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Maximal and Riesz Potential Operators in Double Phase Lorentz Spaces of Variable Exponents*

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Abstract—In the present note, we discuss the boundedness of maximal and Riesz potential operators in double-phase Lorentz spaces of variable exponents defined by a symmetric decreasing rearrangement in the sense of Almut [1].

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1. INTRODUCTION

Recently, the study of double phase problems is very active in the field of Harmonic Analysis, Variable Exponent Analysis and PDE's. The double phase functional was introduced by Zhikov [2]. Regarding regularity theory of differential equations, Mingione and collaborators [3]–[5] investigated a double phase functional

$$\Phi(x,t) = t^p + a(x)t^q, \qquad x \in \mathbf{R}^N, \quad t \geq 0,$$

where $1 < p < q$, a is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$. Regularity properties for general functionals was studied, under the condition $q \leq (1+\theta/N)p$, in [6]. We refer to, e.g., [7] and [8] for Calderón-Zygmund estimates, [9] for the eigenvalue problem, and [10] for the boundedness of the maximal operator.

In [11], relaxing the continuity of $a(\cdot)$, we considered the double phase functional

$$\tilde{\Phi}(x,t) = t^{p(x)} + (b(x)t)^{q(x)},$$

where p, q are log-Hölder continuous and b is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$. We showed the boundedness of the maximal operator and Sobolev's inequality for double phase functionals with variable exponents. See also [12]. For other recent works, see [13], [16], etc.

Let $B(x,r)$ denote the open ball centered at x of radius $r > 0$. The volume of a measurable set $E \subset \mathbf{R}^N$ is written as $|E|$.

For a measurable function f on $\mathbf{R}^N$, we define the symmetric decreasing rearrangement of f by

$$f^\star(x) = \int_0^\infty \chi_{E_f(t)^\star}(x)\, dt,$$

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where $E^\star = \{x : |B(0,|x|)| < |E|\}$ and $E_f(t) = \{y : |f(y)| > t\}$ (see [1]). Note here that

$$f^*(|B(0,|x|)|) = f^\star(x),$$

where f^* is the usual decreasing rearrangement of f. The fundamental fact of the symmetric decreasing rearrangement of f is that

$$|E_f(t)| = |E_{f^\star}(t)|$$

for all $t \ge 0$. This readily gives the rearrangement preserving L^p-norm property such as

$$\|f\|_{L^p(\mathbf{R}^N)} = \|f^\star\|_{L^p(\mathbf{R}^N)}$$

for $1 \le p \le \infty$. For fundamental properties of the symmetric decreasing rearrangement, see Almut [1]. We also refer to his papers [17], [18] and [19, Chap. 4].

For variable exponents p and s, the Lorentz space $\mathcal{L}^{s,p}(\mathbf{R}^N)$ is defined as the set of all measurable functions f on $\mathbf{R}^N$ with

$$\|f\|_{\mathcal{L}^{s,p}(\mathbf{R}^N)} = \inf\left\{\lambda > 0 : \int_{\mathbf{R}^N} |f^\star(x)/\lambda|^{p(x)} |x|^{N(p(x)/s(x)-1)}\, dx \le 1\right\} < \infty.$$

See the paper by Ephremidze, Kokilashvili and Samko [20] when p and q are radial. In [21], the boundedness of the maximal operator and Sobolev's inequality in the Lorentz space of variable exponents were studied.

Our first aim in this note is to establish the boundedness of the maximal operator in double-phase Lorentz spaces of variable exponents (Theorem 3 and Corollary 3), as an extension of [21]. We also give Sobolev's inequality in double-phase Lorentz spaces of variable exponents (Theorem 4 and Corollary 4).

Throughout this paper, let C denote various constants independent of the variables in question.

2. SYMMETRIC DECREASING REARRANGEMENT AND LORENTZ SPACES OF VARIABLE EXPONENTS

The (centered) maximal function Mf of a measurable function f on $\mathbf{R}^N$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

Lemma 1 [21, Lemma 2.2]. *For all measurable functions f on $\mathbf{R}^N$,*

$$(Mf)^\star(x) \le C \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f^\star(y)\, dy \le C Mf^\star(x),$$

where C is a positive constant independent of f.

