to \mathbf{U}(\mathbb{T}_{x_0}(W)).$$

The image group $\mathbf{Im}(ptr) \subset \mathbf{U}(\mathbb{T}_{x_0}(W))$ is denoted by $\mathbf{Hol}_\nabla(x_0, W))$ and called the *holonomy group* of the connection ∇ on the manifold W. Since $\nabla\omega = 0$, we see that the holonomy group $\mathbf{\Pi}^h_\nabla(x_0, W)$ is naturally extended by the action on the Lagrangian Grassmannian $\mathcal{LG}(\mathbb{T}_xW)$,

$$\mathbf{Hol}_\nabla(x_0, W) \times \mathcal{LG}(\mathbb{T}_xW) \to \mathcal{LG}(\mathbb{T}_xW).$$

With this action, we can define the so-called reduced Lagrangian Grassmannian as the quotient space

$$\Pi\mathcal{LG}(\mathbb{T}_xW) = \mathcal{LG}(\mathbb{T}_xW)/\mathbf{Hol}_\nabla(x_0, W)$$

and hence also the map of the Lagrangian submanifold $N^n \subset W^{2n}$,

$$ptr\colon N^n \to \Pi\mathcal{LG}(\mathbb{T}_xW),$$

which defines a homomorphism on cohomology

$$(ptr)^*\colon H^*(\Pi\mathcal{LG}(\mathbb{T}_xW)) \to H^*(N^n),$$

i.e., the Maslov–Trofimov characteristic class

$$\alpha(N^n) = (ptr)^*(\alpha) \in H^*(N^n), \qquad \alpha \in H^*(\Pi\mathcal{LG}(\mathbb{T}_xW)).$$

3. DESCRIPTION OF BUNDLES OF LAGRANGIAN GRASSMANNIANS

Taking into account the existence of consistent structures, we assume that the manifold W is provided with an almost complex structure (W, J); i.e., the tangent bundle $\mathbb{T}W$ is a complex vector bundle with structure group $\mathbb{U}(n)$. The bundle of Lagrangian Grassmannians $\mathcal{LG}(\mathbb{T}W)$ is constructed as follows:

1) We take the principal bundle with structure group $\mathbb{U}(n)$ associated with the tangent bundle $\mathbb{T}W$. This bundle is denoted by

$$\begin{tikzcd} P_{\mathbb{U}(n)}(\mathbb{T}W) \arrow[d, "\pi_{P_{\mathbb{U}(n)}}"] \\ W. \end{tikzcd}$$

The total space of this bundle can be described explicitly as the set of orthonormal complex bases $(e_1, \dots, e_n)$ in the fibers of the tangent bundle

$$P_{\mathbb{U}(n)}(\mathbb{T}W) = \{(e_1, \dots, e_n) \subset \mathbb{T}_xW : x \in W,\ H(e_i, e_j) = \delta_{i,j}\}.$$

Set $\pi_{P_{\mathbb{U}(n)}}(e_1, \dots, e_n) = x \in W$. The group $\mathbb{U}(n)$ acts on the right freely along fibers on the space $P_{\mathbb{U}(n)}(\mathbb{T}W)$ by the formula

$$P_{\mathbb{U}(n)}(\mathbb{T}W) \times \mathbb{U}(n) \to P_{\mathbb{U}(n)}(\mathbb{T}W),$$

$$(e_1, \dots, e_n) \subset \mathbb{T}_xW, \quad U \in \mathbb{U}(n), \quad U = \begin{pmatrix} u^1_1, & \cdots, & u^1_n \\ \vdots & & \vdots \\ u^n_1, & \cdots, & u^n_n \end{pmatrix},$$

$$((e_1,\dots,e_n),U)\mapsto(e_1,\dots,e_n)\begin{pmatrix}u_1^1, & \cdots, & u_n^1\\ \vdots & & \vdots\\ u_1^n, & \cdots, & u_n^n\end{pmatrix}.$$

Thus,

$$\begin{tikzcd} P_{\mathbb{U}(n)}(\mathbb{T}W) \arrow[r, equal] \arrow[d, "\pi_{P_{\mathbb{U}(n)}}"] & P_{\mathbb{U}(n)}(\mathbb{T}W) \arrow[d, "\pi_{P_{\mathbb{U}(n)}}"] \\ W \arrow[r, equal] & P_{\mathbb{U}(n)}(\mathbb{T}W)/\mathbb{U}(n). \end{tikzcd}$$

