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On the $D(2)$-Vertex Distinguishing Total Coloring of Graphs with $\Delta = 3$*

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Abstract—A $D(2)$-*vertex-distinguishing total coloring* of a graph G is a proper total coloring such that no pair of vertices, within distance two, has the same set of colors, and the minimum number of colors required for such a coloring is called $D(2)$-*vertex-distinguishing total chromatic number* of G, and denoted by $\chi_{2vt}(G)$. In this paper, we prove that $\chi_{2vt}(G) \leq 11$ for any graph G with $\Delta(G) = 3$.

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1. INTRODUCTION

All graphs considered in this paper are simple, finite, and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $d_G(x)$ (denote by $d(x)$ for short) be the *degree* of a vertex x in G, and $\Delta(G) = \max\{d(x)|x \in V(G)\}$ the maximum degree of G. Let P_n and $C_n (n \geq 3)$ be the *path* and the *cycle* of order n, respectively, and $K_{m,n}$ be a *complete bipartite graph* with bipartition (X, Y), where $|X| = m$ and $|Y| = n$. We call a path $P_{n+1} = u_0u_1 \cdots u_n$ an *internal path* of G if $d(u_0) \geq 3, d(u_n) \geq 3$, and $d(u_i) = 2, i = 1, 2, \cdots, n-1$. Let $G - \{v\}$ be the subgraph of G obtained by deleting the vertex v and its incident edges. Let $G - \{uv\}$ and $G + \{uv\}$ be the graphs obtained from G by deleting the edge uv and by adding a new edge uv, respectively. A *cut edge* of G is an edge uv such that $\omega(G - \{uv\}) > \omega(G)$, where $\omega(G)$ refers to the number of components in G. The disjoint union $G_1 \dot{\cup} G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. A vertex of degree one is called a *pendent vertex*. The *distance* between two vertices u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. A graph G is r-*regular* if $d(u) = r$ for all $u \in V(G)$. Especially, if a graph G has a 1-regular *spanning subgraph*, then the spanning subgraph is called a *perfect matching* of G. The *square* G^2 of a graph G can be obtained from G by adding all edges between two vertices of distance two in G. For two sets A and B, $A \backslash B$ represents the difference set of A and B. The terminologies and notations used but undefined in this paper can be found in [1].

A proper k-total coloring of G is a mapping $f : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ such that any two adjacent or incident elements in $V(G) \cup E(G)$ receive different colors. We use

$$S_f(x) = \{f(x)\} \cup \{f(xy)|xy \in E(G)\}$$

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to denote the set of colors assigned to a vertex x and those edges incident with x. A proper k-total coloring f of G is said to be a *k-$D(\beta)$-vertex distinguishing total coloring* [2] (write as k-$D(\beta)$-VDTC for short) if $S_f(u) \neq S_f(v)$ for any two vertices $u, v \in V(G)$ with $d(u,v) \leq \beta$, and denote by $\chi_{\beta vt}(G) = \min\left\{k|G \text{ has a } k\text{-}D(\beta)\text{-VDTC}\right\}$ the $D(\beta)$-vertex distinguishing total chromatic number of G. Especially, if $\beta = 1$, then a $D(1)$-vertex distinguishing total coloring is just the *adjacent vertex distinguishing total coloring* of G (see [3]); if $\beta = 2$, then $D(\beta)$-vertex distinguishing total coloring is also called *$D(2)$-vertex distinguishing total coloring* of G, and the corresponding chromatic number is written as $\chi_{2vt}(G)$. Besides, let f be a total coloring of G. If any two vertices $u, v \in V(G)$ with $d(u,v) \leq 2$ satisfy $S_f(u) \neq S_f(v)$, then we call that u and v are *$D(2)$-vertex distinguishable*.

Graph coloring has always been classic topic in graph theory, and it's widely applied in practice (see [1]). In recent decades, many scholars have conducted a series of researches around it [4]–[7]. The adjacent vertex distinguishing total coloring of graphs was first introduced by Zhang et al. in [3]. They conjectured that the adjacent vertex distinguishing total chromatic number $\chi_{at}(G)$ of any graph G, with order at least 2, satisfies $\chi_{at}(G) \leq \Delta(G) + 3$, and further showed that the conjecture holds for *complete graph*, *complete bipartite graph*, *tree*, and so on. After then, scholars have carried out a lot of research to the conjecture. Wang [8] and Chen [9] independently proved that if G is a graph with maximum degree $\Delta(G) = 3$, then $\chi_{at}(G) \leq 6$. Cheng et al. [10] showed that for a planar graph G with $\Delta(G) \geq 10$, $\chi_{at}(G) \leq \Delta(G) + 3$. Huang et al. [11] verified that $\chi_{at}(G) \leq 2\Delta(G)$ for any graph G with $\Delta(G) \geq 3$. Later, B. Vučković [12] improved this bound, and got that $\chi_{at}(G) \leq 2\Delta(G) - 1$ if $\Delta(G) \geq 4$.