For $f \in L^1_{loc}(\mathbf{R}^N)$, we define the Riesz potential of order α $(0 < \alpha < N)$ by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy.$$

Lemma 2 [21, Lemma 2.4]. *For all nonnegative measurable functions f on $\mathbf{R}^N$,*

$$(I_\alpha f)^\star(x) \le C \int_{\mathbf{R}^N} (|x|+|y|)^{\alpha-N} f^\star(y)\, dy \le C(I_\alpha f^\star)(x),$$

where C is a positive constant independent of f.

For the fundamental properties of symmetric decreasing rearrangements, see Almut [1].

A function p on $\mathbf{R}^N$ is said to be log-Hölder continuous if

(P1) p is locally log-Hölder continuous, namely,

$$|p(x) - p(y)| \le \frac{C_0}{\log(1/|x-y|)} \qquad \text{for} \quad |x-y| \le \frac{1}{e}$$

with a constant $C_0 \ge 0$;

(P2) p is log-Hölder continuous at infinity, namely,

$$|p(x) - p(\infty)| \le \frac{C_\infty}{\log(e+|x|)}$$

with constants $C_\infty \ge 0$ and $p(\infty)$.

Let $\mathcal{P}(\mathbf{R}^N)$ be the class of all log-Hölder continuous functions p on $\mathbf{R}^N$. If in addition p satisfies

(P3) $1 < p^- := \inf_{x\in\mathbf{R}^N} p(x) \le \sup_{x\in\mathbf{R}^N} p(x) =: p^+ < \infty$,

then we write $p \in \mathcal{P}_1(\mathbf{R}^N)$.

Definition 1. For $p \in \mathcal{P}_1(\mathbf{R}^N)$ and $\tau \in \mathcal{P}(\mathbf{R}^N)$, $L^{\tau,p}(\mathbf{R}^N)$ denotes the weighted $L^{p(\cdot)}$ space of all functions f with

$$\|f\|_{L^{\tau,p}(\mathbf{R}^N)} = \inf\left\{\lambda > 0 : \int_{\mathbf{R}^N} |f(x)/\lambda|^{p(x)} |x|^{\tau(x)} dx \le 1\right\} < \infty.$$

We write $L^{0,p}(\mathbf{R}^N) = L^{p(\cdot)}(\mathbf{R}^N)$ and

$$\|f\|_{L^{0,p}(\mathbf{R}^N)} = \|f\|_{L^{p(\cdot)}(\mathbf{R}^N)}.$$

Definition 2. For $s, p \in \mathcal{P}_1(\mathbf{R}^N)$, denote by $\mathcal{L}^{s,p}(\mathbf{R}^N)$ the set of all measurable functions f such that

$$\|f\|_{\mathcal{L}^{s,p}(\mathbf{R}^N)} = \inf\left\{\lambda > 0 : \int_{\mathbf{R}^N} |f^\star(x)/\lambda|^{p(x)} |x|^{N(p(x)/s(x)-1)} dx \le 1\right\} < \infty.$$

3. THE BOUNDEDNESS OF MAXIMAL AND POTENTIAL OPERATORS IN LORENTZ SPACES OF VARIABLE EXPONENTS

We know the following boundedness of the maximal operator in the weighted $L^{p(\cdot)}$ space, which is an extension of Diening [22] and Cruz-Uribe, Fiorenza and Neugebauer [23].

Theorem 1 [21, Theorem 4.1]. *Let $p \in \mathcal{P}_1(\mathbf{R}^N)$ and $\tau \in \mathcal{P}(\mathbf{R}^N)$. Suppose*

(T1) $-N < \tau(0) < N(p(0) - 1)$ *and* $-N < \tau(\infty) < N(p(\infty) - 1)$.