On each map $U_\alpha\subset W$, we have the following trivialization of the bundle $P_{\mathbb{U}(n)}(\mathbb{T}W)$:

$$\begin{tikzcd} U_\alpha\times\mathbb{U}(n) \arrow[r, hook, "\varphi_\alpha^{P_{\mathbb{U}(n)}}"] \arrow[d] & P_{\mathbb{U}(n)}(\mathbb{T}W) \arrow[d] \\ U_\alpha \arrow[r, hook] & W, \end{tikzcd}$$

which is generated by the trivialization of the tangent bundle $\mathbb{T}W$:

$$\begin{tikzcd} U_\alpha\times\mathbb{C}(n) \arrow[r, hook, "\varphi_\alpha^{\mathbb{T}}"] \arrow[d] & \mathbb{T}(W) \arrow[d] \\ U_\alpha \arrow[r, hook] & W, \end{tikzcd}$$

while the matching functions $\varphi_{\alpha\beta}^{P_{\mathbb{U}(n)}}$ on the intersection of two maps $U_{\alpha\beta}=U_\alpha\cap U_\beta$ take values in the group $\mathbb{U}(n)$, $\varphi_{\alpha\beta}^{P_{\mathbb{U}(n)}}(x)\in\mathbb{U}(n)$, and are the multiplication on the left:

$$\begin{tikzcd} U_{\alpha\beta}\times\mathbb{U}(n) \arrow[r, hook] \arrow[d] \arrow[rrrr, bend left, "\varphi_{\alpha\beta}^{P_{\mathbb{U}(n)}}=(\varphi_\alpha^{P_{\mathbb{U}(n)}})^{-1}\varphi_\beta^{P_{\mathbb{U}(n)}}"] & U_\beta\times\mathbb{U}(n) \arrow[r, "\varphi_\beta^{P_{\mathbb{U}(n)}}"] \arrow[d] & P_{\mathbb{U}(n)}(\mathbb{T}W) \arrow[d] & U_\alpha\times\mathbb{U}(n) \arrow[l, "\varphi_\alpha^{P_{\mathbb{U}(n)}}"'] \arrow[d] & U_{\alpha\beta}\times\mathbb{U}(n) \arrow[l, hook'] \arrow[d] \\ {[2mm]}U_{\alpha\beta} \arrow[r, hook] & U_\beta \arrow[r, hook] & W & U_\alpha \arrow[l, hook'] & U_{\alpha\beta} \arrow[l, hook'] \end{tikzcd}$$

$$\varphi_{\alpha\beta}^{P_{\mathbb{U}(n)}}(x,A)=(x,\varphi_{\alpha\beta}^{P_{\mathbb{U}(n)}}(x)\cdot A),\quad x\in U_{\alpha\beta},\qquad \varphi_{\alpha\beta}^{P_{\mathbb{U}(n)}}(x),\quad A\in\mathbb{U}(n);$$

$$\varphi_{\alpha\beta}^{P_{\mathbb{U}(n)}}(x)=\varphi_{\alpha\beta}^{\mathbb{T}}(x),\quad x\in U_{\alpha\beta}.$$

2) Further, we use the right action of the group $\mathbb{U}(n)$ in the total space $P_{\mathbb{U}(n)}(\mathbb{T}W)$ and factor the space $P_{\mathbb{U}(n)}(\mathbb{T}W)$ with respect to the subgroup $\mathbb{O}(n)\subset\mathbb{U}(n)$. We obtain the bundle isomorphism

$$\begin{tikzcd} (P_{\mathbb{U}(n)}(\mathbb{T}W))/\mathbb{O}(n) \arrow[r, "f"] \arrow[d] & \mathcal{LG}(\mathbb{T}W) \arrow[d] \\ W \arrow[r, equal] & W \end{tikzcd}$$

by the following formula: for $(e_1,\dots,e_n)\subset\mathbb{T}_xW$, $H(e_i,e_j)=\delta_{i,j}$, we set

$$f(e_1,\dots,e_n)=L_{\mathbb{R}}(e_1,\dots,e_n)\subset\mathbb{T}_xW,$$

and $\omega_{|L_{\mathbb{R}}(e_1,\dots,e_n)} = 0$.

Since f is a bundle isomorphism, it follows that there exists an inverse mapping

$$\begin{tikzcd} \mathcal{LG}(\mathbb{T}W) \arrow[r, "u=f^{-1}"] \arrow[d] & (P_{\mathbb{U}(n)}(\mathbb{T}W))/\mathbb{O}(n) \arrow[d] \\ W \arrow[r, equal] & W, \end{tikzcd}$$

which takes each Lagrangian plane $L \in \mathcal{LG}(\mathbb{T}W)$ such that $L \subset \mathbb{T}_x W$ to a basis $(e_1,\dots,e_n) \subset L$,

$$u(L) = (e_1,\dots,e_n).$$

The map u is explicitly given by the differential of the embedding $h\colon \Lambda \to \mathcal{LG}(\mathbb{T}W)$. In each chart $U_\alpha \subset W$, this isomorphism f is of the form

$$\begin{tikzcd} eU_\alpha \times \mathbb{U}(n)/\mathbb{O}(n) \arrow[r, hook] \arrow[d] \arrow[rrr, bend left, "f_\alpha"] & (P_{\mathbb{U}(n)}(\mathbb{T}W))/\mathbb{O}(n) \arrow[r, "f"] \arrow[d] & \mathcal{LG}(\mathbb{T}W) \arrow[d] & \mathcal{LG}(\mathbb{T}U_\alpha) \arrow[l, hook'] \arrow[d] \\ U_\alpha \arrow[r, hook] & W \arrow[r, equal] & W & U_\alpha. \arrow[l, hook'] \end{tikzcd}$$