In 2006, Zhang et al. [2] proposed the concept entitled *$D(\beta)$-vertex distinguishing total coloring of graphs*, and got the $D(\beta)$-vertex distinguishing total chromatic number of some simple graphs such as *path*, *cycle*, and *complete graph* for $\beta \geq 2$. Based on the above, they presented the conjecture given below:

Conjecture 1 (see [2]). *Let G be a connected graph on $n \geq 2$ vertices. Then*

$$\chi_{\beta vt}(G) \leq \mu_\beta(G) + 1.$$

where $\mu_\beta(G) = \min\left\{\theta : \binom{\theta}{i+1} \geq n_i,\ \delta \leq i \leq \Delta\right\}$, $n_i = \max\left\{|S| : S \subseteq V(G)\right\}$, *and S is composed of the vertices with degree i and their distance is no more than β.*

From conjecture 1, if $\beta = 2$, then we see that the following conjecture.

Conjecture 2 (see [2]). *Let G be a connected graph on $n \geq 2$ vertices. Then*

$$\chi_{2vt}(G) \leq \mu_2(G) + 1.$$

Meanwhile, Zhang et al.[2] also showed that the conjecture holds for some simple graphs. In this paper, we aim at Conjecture 2 to consider the $D(2)$-vertex distinguishing total coloring of a graph G, and prove that $\chi_{2vt}(G) \leq 11$ for any graph G with maximum degree $\Delta(G) = 3$.

2. PRELIMINARIES

Lemma 1 (see [2, Theorem 2.2]). *Let G be a graph with n components, i.e., $G = G_1 \cup G_2 \cup \cdots \cup G_n$. Then $\chi_{2vt}(G) = \max\left\{\chi_{2vt}(G_i)|i = 1, 2, \cdots, n\right\}$.*

Lemma 2 (see [2, Theorem 2.4]). *For a path P_n with order $n \geq 2$, $\chi_{2vt}(P_n) \leq 4$.*

Lemma 3 (see [2, Theorems 2.7, 2.8, 2.9]). *Let C_n be a cycle on $n \geq 3$ vertices. Then $\chi_{2vt}(C_n) \leq 5$.*

Lemma 4 (see [13, pp.243], [14, Lemma 2.4.6]). *For a 3-regular graph G, if G contains no perfect matching, then G has at least 3 cut edges.*

A *list assignment* of G is a function L which assigns to each vertex $x \in V(G)$ a list $L(x)$ of colors. A *list coloring* of G with list assignment L is a mapping $f : V(G) \to \bigcup_{x \in V(G)} L(x)$ such that $f(x) \in L(x)$ for all $x \in V(G)$ and $f(u) \neq f(v)$ for any adjacent vertices u and v, and we say that G is L-colorable. A graph is called k-*choosable* if G is L-colorable whenever all lists have size k. The *list chromatic number* $\chi_l(G)$ is the minimum k such that G is k-choosable. In [15], the list chromatic number of the square G^2 of a graph G with $\Delta(G) \leq 3$ is given in the following.

Lemma 5 (see [15, Theorem 1]). *For a graph G with $\Delta(G) \leq 3$, if G is not isomorphic to the Petersen graph, then $\chi_l(G^2) \leq 8$.*

A *proper vertex coloring* of G is a mapping $f : V(G) \to \{1,\ 2,\ \cdots,\ k\}$ such that for any two adjacent vertices u and v, $f(u) \neq f(v)$, and denote by $\chi(G)$ the proper vertex chromatic number of G. By the definition of *list-coloring* of G, we can see that a list-coloring of G is also a proper vertex coloring of the graph, and $\chi(G) \leq \chi_l(G)$. A 2-*distance coloring* of a graph G is a proper vertex coloring such that no two vertices, within distance 2 in G, are assigned the same color, and we denote by $\chi_2(G)$ the 2-*distance chromatic number* of G. It is obvious that for any a simple graph G and its square graph G^2, $\chi_2(G) = \chi(G^2)$. Hence, we can deduce that $\chi_2(G) = \chi(G^2) \leq \chi_l(G^2)$. Furthermore, the following corollary can be deduced by Lemma 5.

Corollary 1. *If G is a graph with $\Delta(G) \leq 3$, but not isomorphic to the Petersen graph, then*

$$\chi_2(G) \leq 8.$$

Lemma 6. *If G is the Petersen graph, then $\chi_{2vt}(G) = 6$.*

Proof. Suppose that G has vertex-set $V(G) = \{u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ v_1,\ v_2,\ v_3,\ v_4,\ v_5\}$ and edge-set $E(G) = \{u_i u_{i+1},\ u_i v_i,\ v_i v_{i+2}\}$, where $i = 1, 2, \cdots, 5$. According to the definition of $D(2)$-VDTC, $\chi_{2vt}(G) \geq 6$ due to $\binom{6}{4} = 15 > 10$ and $\binom{5}{4} = 5 < 10$. Let $f : V(G) \cup E(G) \to \{1,\ 2,\ \cdots,\ 6\}$ be a total coloring of G. Suppose $f(u_i v_i) = 6$ for $i = 1, 2, 3, 4, 5$. Then we color the vertices u_1, u_2, u_3, u_4, and u_5 with $1, 2, 3, 4$, and 5 respectively, and color the vertices v_1, v_2, v_3, v_4, and v_5 with $5, 1, 2, 3$, and 4 respectively. Meanwhile, the edges u_1u_2, u_2u_3, u_3u_4, u_4u_5, and u_5u_1 are colored by $3, 4, 5, 1$, and 2 respectively. After that, the edges v_1v_3, v_2v_4, v_3v_5, v_4v_1, and v_5v_2 are colored by $3, 4, 5, 1$, and 2 respectively. Clearly, f is a proper total coloring of G, and it is easy to see that