Since the complex basis $(e_1,\dots,e_n)$ is orthonormal, it follows that, in particular, the subspace $L_{\mathbb{R}}(e_1,\dots,e_n)$ is a Lagrangian plane independent of the choice of the real basis in it. In the chart U_α, we set the function

$$\det\nolimits^2_\alpha\colon \mathcal{LG}(\mathbb{T}U_\alpha) \to \mathbb{C}\setminus\{\mathbf{0}\} \sim \mathbb{S}^1$$

by the formula

$$\det\nolimits^2_\alpha(L_{\mathbb{R}}(e_1,\dots,e_n)) = \det\nolimits^2(f_\alpha^{-1}(L_{\mathbb{R}}(e_1,\dots,e_n))).$$

The value of the function $\det^2_\alpha$ coincides with the Maslov index on each Lagrangian manifold lying in the space $\mathbb{T}U_\alpha$ (see Vasiliev [3, Theorem 6.2.4, p. 72] (2000)).

It is also useful to consider the function

$$\det\nolimits^{2k}_\alpha\colon \Lambda Gr(\mathbb{T}U_\alpha) \to \mathbb{C}$$

using the formula

$$\det\nolimits^{2k}_\alpha(L_{\mathbb{R}}(e_1,\dots,e_n)) = (\det\nolimits^2(f^{-1}(L_{\mathbb{R}}(e_1,\dots,e_n))))^k.$$

The corresponding Maslov class is written as the one-dimensional cohomology class generated by the differential form $\mathbf{M}^a \in H^1(\mathcal{LG}(\mathbb{R}^{2n}))$,

$$\mathbf{M}^a = f^*\left(\frac{dz^k}{2\pi i z^k}\right) = kf^*\left(\frac{dz}{2\pi i z}\right) \in H^1(\mathcal{LG}(\mathbb{R}^{2n})).$$

4. CONSTRUCTION OF THE MASLOV INDEX ON THE TOTAL SPACE OF BUNDLES OF LAGRANGIAN GRASSMANNIANS

The problem is to find conditions on the symplectic manifold W under which the Maslov index can be constructed on the whole symplectic manifold W, i.e., under which the function $\det^2_\alpha$ is independent of the choice of the chart U_α.

If the matching functions of the tangent bundle $\mathbb{T}W$ take values in an orthogonal subgroup $\varphi^{\mathbb{T}}_{\alpha\beta}(w) \in \mathbb{O}(n)$, then, in each fiber of the bundle $\Lambda Gr(\mathbb{T}W)$, the function $\det^2_\alpha$ does not depend on the choice of trivialization:

$$\det\nolimits^2(\varphi^{\mathbb{T}}_{\alpha\beta}(x)\cdot A) = \det\nolimits^2(\varphi^{\mathbb{T}}_{\alpha\beta}(x))\cdot\det\nolimits^2(A) = \det\nolimits^2(A).$$

In fact, for the definition of $\det^2_\alpha$ to be independent of the trivialization, the constraint $\varphi^{\mathbb{T}}_{\alpha\beta}(w) \in \mathbb{O}(n)$ is too burdensome. It suffices to assume that the structure group $\mathbb{U}(n)$ reduces to the subgroup $\mathbb{SU}(n) \subset \mathbb{U}(n)$. Certainly, the subgroup $\mathbb{O}(n) \not\subset \mathbb{SU}(n)$ creeps a little out of the subgroup $\mathbb{SU}(n)$. But this is easy to fix: consider the group homomorphism

$$\det\colon \mathbb{U}(n) \to \mathbb{S}^1 \approx \mathbb{U}(1)$$

and the finite subgroup $\mathbb{H} \subset \mathbb{S}^1$ giving the exact sequence

$$\mathbf{1} \longrightarrow \mathbb{H} \longrightarrow \mathbb{S}^1 \longrightarrow \mathbb{S}^{\mathbb{H}} \longrightarrow \mathbf{1}.$$

Set

$$\mathbb{S}^{\mathbb{H}}\mathbb{U}(n) = \det^{-1}(\mathbb{H}) \subset \mathbb{U}(n).$$

In the case where $\mathbb{H} = \mathbf{1}$, we obtain $\mathbb{SU}(1)$. In the case where $\mathbb{H} = \{0,1\} = \mathbb{Z}_2$, we obtain the subgroup $\mathbb{S}^{Z_2}\mathbb{U}(n) \supset \mathbb{O}(n)$. Therefore, the problem of constructing the function $\det^2_\alpha$ independent of the choice of trivialization, reduces to the following theorem.

Theorem 2. *The function $\det^{2k}_\alpha$ is well defined on the total space of the bundle of Lagrangian Grassmannians, i.e., it is independent of the choice of the chart U_α, if the structure group of the complex tangent bundle $\mathbb{T}W$ reduces to the subgroup $\mathbb{S}^{\mathbb{H}}\mathbb{U}(n) \subset \mathbb{U}(n)$, where $\mathbb{H} \subset \mathbb{U}(1)$ is a finite subgroup of order k.*

Proof. If the matching functions of the tangent bundle $\mathbb{T}W$ take values in the subgroup $\mathbb{S}^H\mathbb{U}(n) \subset \mathbb{U}(n)$, $\varphi^{\mathbb{T}}_{\alpha\beta}(w) \in \mathbb{S}^H\mathbb{U}(n)$, then, in each fiber of the bundle $\Lambda Gr(\mathbb{T}W)$, the function $\det^{2k}_\alpha$ is independent of the choice of trivialization:

$$\det{}^{2k}(\varphi^{\mathbb{T}}_{\alpha\beta}(x)\cdot A) = (\det(\varphi^{\mathbb{T}}_{\alpha\beta}(x)))^{2k} \cdot (\det(A))^{2k} = \det{}^{2k}(A),$$

because $\det(\varphi^{\mathbb{T}}_{\alpha\beta}(x)) \subset H$, and therefore, $(\det(\varphi^{\mathbb{T}}_{\alpha\beta}(x)))^{2k} = \mathbf{1}$. $\square$

When does the structure group $\mathbb{U}(n)$ reduce to the subgroup $\mathbb{S}^{\mathbb{H}}\mathbb{U}(n) = \det^{-1}(\mathbb{H}) \subset \mathbb{U}(n)$? The answer to this question is as follows.

Theorem 3. *The structure group $\mathbb{U}(n)$ of the complex tangent bundle $\mathbb{T}W$ reduces to the subgroup $\mathbb{S}^{\mathbb{H}}\mathbb{U}(n) = \det^{-1}(\mathbb{H}) \subset \mathbb{U}(n)$ when the first Chern class $c_1(TW) \in H^2(W,\mathbb{Z})$ is of finite order $k = \#(\mathbb{H})$.*

Proof. Consider the composition of the maps

$$\varphi_{\mathbb{H}}\colon \mathbb{U}(n) \xrightarrow{\det} \mathbb{S}^1 \to \mathbb{S}^1/\mathbb{H} \approx \mathbb{S}^{\mathbb{H}} \approx \mathbb{S}^1,$$

We obtain the exact sequence

$$\mathbf{1} \longrightarrow \mathbb{S}^{\mathbb{H}}\mathbb{U}(n) \hookrightarrow \mathbb{U}(n) \xrightarrow{\varphi_H} \mathbb{S}^{\mathbb{H}} \longrightarrow \mathbf{1}.$$

Since the tangent bundle $\mathbb{T}(W)$ admits the structure of a complex bundle with structure group $\mathbb{U}(n)$, it follows that the principal bundle $St(\mathbb{T}W)$ associated with the bundle $\mathbb{T}(W)$ is the inverse image of the canonical bundle classifying the space of the structure group $\mathbb{U}(n)$:

$$\begin{tikzcd} St(\mathbb{T}W) \arrow[r, "f^*"] \arrow[d, "/\mathbb{U}(n)"'] & E_{\mathbb{U}(n)} \arrow[d, "/\mathbb{U}(n)"] \\ W \arrow[r, "f"'] & B_{\mathbb{U}(n)}. \end{tikzcd}$$

If the structure group $\mathbb{U}(n)$ reduces to the subgroup $\mathbb{S}^H\mathbb{U}(n) \subset \mathbb{U}(n)$, then this means that the map f rises to the map g:

$$\begin{tikzcd}
St(\mathbb{T}W) \arrow[r, "f^*"] \arrow[dd, "/\mathbb{U}(n)"'] & E_{\mathbb{U}(n)} \arrow[d, "/\mathbb{S}^{\mathbb{H}}\mathbb{U}(n)"'] \arrow[dd, bend left=60, "/\mathbb{U}(n)"] \\
 & B_{\mathbb{S}^H\mathbb{U}(n)} \arrow[d, "/\mathbb{S}^{\mathbb{H}}"'] \\
W \arrow[r, "f"'] \arrow[ur, dashed, "s"] & B_{\mathbb{U}(n)}.
\end{tikzcd}$$