$$S(u_1) = \{1,\ 2,\ 3,\ 6\},\quad S(u_2) = \{2,\ 3,\ 4,\ 6\},\quad S(u_3) = \{3,\ 4,\ 5,\ 6\},$$
$$S(u_4) = \{1,\ 4,\ 5,\ 6\},\quad S(u_5) = \{1,\ 2,\ 5,\ 6\},\quad S(v_1) = \{1,\ 3,\ 5,\ 6\},$$
$$S(v_2) = \{1,\ 2,\ 4,\ 6\},\quad S(v_3) = \{2,\ 3,\ 5,\ 6\},\quad S(v_4) = \{1,\ 3,\ 4,\ 6\}, \text{ and}$$
$$S(v_5) = \{2,\ 4,\ 5,\ 6\}.$$

Therefore, f is a 6-D(2)-VDTC of G, and thus $\chi_{2vt}(G) = 6$. □

Lemma 7. *For a 3-regular graph G with a perfect matching, $\chi_{2vt}(G) \leq 11$.*

Proof. If G is the Petersen graph, then $\chi_{2vt}(G) = 6 < 11$ by Lemma 6. Otherwise, we consider that G is a 3-regular graph but not isomorphic to the Petersen graph. We decompose G as a perfect matching M and a union of some cycles, denoted by $C_{n_i}^{(i)} = x_1^i x_2^i \cdots x_{n_i}^i x_1^i$ for $i = 1, 2, \cdots, t$, where n_i is the length of $C_{n_i}^{(i)}$. According to Corollary 1, one can see that G has a 2-distance vertex coloring with 8 colors, suppose that $f_1 : V(G) \to \{1,\ 2,\ \cdots,\ 8\}$ is a 2-distance vertex coloring of G. Let $f_2 : E(G) \to \{1,\ 2,\ \cdots,\ 11\}$ be a proper edge coloring of G. Without loss of generality, we may suppose that all the edges in M are colored by 9. Then we will color the edges of cycles $C_{n_i}^{(i)}$ where $i = 1, 2, \cdots, t$. For clarity, we may suppose that $C_{n_1}^{(1)}$, $C_{n_2}^{(2)}$, $\cdots$, $C_{n_r}^{(r)}$ are even cycles, and $C_{n_{r+1}}^{(r+1)}$, $C_{n_{r+2}}^{(r+2)}$, $\cdots$, $C_{n_s}^{(s)}$, $C_{n_{s+1}}^{(s+1)}$, $\cdots$, $C_{n_t}^{(t)}$ are odd cycles, in which $C_{n_{r+1}}^{(r+1)}$, $C_{n_{r+2}}^{(r+2)}$, $\cdots$, $C_{n_s}^{(s)}$ satisfy that each vertex of one odd cycle is not adjacent to any vertex of the others, meanwhile, $C_{n_{s+1}}^{(s+1)}$, $C_{n_{s+2}}^{(s+2)}$, $\cdots$, $C_{n_t}^{(t)}$ satisfy that some vertices of one cycle are adjacent to some vertices of the others, where $r \leq s \leq t$.

Suppose that the edges of $C^{(1)}_{n_1}, C^{(2)}_{n_2}, \cdots, C^{(r)}_{n_r}$ are colored by 10 and 11 alternately. Now we color the edges of the cycles $C^{(r+1)}_{n_{r+1}}, C^{(r+2)}_{n_{r+2}}, \cdots, C^{(s)}_{n_s}$. For each cycle $C^{(i)}_{n_i}$ with $i = r+1, r+2, \cdots, s$, the edges $x^i_1x^i_2, x^i_2x^i_3, \cdots, x^i_{n_i-1}x^i_{n_i}$ of $C^{(i)}_{n_i}$ are colored alternately by 10 and 11, and the edge $x^i_{n_i}x^i_1$ is colored by a color α, where $\alpha \in \{1,\ 2,\ \cdots,\ 8\}\backslash\{f_1(x^i_1),\ f_1(x^i_{n_i})\}$.

After that, we will color the edges of cycles $C^{(s+1)}_{n_{s+1}},\ C^{(s+2)}_{n_{s+2}},\ \cdots,\ C^{(t)}_{n_t}$. We construct a new graph H with vertex set $V(H) = \{C^{(s+1)}_{n_{s+1}},\ C^{(s+2)}_{n_{s+2}},\ \cdots,\ C^{(t)}_{n_t}\}$ and edge set $E(H)$, where $E(H)$ equals to the set of all edges which connect the vertices of $C^{(i)}_{n_i}$ with the vertices of $C^{(j)}_{n_j}$ in G, where $i \neq j$ and $i, j = s+1, s+2, \cdots, t$. It is easy to see that H may be disconnected. Let $H = H_1 \cup H_2 \cup \cdots \cup H_m$, where $H_i (i = 1, 2, \cdots, m)$ are components of H.