This diagram expands to the following diagram:

$$\begin{tikzcd}
St(\mathbb{T}W) = f^*(E_{\mathbb{U}(n)}) \arrow[r, "f^*"] \arrow[d, "/\mathbb{S}^{\mathbb{H}}\mathbb{U}(n)"] \arrow[dd, bend right=60, "/\mathbb{U}(n)"'] & E_{\mathbb{U}(n)} \arrow[d, "/\mathbb{S}^{\mathbb{H}}\mathbb{U}(n)"'] \arrow[dd, bend left=60, "/\mathbb{U}(n)"] \\
f^*(E_{\mathbb{U}(n)})/\mathbb{S}^{\mathbb{H}}\mathbb{U}(n) \arrow[r, "f'"] \arrow[d, "/\mathbb{S}^{\mathbb{H}}"] & B_{\mathbb{S}^H\mathbb{U}(n)} \arrow[d, "/\mathbb{S}^{\mathbb{H}}"'] \\
W \arrow[r, "f"'] \arrow[u, dashed, bend left, "s'"] \arrow[ur, dashed, "s"'] & B_{\mathbb{U}(n)}.
\end{tikzcd}$$

The condition for the existence of the map s is equivalent to the existence of the section s' in the principal bundle ξ_W,

$$\xi_W : \xrightarrow[f]{} \xi_{B_{\mathbb{U}(n)}} : \xrightarrow[g]{} \xi_{B_{\mathbb{S}^{\mathbb{H}}}} :$$

$$\begin{tikzcd}
f^*(E_{\mathbb{U}(n)})/\mathbb{S}^{\mathbb{H}}\mathbb{U}(n) \arrow[r, "f'"] \arrow[d, "/\mathbb{S}^{\mathbb{H}}"] & B_{\mathbb{S}^H\mathbb{U}(n)} \arrow[r, "g'"] \arrow[d, "/\mathbb{S}^{\mathbb{H}}"'] & E_{\mathbb{S}^{\mathbb{H}}} \arrow[d, "/\mathbb{S}^{\mathbb{H}}"'] \\
W \arrow[r, "f"'] \arrow[u, dashed, bend left, "s'"] & B_{\mathbb{U}(n)}. \arrow[r, "g"'] & B_{\mathbb{S}^{\mathbb{H}}}.
\end{tikzcd}$$

The existence of the section s' in the principal bundle ξ_W implies that the principal bundle ξ_W is trivial. This means that the map $g \circ f$ is homotopic to the trivial map. The spectral sequences for the bundles ξ_W, $\xi_{B_{\mathbb{U}(n)}}$, and $\xi_{B_{\mathbb{S}^{\mathbb{H}}}}$ are of the form

- for ξ_W,

$$E_2^{p,q}(\xi_W) = H^p(W; H^q(\mathbb{S}^{\mathbb{H}}; \mathbb{Z})), \qquad d_2^W : E_2^{0,1}(\xi_W) \to E_2^{2,0}(\xi_W),$$
$$E_2^{0,1}(\xi_W) = H^1(\mathbb{S}^{\mathbb{H}}; \mathbb{Z}) = \mathbb{Z} \ni a_{\mathbb{H}}, \qquad E_2^{2,0}(\xi_W) = H^2(W; \mathbb{Z}) \ni d_2^W(a_{\mathbb{H}});$$

- for $\xi_{B_{\mathbb{U}(n)}}$,

$$E_2^{p,q}(\xi_{B_{\mathbb{U}(n)}}) = H^p(B_{\mathbb{U}(n)}; H^q(\mathbb{S}^{\mathbb{H}}; \mathbb{Z})), \qquad d_2^{B_{\mathbb{U}(n)}} : E_2^{0,1}(\xi_{B_{\mathbb{U}(n)}}) \to E_2^{2,0}(\xi_{B_{\mathbb{U}(n)}}),$$
$$E_2^{0,1}(\xi_{B_{\mathbb{U}(n)}}) = H^1(\mathbb{S}^{\mathbb{H}}; \mathbb{Z}) = \mathbb{Z} \ni a_{\mathbb{H}}, \qquad E_2^{2,0}(\xi_{B_{\mathbb{U}(n)}}) = H^2(B_{\mathbb{U}(n)}; \mathbb{Z}) \ni d_2^{B_{\mathbb{U}(n)}}(a_{\mathbb{H}});$$