We first color the edges of odd cycles corresponding to the vertices of H_1. For any $y \in V(H_1)$, let $C^{(y)}_{n_y} = x^y_1x^y_2\cdots x^y_{n_y}x^y_1$ be the odd cycle corresponding to y in G. Given a vertex $v \in V(H_1)$, then the corresponding odd cycle is $C^{(v)}_{n_v} = x^v_1x^v_2\cdots x^v_{n_v}x^v_1$. We alternately color the edges $x^v_1x^v_2, x^v_2x^v_3, \cdots, x^v_{n_v-1}x^v_{n_v}$ by 10 and 11, and color the edge $x^v_{n_v}x^v_1$ by the color α. For any $uv \in E(H_1)$, we also color the edges $x^u_1x^u_2, x^u_2x^u_3, \cdots, x^u_{n_u-1}x^u_{n_u}$ of $C^{(u)}_{n_u}$ by 10 and 11 alternately, and color $x^u_{n_u}x^u_1$ by a color β, where $\beta \in \{1,\ 2,\ \cdots,\ 8\}\backslash\{f_1(x^u_{n_u}),\ f_1(x^u_1),\ f_1(x^v_{n_v}),\ f_1(x^v_1),\ \alpha\}$. For any $w \in V(H_1)$ with $d_{H_1}(v, w) \geq 2$, if $d_{H_1}(v, w) = 2$, then we color the edges of $C^{(w)}_{n_w}$ by the coloring function that has dyed the edges of $C^{(v)}_{n_v}$; if $d_{H_1}(v, w) = 3$, then we color the edges of $C^{(w)}_{n_w}$ by the coloring function that has dyed the edges of $C^{(u)}_{n_u}$. As an analogy, all edges of $C^{(w)}_{n_w}$, with $d_{H_1}(v, w) \geq 4$, can be colored in the same way, meanwhile, one can color the edges of odd cycles corresponding to the vertices of $H_2, H_3, \cdots, H_m$ by the coloring function which has dyed the edges of H_1. So far, all the edges of cycles $C^{(s+1)}_{n_{s+1}}, C^{(s+2)}_{n_{s+2}}, \cdots, C^{(t)}_{n_t}$ have been colored.

Thus, for any vertex of G, there are 5 color sets under f_2, that is, $\{9,\ 10,\ 11\}, \{\alpha,\ 9,\ 10\}, \{\alpha,\ 9,\ 11\}, \{\beta,\ 9,\ 10\}$, and $\{\beta,\ 9,\ 11\}$, respectively.

Finally, we construct a total coloring $f : V(G) \cup E(G) \to \{1,\ 2,\ \cdots,\ 11\}$ of G as follows:

$$f(z) = \begin{cases} f_1(z), & z \in V(G), \\ f_2(z), & z \in E(G). \end{cases} \tag{1}$$

Obviously, f is a proper total coloring of G. Under the coloring f, we show that any two vertices of G with distance no more than 2 are $D(2)$-vertex distinguishable.

Let $x^i_k \in V(C^{(i)}_{n_i})$ and $x^j_l \in V(C^{(j)}_{n_j})$ be two vertices with $d(x^i_k, x^j_l) \leq 2$, where i and j are not necessary distinct, and $i, j = 1, 2, \cdots, t$. Let $S_f(x^i_k)$ and $S_f(x^j_l)$ be the color sets that correspond to x^i_k and x^j_l, respectively. It is evident that $S_f(x^i_k) = \{f_1(x^i_k)\} \cup S_{f_2}(x^i_k)$ and $S_f(x^j_l) = \{f_1(x^j_l)\} \cup S_{f_2}(x^j_l)$. Moreover, we notice that $d(x^i_k, x^j_l) \leq 2$, from the coloring function f_2, one can see that

$$S_{f_2}(x^i_k), S_{f_2}(x^j_l) \in \Big\{\{9,\ 10,\ 11\},\ \{\alpha,\ 9,\ 10\},\ \{\alpha,\ 9,\ 11\},\ \{\beta,\ 9,\ 10\},\ \{\beta,\ 9,\ 11\}\Big\},$$

but $S_{f_2}(x^i_k) = S_{f_2}(x^j_l) \notin \Big\{\{\alpha,\ 9,\ 10\},\ \{\alpha,\ 9,\ 11\},\ \{\beta,\ 9,\ 10\},\ \{\beta,\ 9,\ 11\}\Big\}$. Now we consider four cases in the following:

- for $S_{f_2}(x^i_k) = \{9,\ 10,\ 11\}$, if $S_{f_2}(x^j_l) = \{9,\ 10,\ 11\}$, then one can see that $S_f(x^i_k) \neq S_f(x^j_l)$ since $f_1(x^i_k) \neq f_1(x^j_l)$; if $S_{f_2}(x^j_l) \in \Big\{\{\alpha,\ 9,\ 10\},\ \{\alpha,\ 9,\ 11\},\ \{\beta,\ 9,\ 10\},\ \{\beta,\ 9,\ 11\}\Big\}$, then we obtain $\{f_1(x^i_k)\} \cup S_{f_2}(x^i_k) \neq \{f_1(x^j_l)\} \cup S_{f_2}(x^j_l)$ since $\{\alpha,\ \beta,\ f_1(x^i_k),\ f_1(x^j_l)\} \subset \{1,\ 2,\ \cdots,\ 8\}$ and, consequently, $S_f(x^i_k) \neq S_f(x^j_l)$;