- for $\xi_{B_{\mathbb{S}^{\mathbb{H}}}}$,

$$E_2^{p,q}(\xi_{B_{\mathbb{S}^{\mathbb{H}}}}) = H^p(B_{\mathbb{S}^{\mathbb{H}}}; H^q(\mathbb{S}^{\mathbb{H}}; \mathbb{Z})), \qquad d_2^{B_{\mathbb{S}^{\mathbb{H}}}} : E_2^{0,1}(\xi_{B_{\mathbb{S}^{\mathbb{H}}}}) \to E_2^{2,0}(\xi_{B_{\mathbb{S}^{\mathbb{H}}}}),$$
$$E_2^{0,1}(\xi_{B_{\mathbb{S}^{\mathbb{H}}}}) = H^1(\mathbb{S}^{\mathbb{H}}; \mathbb{Z}) = \mathbb{Z} \ni a_{\mathbb{H}}, \qquad E_2^{2,0}(\xi_{B_{\mathbb{S}^{\mathbb{H}}}}) = H^2(B_{\mathbb{S}^{\mathbb{H}}}; \mathbb{Z}) = \mathbb{Z} \ni d_2^{B_{\mathbb{S}^{\mathbb{H}}}}(a_{\mathbb{H}}).$$

The commutativity of the diagrams of the bundles ξ_W, $\xi_{B_{\mathbb{U}(n)}}$, and $\xi_{B_{\mathbb{S}^{\mathbb{H}}}}$ implies

$$d_2^W(a_{\mathbb{H}}) = f^*(g^*(d_2^{B_{\mathbb{S}^{\mathbb{H}}}}(a_{\mathbb{H}}))) = (g\circ f)^*(d_2^{B_{\mathbb{S}^{\mathbb{H}}}}(a_{\mathbb{H}})) = 0.$$

Therefore,

$$f^*(d_2^{B_{\mathbb{U}(n)}}(a_{\mathbb{H}})) = 0.$$

Thus, the cohomology class

$$d_2^{B_{\mathbb{U}(n)}}(a_{\mathbb{H}}) \in H^2(B_{\mathbb{U}(n)};\mathbb{Z})$$

is expressed in terms of the first Chern class as follows:

$$d_2^{\xi_{B_{\mathbb{U}(n)}}}(a_{\mathbb{H}}) = \lambda^{\mathbb{H}}\cdot c_1 \in H^2(B_{\mathbb{U}(n)};\mathbb{Z}).$$

Therefore, $d_2^{\xi_W}(a_{\mathbb{H}}) = \lambda^{\mathbb{H}}\cdot c_1(TW)$.

This implies that the structure group $\mathbb{U}(n)$ reduces to the subgroup

$$\mathbb{S}^{\mathbb{H}}\mathbb{U}(n) = \det{}^{-1}(\mathbb{H}) \subset \mathbb{U}(n)$$

if the first Chern class $c_1(TW)\in H^2(W,\mathbb{Z})$ is of finite order $\lambda^{\mathbb{H}}$.

It remains to verify that $\lambda^{\mathbb{H}} = k = \#(\mathbb{H})$.

To this end, consider two bundles

$$\begin{tikzcd}
 & E_{\mathbb{U}}(n) \arrow[r, equal] \arrow[d, "/\mathbb{S}\mathbb{U}(n)"] & E_{\mathbb{U}}(n) \arrow[d, "/\mathbb{S}^H\mathbb{U}(n)"] & \\
\xi_1 : \arrow[rrr, dashed, bend left=80] & B_{\mathbb{S}\mathbb{U}(n)} \arrow[r, "h'"] \arrow[d, "/\mathbb{S}^1"] & B_{\mathbb{S}^H\mathbb{U}(n)} \arrow[d, "/\mathbb{S}^{\mathbb{H}}"] & : \xi_{B_{\mathbb{U}(n)}} \\
 & B_{\mathbb{U}(n)} \arrow[r, equal, "h=\mathbf{Id}"] & B_{\mathbb{U}(n)}. &
\end{tikzcd}$$

and the corresponding spectral sequences

$$\begin{aligned}
&\xi_1 :\\
&E_2^{p,q}(\xi_1) = H^p(B_{\mathbb{U}(n)};H^q(\mathbb{S}^1;\mathbb{Z})), && d_2^{\xi_1} : E_2^{0,1}(\xi_1)\to E_2^{2,0}(\xi_1),\\
&E_2^{0,1}(\xi_1) = H^1(\mathbb{S}^1;\mathbb{Z}) = \mathbb{Z}\ni a_1, && E_2^{2,0}(\xi_1) = H^2(B_{\mathbb{U}(n)};\mathbb{Z})\ni d_2^{\xi_1}(a_1) = c_1,
\end{aligned}$$

$$\begin{aligned}
&\xi_{B_{\mathbb{U}(n)}} :\\
&E_2^{p,q}(\xi_{B_{\mathbb{U}(n)}}) = H^p(B_{\mathbb{U}(n)};H^q(\mathbb{S}^{\mathbb{H}};\mathbb{Z})), && d_2^{B_{\mathbb{U}(n)}} : E_2^{0,1}(\xi_{B_{\mathbb{U}(n)}})\to E_2^{2,0}(\xi_{B_{\mathbb{U}(n)}}),\\
&E_2^{0,1}(\xi_{B_{\mathbb{U}(n)}}) = H^1(\mathbb{S}^{\mathbb{H}};\mathbb{Z}) = \mathbb{Z}\ni a_{\mathbb{H}}, && E_2^{2,0}(\xi_{B_{\mathbb{U}(n)}}) = H^2(B_{\mathbb{U}(n)};\mathbb{Z})\ni d_2^{\xi_{B_{\mathbb{U}(n)}}}(a_{\mathbb{H}}).
\end{aligned}$$

The map $\mathbb{S}^1 \xrightarrow{h'} \mathbb{S}^{\mathbb{H}}$ gives $(h')^*(a_{\mathbb{H}}) = k\cdot a_1$. Since the diagrams of two bundles ξ_1 and $\xi_{B_{\mathbb{U}(n)}}$ are commutative, it follows that their spectral sequences commute, and hence

$$h^*(d_2^{\xi_{B_{\mathbb{U}(n)}}}(a_{\mathbb{H}})) = d_2^{\xi_1}((h')^*(a_{\mathbb{H}})),$$

$$d_2^{\xi_{B_{\mathbb{U}(n)}}}(a_{\mathbb{H}}) = d_2^{\xi_1}(k\cdot a_1) = k\cdot d_2^{\xi_1}(a_1) = k\cdot c_1.$$

Thus, $d_2^{\xi_W}(a_{\mathbb{H}}) = k\cdot c_1(TW)$, i.e., $\lambda^{\mathbb{H}} = k = \#(\mathbb{H})$. $\square$

SUPPLEMENT BY A. T. FOMENKO. "CONSTRUCTING THE GENERALIZED MASLOV CLASS FOR THE TOTAL SPACE $W = \mathbb{T}^*(M)$ OF THE COTANGENT BUNDLE"

The construction of the generalized Maslov class was presented in the book of Trofimov and Fomenko (1995) [2].

Let M be a smooth manifold, and let ω be a symplectic structure on the total space $\mathbb{T}^*M$. Every tangent space $\mathbb{T}_z(\mathbb{T}^*M)$, $z \in \mathbb{T}^*M$, is a symplectic vector space, and in it we can take the Lagrangian subspace V_z tangent to the vertical, i.e., consisting of tangent vectors ξ such that $d\pi_z\xi = 0$, where the letter π denotes the standard projection $\pi\colon \mathbb{T}^*M \to M$. The choice of the Riemannian metric on M induces a positive definite inner product on V_z, making it possible to identify $\mathcal{L}G(\mathbb{T}_z(\mathbb{T}^*M))$ with $\mathbb{U}(n)/\mathbb{O}(n)$. This identification is ambiguous but allows us to find the well defined differential form $(\det^2)^*(dz/2\pi iz)$ on the total space of the bundle $\mathcal{L}G(\mathbb{T}^*M)$ over $\mathbb{T}^*M$, whose fiber above over the point $z \in \mathbb{T}^*M$ consists of all Lagrangian subspaces in $\mathbb{T}_z(\mathbb{T}^*M)$.

If N is a Lagrangian submanifold in $\mathbb{T}^*M$, then, for any curve γ on N, the curve on $\mathcal{L}G(\mathbb{T}^*M)$ is defined naturally. In this case, using the formula

$$l = \oint_\gamma (\det^2)^* \frac{dz}{2\pi iz}\,,$$

we define an integer; thus, we obtain an element in $H^1(N;\mathbb{Z})$, which is called a *generalized Maslov class* of submanifolds N. This class does not depend on the choice of the Riemannian metric.

In fact, the space $\mathcal{L}G(\mathbb{T}^*M)$ can be represented as the total space of the principal bundle with structure group $\mathbb{U}(n)$, right factored with respect to the subgroup $\mathbb{O}(n)$.

Let $W = \mathbb{T}(M)$. We introduce a natural almost-complex structure J on the manifold W as follows:

$$J\colon \mathbb{T}(\mathbb{T}^*(M)) \to \mathbb{T}(\mathbb{T}^*(M)).$$

The local coordinates on the manifold $W = \mathbb{T}(M)$ are of the form

$$(x_\alpha, p^\alpha) = (x^j_\alpha, p^\alpha_k)^n_{j,k=1}.$$

The symplectic structure is given by the differential form

$$\omega = dp^\alpha \wedge dx_\alpha = \sum dp^\alpha_j \wedge dx^j_\alpha.$$

See [1] for details.