- for $S_{f_2}(x_k^i) = \{\alpha,\ 9,\ 10\}$, if $S_{f_2}(x_l^j) = \{\alpha,\ 9,\ 11\}$, then $\{f_1(x_k^i),\ \alpha,\ 9,\ 10\} \neq \{f_1(x_l^j),\ \alpha,\ 9,\ 11\}$ due to $\{f_1(x_k^i),\ f_1(x_l^j)\} \subset \{1,\ 2,\ \cdots,\ 8\}$, i.e., $S_f(x_k^i) \neq S_f(x_l^j)$; if
$$S_{f_2}(x_l^j) \in \Big\{\{\beta,\ 9,\ 10\},\ \{\beta,\ 9,\ 11\}\Big\},$$
since $\beta \in \{1,\ 2,\ \cdots,\ 8\}\backslash\{f_1(x_k^i),\ \alpha\}$ we see that $\beta \notin S_f(x_k^i)$, and thus $S_f(x_k^i) \neq S_f(x_l^j)$;

- for $S_{f_2}(x_k^i) = \{\alpha,\ 9,\ 11\}$, if $S_{f_2}(x_l^j) \in \Big\{\{\beta,\ 9,\ 10\},\ \{\beta,\ 9,\ 11\}\Big\}$ then $S_f(x_k^i) \neq S_f(x_l^j)$ as $\beta \notin S_f(x_k^i)$;

- for $S_{f_2}(x_k^i) = \{\beta,\ 9,\ 10\}$, if $S_{f_2}(x_l^j) = \{\beta,\ 9,\ 11\}$, then we see that
$$\{f_1(x_k^i),\ \beta,\ 9,\ 10\} \neq \{f_1(x_l^j),\ \beta,\ 9,\ 11\}$$
because $\{f_1(x_k^i),\ f_1(x_l^j)\} \subset \{1,\ 2,\ \cdots,\ 8\}$, and hence, $S_f(x_k^i) \neq S_f(x_l^j)$.

Therefore, f is an 11-$D(2)$-VDTC of G. The proof completes. □

3. MAIN RESULTS

Theorem 1. *Let G be a graph with maximum degree $\Delta(G) \leq 3$. Then $\chi_{2vt}(G) \leq 11$.*

Proof. Let G be a graph with $\Delta(G) \leq 3$. If $\Delta(G) \leq 2$, then G is a path, or a cycle, or a union of paths and cycles. From Lemmas 2 and 3, both the path and the cycle have a 5-$D(2)$-VDTC, and it follows from Lemma 1 that the union of paths and cycles also has a 5-$D(2)$-VDTC; if $\Delta(G) = 3$, without loss of generality, by Lemma 1, we may suppose that G is connected. Then we will prove that G has an 11-$D(2)$-VDTC by induction on $|E(G)|$.

When $|E(G)| = 3$, then $G \cong K_{1,3}$, it is easy to see that G has a 7-$D(2)$-VDTC, however $7 < 11$, the conclusion holds; when $|E(G)| \geq 4$, suppose that for any connected graph G' with $\Delta(G') \leq 3$ and $|E(G')| < |E(G)|$, G' has an 11-$D(2)$-VDTC. Now we consider two cases as follows.

Case 1. G has a cut edge uv. Since G is a connected graph with $\Delta(G) = 3$, there is at most one pendant vertex in $\{u,\ v\}$. Thus, the following two subcases are considered.

Subcase 1.1. There is just one pendant vertex in u and v. We may suppose that $d_G(v) = 1$. Then $2 \leq d_G(u) \leq 3$. Let u_1 and u_2 (if exists) be the neighbors of u different from v, and u_{i1} and u_{i2} (if exists) be the neighbors of u_i different from u. Let $G' = G - \{v\}$. Then G' has an 11-D(2)-VDTC f' by induction. We will construct an 11-D(2)-VDTC f of G by coloring uv and v.

Setting $f(x) = f'(x)$ for $\forall x \in V(G') \cup E(G')$, we first color the edge uv. We notice that the color set of u should be distinguished from the color sets of vertices within distance 2, there are 6 vertices at most, that is, $\{u_1, u_2, u_{11}, u_{12}, u_{21}, u_{22}\}$ (if exists). In addition, by the definition of proper total coloring of graphs, there are at least 8 colors in $\{1,\ 2,\ \cdots,\ 11\}$ which can be used to color uv. Since $8 > 6$ we can select one color from $\{1,\ 2,\ \cdots,\ 11\}\backslash\{f(u),\ f(uu_1),\ f(uu_2)\}$ to dye uv so that $S_f(u) \neq S_f(u_i)$ and $S_f(u) \neq S_f(u_{ij})$, where $i, j = 1, 2$. Next, we color the vertex v. Since there are 9 colors which can be used to dye v properly, and the color set of v should be distinguished from the color sets of 3 vertices at most, that is, $\{u,\ u_1,\ u_2\}$ (if exists), there exists one color in $\{1,\ 2,\ \cdots,\ 11\}\backslash\{f(u),\ f(uv)\}$ for v, such that $S_f(v) \neq S_f(u)$ and $S_f(v) \neq S_f(u_i)$ for $i = 1, 2$. Besides those, the other vertices of G are $D(2)$-vertex distinguishable. Thus, f is an 11-D(2)-VDTC of G.