The Euclidean structure is given by the Euclidean form

$$G = g^\alpha_{j,k}\, dx^j_\alpha\, dx^k_\alpha + g^{j,k}_\alpha\, dp^\alpha_j\, dp^\alpha_k.$$

The almost complex structure J is given by the formula

$$J\biggl(\frac{\partial}{\partial x^j_\alpha}\biggr) = \frac{\partial}{\partial p^\alpha_j}\,, \qquad J\biggl(\frac{\partial}{\partial p^\alpha_j}\biggr) = -\frac{\partial}{\partial x^j_\alpha}\,.$$

Under the change of coordinates, the components of the tangent space $\mathbb{T}(\mathbb{T}^*(M))$ change by the tensor law:

$$\frac{\partial}{\partial x^j_\alpha} = \frac{\partial x^k_\beta}{\partial x^j_\alpha}\,\frac{\partial}{\partial x^k_\beta}\,, \qquad \frac{\partial}{\partial p^\alpha_j} = \frac{\partial x^j_\alpha}{\partial x^k_\beta}\,\frac{\partial}{\partial p^\beta_k}\,.$$

Then

$$J\biggl(\frac{\partial}{\partial x^j_\alpha}\biggr) = \frac{\partial x^k_\beta}{\partial x^j_\alpha} J\biggl(\frac{\partial}{\partial x^k_\beta}\biggr) = \frac{\partial x^k_\beta}{\partial x^j_\alpha}\,\frac{\partial}{\partial p^\beta_k} = \frac{\partial}{\partial p^\alpha_j}\,;$$

i.e., the operator J is independent of the choice of the chart. This means that the bundle $\mathbb{T}(\mathbb{T}^*(M))$ is a complex bundle with structure group $\mathbb{U}(n)$, which reduces to the subgroup $\mathbb{O}(n) \subset \mathbb{U}(n)$.

We replace the bundle $\mathbb{T}(\mathbb{T}^*(M))$ by the principal $\mathbb{U}(n)$-bundle $P_{U(n)}(\mathbb{T}^*M)$ with fiber $\mathbb{U}(n)$. On the total space of the principal $\mathbb{U}(n)$-bundle $P_{U(n)}(\mathbb{T}^*M)$, the group $\mathbb{U}(n)$ acts freely on the right; in particular, the subgroup $\mathbb{O}(n) \subset \mathbb{U}(n)$ acts freely on the right as well. Factoring the total space $P_{U(n)}(\mathbb{T}^*M)$ by the right action of the group $\mathbb{O}(n)$, we obtain the bundle $P\mathbb{U}(\mathbb{T}^*M)/\mathbb{O}(n)$ with fiber $\mathbb{U}(n)/\mathbb{O}(n)$ and structure group $\mathbb{U}(n)$, which acts on the fiber $\mathbb{U}(n)/\mathbb{O}(n)$ by left multiplication. The bundle of Lagrangian Grassmannians $\mathcal{L}G(\mathbb{T}^*M)$ is isomorphic to the principal bundle $(P_{\mathbb{U}(n)}(\mathbb{T}^*M))/\mathbb{O}(n)$:

$$\begin{tikzcd}
\mathcal{LG}(\mathbb{T}^*M) \arrow[r, "u"] \arrow[d] & (P_{\mathbb{U}(n)}(\mathbb{T}^*M))/\mathbb{O}(n) \arrow[d] \\
\mathbb{T}^*M \arrow[r, equal] & \mathbb{T}^*M.
\end{tikzcd}$$

The map u is given by the equivariant map of one fiber:

$$u\colon \Lambda(n) \to \mathbb{U}(n)/\mathbb{O}(n),$$

where $\Lambda(n)$ is a Lagrangian Grassmannian, $\Lambda(n) = \{L \subset \mathbb{C}^n : \omega_{|L} = 0\}$.

The structure group $\mathbb{U}(n)$ of the bundle $(P_{\mathbb{U}(n)}(\mathbb{T}^*M))/\mathbb{O}(n)$ reduces to a subgroup $\mathbb{O}(n) \subset \mathbb{U}(n)$, and hence the map

$$\det{}^2\colon \mathbb{U}(n)/\mathbb{O}(n) \to \mathbb{S}^1$$

extends to the map of the total spaces of bundles

$$\begin{tikzcd}
\mathcal{LG}(\mathbb{T}^*M) \arrow[r, "u"] \arrow[d] \arrow[rr, bend left, dashed, "\det^2 \cdot u"] & (P_{\mathbb{U}(n)}(\mathbb{T}^*M))/\mathbb{O}(n) \arrow[r, "\det^2"] \arrow[d] & \mathbb{S}^1 \; . \\
\mathbb{T}^*M \arrow[r, equal] & \mathbb{T}^*M. &
\end{tikzcd}$$

The generalized Maslov class is constructed as an inverse image by the mapping:

$$(\det{}^2 \cdot u)^*(dz/2\pi iz) \in H^1(\mathcal{LG}(\mathbb{T}^*M)).$$

ACKNOWLEDGMENTS

This research was initiated by discussions with A. T. Fomenko and V. E. Nazaikinskii, to both of whom I express my deep gratitude.

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