Subcase 1.2. There is no pendent vertex in u and v. Clearly, $2 \leq d_G(u) \leq 3$ and $2 \leq d_G(v) \leq 3$. Let u_1 and u_2 (if exists) be the neighbors of u different from v, and v_1 and v_2 (if exists) be the neighbors of v different from u. We suppose that $G - \{uv\} = G_1 \dot{\cup} G_2$, where $u \in V(G_1)$ and $v \in V(G_2)$. Let

$G^{'} = G_1 + \{uv\}$ and $G^{''} = G_2 + \{uv\}$. Then by the induction hypothesis, $G^{'}$ has an 11-$D(2)$-VDTC f_1, and $G^{''}$ has an 11-$D(2)$-VDTC f_2.

Without loss of generality, we let $f_1(u) = f_2(u) = 1$, $f_1(uv) = f_2(uv) = 2$, and $f_1(v) = f_2(v) = 3$. Note that if $f_1(v) \in S_{f_1}(u)$, then one can recolor $v \in V(G^{'})$ to yield $f_1(v) \notin S_{f_1}(u)$ since there are at least 7 colors in $\{1,\ 2,\ \cdots,\ 11\}\backslash S_{f_1}(u)$ which can be used to dye v so that the color set of v is distinguished from the color sets of 3 vertices at most, that is, $\{u,\ u_1,\ u_2\}$ (if exists); if $f_2(u) \in S_{f_2}(v)$, similarly, the vertex $u \in V(G^{''})$ can also be recolored with $f_2(u) \notin S_{f_2}(v)$, and thus, we may suppose that $f_1(v) \notin S_{f_1}(u)$ and $f_2(u) \notin S_{f_2}(v)$. Hence, we can assume that the colors assigned on $uu_1, uu_2 \in E(G^{'})$ are 4 and 5, respectively, and the colors assigned on $vv_1, vv_2 \in E(G^{''})$ are 6 and 7, respectively.

Now, we construct a proper total coloring f^* of G as follows: for any $z \in V(G) \cup E(G)$, define

$$f^*(z) = \begin{cases} f_1(z), & z \in V(G^{'}) \cup E(G^{'}), \\ f_2(z), & z \in V(G^{''}) \cup E(G^{''}). \end{cases} \tag{2}$$

Since $6 \notin S_{f^*}(u)$, $7 \notin S_{f^*}(u)$, $6 \in S_{f^*}(v) \cap S_{f^*}(v_1)$, and $7 \in S_{f^*}(v) \cap S_{f^*}(v_2)$, we see that

$$S_{f^*}(u) \neq S_{f^*}(v) \qquad \text{and} \qquad S_{f^*}(u) \neq S_{f^*}(v_i) \quad \text{for} \ \ i = 1, 2.$$

Since $4 \notin S_{f^*}(v)$, $5 \notin S_{f^*}(v)$, $4 \in S_{f^*}(u) \cap S_{f^*}(u_1)$, and $5 \in S_{f^*}(u) \cap S_{f^*}(u_2)$, we obtain

$$S_{f^*}(v) \neq S_{f^*}(u_i), \qquad \text{where} \quad i = 1, 2.$$

Besides these relations, it is clear that the other vertices of G are $D(2)$-vertex distinguishable. Therefore, f^* is an 11-D(2)-VDTC of G.

Case 2. G has no cut edge. It follows that G has no pendent vertex, that is, $2 \leq d(x) \leq 3$ for any $x \in V(G)$. If G doesn't contain vertex of degree-2, then G should be a 3-regular graph. Since G has no cut edge, by Lemma 4, G must have a perfect matching. Therefore, it follows from Lemma 7 that $\chi_{2vt}(G) \leq 11$; if G contains vertex of degree-2, let's consider the following two subcases in terms of the distance between any two such vertices in G.

Subcase 2.1. There exist two vertices of degree-2 within distance 2 in G.

Subcase 2.1.1. The two vertices of degree-2 are adjacent in G. Let $P_{n+1} = u_0u_1 \cdots u_n (n \geq 3)$ be an internal path in G including at least two vertices of degree-2. Let $u_0^{'}$ and $u_0^{''}$ be the neighbors of u_0 different from u_1, and let $u_n^{'}$ and $u_n^{''}$ be the neighbors of u_n different from u_{n-1}. Now, we suppose that $G^{'}$ is the graph obtained by contracting $P_{n+1}(= u_0u_1 \cdots u_n)$ to u_0vu_n, see F_1 in Fig. 1. By the induction hypothesis, $G^{'}$ has an 11-$D(2)$-VDTC $\varphi^{'}$. Without loss of generality, we suppose that $\varphi^{'}(u_0v) = 1$, $\varphi^{'}(v) = 11$, and $\varphi^{'}(u_nv) = 2$. Next, we construct an 11-$D(2)$-VDTC φ of G.

For $3 \leq n \leq 4$, let $\varphi(u_0u_1) = 1$, $\varphi(u_1) = 11$, and $\varphi(u_{n-1}u_n) = 2$. If $n = 3$, we first color the edge u_1u_2. Note that there are just 8 colors for u_1u_2 to obtain a proper total coloring, meanwhile, the color set of u_1 should be distinguished from the color sets of at most 4 vertices, that is, $\{u_0,\ u_0^{'},\ u_0^{''},\ u_3\}$. Since $8 > 4$ we can select one color from $\{1,\ 2,\ \cdots,\ 11\}\backslash\{\varphi(u_0u_1),\ \varphi(u_1),\ \varphi(u_2u_3)\}$ to dye u_1u_2 such that u_1 and $\{u_0,\ u_0^{'},\ u_0^{''},\ u_3\}$ are $D(2)$-vertex-distinguishable. Next, we color the vertex u_2. Since there are at least 7 colors in $\{1,\ 2,\ \cdots,\ 11\}$ which can be used to dye u_2 properly, and the color set of u_2 should be distinguished from the color sets of at most 5 vertices, that is, $\{u_0,\ u_1,\ u_3,\ u_3^{'},\ u_3^{''}\}$, we can select one color from $\{1,\ 2,\ \cdots,\ 11\}\backslash\{\varphi(u_1u_2), \varphi(u_2u_3),\ \varphi(u_1),\ \varphi(u_3)\}$ for u_2, where $\varphi(u_3) = \varphi^{'}(u_3)$, such that u_2 and $\{u_0,\ u_1,\ u_3,\ u_3^{'},\ u_3^{''}\}$ are $D(2)$-vertex-distinguishable; if $n = 4$, by the same way, we can color u_1u_2, u_2, u_2u_3, and u_3 such that the vertices u_0, u_1, u_2, u_3, u_4, $u_0^{'}$, $u_0^{''}$, $u_4^{'}$, and $u_4^{''}$ are $D(2)$-vertex distinguishable.

For $n \geq 5$, let $\varphi(u_0u_1) = \varphi(u_{n-2}u_{n-1}) = 1$, $\varphi(u_1) = \varphi(u_{n-1}) = 11$, and let

$$\varphi(u_1u_2) = \varphi(u_{n-1}u_n) = 2.$$

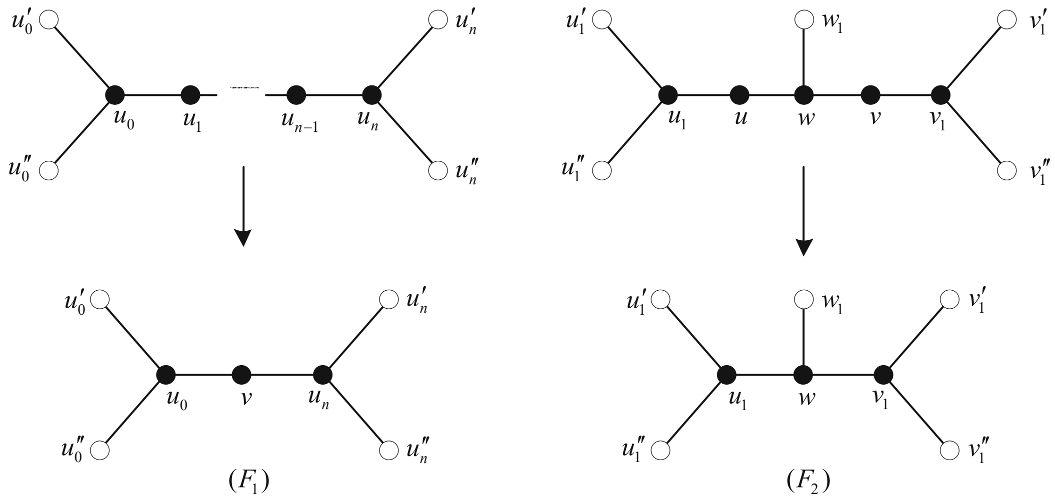

Fig. 1. The configuration in the proof of subcase 2.1, where "•" refers to the vertex whose degree is certain, and "○" refers to the vertex whose degree is uncertain and is at least 1.

We then circularly color u_2, u_2u_3, u_3, $\cdots$, $u_{n-3}u_{n-2}$ and u_{n-2} by $3, 4, 5, 6$, and 7. It is evident that the vertices of degree-2 and the vertices of degree-3 are $D(2)$-vertex distinguishable. Since $S_\varphi(u_1) = S_\varphi(u_{n-1}) = S_{\varphi'}(v) = \{1,\ 2,\ 11\}$ we see that $S_\varphi(u_1) \neq S_\varphi(u_0^{'})$, $S_\varphi(u_1) \neq S_\varphi(u_0^{''})$, $S_\varphi(u_{n-1}) \neq S_\varphi(u_n^{'})$, and $S_\varphi(u_{n-1}) \neq S_\varphi(u_n^{''})$. Moreover, it follows from Lemma 2 that the vertices of P_{n+1} are also $D(2)$-vertex distinguishable. Therefore, φ is an 11-D(2)-VDTC of G.

Subcase 2.1.2. The distance between any two vertices of degree-2 in G is equal to 2. Let $u, v \in V(G)$, $d(u) = d(v) = 2$, and $d(u, v) = 2$. Suppose that u_1 and w are the neighbors of u, and v_1 and w are the neighbors of v. Then $d(u_1) = d(v_1) = d(w) = 3$. Let $u_1^{'}$ and $u_1^{''}$ be the neighbors of u_1 different from u, and let $v_1^{'}$ and $v_1^{''}$ be the neighbors of v_1 different from v, and w_1 the neighbor of w different from u and v. Note that $2 \leq d(w_1) \leq 3$ (if exists), we may suppose that $w_1^{'}$ and $w_1^{''}$ are the neighbors of w_1 different from w. Let $G^{'}$ be the graph obtained by contracting u_1uwvv_1 to u_1wv_1, see F_2 in Figure 1. Then by induction hypothesis, $G^{'}$ has an 11-$D(2)$-VDTC $\psi^{'}$. Let $\psi^{'}(ww_1) = 1$, $\psi^{'}(wu_1) = 2$, $\psi^{'}(wv_1) = 3$, and $\psi^{'}(w) = 4$. Next, we will construct an 11-$D(2)$-VDTC ψ of G by coloring uu_1, u, uw, wv, v, and vv_1.

Let $\psi(uu_1) = 2$ and $\psi(vv_1) = 3$. We then color uw and wv in turn. By the definition of proper total coloring of graphs, there are at least 8 colors which can be used to dye uw, and 7 colors that can be used to dye wv. We notice that the color set of w should be distinguished from the color sets of 7 vertices at most, that is, $\{u,\ u_1,\ v,\ v_1,\ w_1,\ w_1^{'},\ w_1^{''}\}$ (if exists). Since $d(u) = d(v) = 2$ and $d(w) = 3$, any proper total coloring of uw and wv would enable that w and $\{u,\ v\}$ are $D(2)$-vertex distinguishable. Thus, we only consider the color set of w should be distinguished from the color sets of at most 5 vertices, that is, $\{u_1,\ v_1,\ w_1,\ w_1^{'},\ w_1^{''}\}$ (if exists). Since there are 8×7 assignments that can be colored the edges uw and wv, and at most $5 \times 2!$ combinations of them such that the color set of w equals to the color sets of the vertices in $\{u_1,\ v_1,\ w_1,\ w_1^{'},\ w_1^{''}\}$, however $8 \times 7 > 5 \times 2!$, we can select two colors in $\{1,\ 2,\ \cdots,\ 11\}$ to dye uw and wv in turn, such that w and $\{u_1,\ v_1,\ w_1,\ w_1^{'},\ w_1^{''}\}$ are $D(2)$-vertex distinguishable. Then we color the vertex u. Note that there are at least 7 colors which can be used to dye u properly, and the color set of u should be distinguished from the color sets of 5 vertices, that is, $\{u_1,\ u_1^{'},\ u_1^{''},\ w,\ w_1\}$ (if exists). Since $7 > 5$ we can select one color from $\{1,\ 2,\ \cdots,\ 11\} \backslash \{\psi(uw),\ \psi(uu_1),\ \psi(w),\ \psi(u_1)\}$ for u so that the vertex u and $\{u_1,\ u_1^{'},\ u_1^{''},\ w,\ w_1\}$ are $D(2)$-vertex distinguishable. We finally color the vertex v. Since there are at least 7 colors which can be used to color v properly, and the color set of v should be distinguished from the color sets of 6 vertices, that is, $\{u,\ w,\ w_1,\ v_1,\ v_1^{'},\ v_1^{''}\}$ (if exists), there exists one color in $\{1,\ 2,\ \cdots,\ 11\}\backslash\{\psi(w),\ \psi(wv),\ \psi(vv_1),\ \psi(v_1)\}$ (note that $\psi(v_1) = \psi^{'}(v_1)$) for v such that v and $\{u,\ w,\ w_1,\ v_1,\ v_1^{'},\ v_1^{''}\}$ are $D(2)$-vertex distinguishable. Hence, we obtain an 11-$D(2)$-VDTC ψ of G.

Subcase 2.2. The distance between any two vertices of degree-2 in G is no less than 3. Let $G^{'}$ be the graph obtained by taking two copies of G and joining their corresponding vertices of degree-2 by an edge. Then $G^{'}$ is a 3-regular graph, and contains at most one cut edge. By Lemma 4, $G^{'}$ has a perfect matching. Hence $G^{'}$ admits an 11-$D(2)$-VDTC $\phi^{'}$ by Lemma 7. We construct a proper total coloring ϕ of G. For any $x \in V(G) \cup E(G)$, set

$$\phi(x) = \phi^{'}(x).$$

Obviously, the vertices of degree-2 and the vertices of degree-3 are $D(2)$-vertex distinguishable if their distance is no more than 2. Since the distance between any two vertices of degree-2 in G is at least 3, one doesn't take the $D(2)$-vertex distinguishable of such vertices into account. In addition, for any two vertices u and v of degree-3, since $S_{\phi^{'}}(u) \neq S_{\phi^{'}}(v)$ while $d(u, v) \leq 2$, we get $S_{\phi}(u) \neq S_{\phi}(v)$. Thus, ϕ is an 11-$D(2)$-VDTC of G, as required.

Summing up the discussions above, the proof is completed. □

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