Then the maximal operator $\mathcal{M} : f \longrightarrow Mf$ is bounded from $L^{\tau,p}(\mathbf{R}^N)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^{\tau,p}(\mathbf{R}^N)} \le C\|f\|_{L^{\tau,p}(\mathbf{R}^N)}$$

for all $f \in L^{\tau,p}(\mathbf{R}^N)$.

This is a special case of [24, Theorem 1.1]. We also refer to [25].

In view of Lemma 1, we obtain the following result.

Corollary 1 [21, Corollary 4.2]. *Let $s, p \in \mathcal{P}_1(\mathbf{R}^N)$. Then the maximal operator $\mathcal{M} : f \longrightarrow Mf$ is bounded from $\mathcal{L}^{s,p}(\mathbf{R}^N)$ into itself, namely, there is a constant $C > 0$ such that*

$$\|Mf\|_{\mathcal{L}^{s,p}(\mathbf{R}^N)} \le C\|f\|_{\mathcal{L}^{s,p}(\mathbf{R}^N)}$$

for all $f \in \mathcal{L}^{s,p}(\mathbf{R}^N)$.

As an application of Theorem 1, we can obtain a Sobolev type inequality for Riesz potentials by using Hedberg's method ([26]).

For $p \in \mathcal{P}_1(\mathbf{R}^N)$, set

$$1/p^\sharp(x) = 1/p(x) - \alpha/N.$$

Theorem 2 [21, Theorem 6.1]. *Let $p \in \mathcal{P}_1(\mathbf{R}^N)$ and $\tau \in \mathcal{P}(\mathbf{R}^N)$. Suppose $p^+ < n/\alpha$ and*

(T2) $\alpha p(0) - N < \tau(0) < N(p(0)-1)$ *and* $\alpha p(\infty) - N < \tau(\infty) < N(p(\infty)-1)$.

Then there is a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{\tau p^\sharp/p, p^\sharp}(\mathbf{R}^N)} \le C \|f\|_{L^{\tau,p}(\mathbf{R}^N)}$$

for all $f \in L^{\tau,p}(\mathbf{R}^N)$.

Using Lemma 2, we obtain the following result.

Corollary 2 [21, Corollary 6.2]. *Let $p, s \in \mathcal{P}_1(\mathbf{R}^N)$, If $p^+ < N/\alpha$ and $s^+ < N/\alpha$, then there is a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{L}^{s^\sharp, p^\sharp}(\mathbf{R}^N)} \le C \|f\|_{\mathcal{L}^{s,p}(\mathbf{R}^N)}$$

for all $f \in \mathcal{L}^{s,p}(\mathbf{R}^N)$.

4. THE BOUNDEDNESS OF MAXIMAL OPERATOR IN DOUBLE-PHASE LORENTZ SPACES OF VARIABLE EXPONENTS

As an extension of [21], we obtain the boundedness of maximal operator in double-phase Lorentz spaces of variable exponents, in view of Theorems 1 and 2.

Recall that b is nonnegative, bounded and Hölder continuous of order $\theta \in (0, 1]$.

Theorem 3. *Let $p, q \in \mathcal{P}_1(\mathbf{R}^N)$ and $\tau, \kappa \in \mathcal{P}(\mathbf{R}^N)$. Suppose*

(D1) $1/q(x) = 1/p(x) - \theta/N$;

(D2) $\kappa(x) = \tau(x)q(x)/p(x)$;

(T3) $\theta p(0) - N < \tau(0) < N(p(0)-1)$ *and* $\theta p(\infty) - N < \tau(\infty) < N(p(\infty)-1)$;

(T4) $-N < \kappa(0) < N(q(0)-1)$ *and* $-N < \kappa(\infty) < N(q(\infty)-1)$.

Then there is a constant $C > 0$ such that

$$\|Mf\|_{L^{\tau,p}(\mathbf{R}^N)} + \|bMf\|_{L^{\kappa,q}(\mathbf{R}^N)} \le C\left(\|f\|_{L^{\tau,p}(\mathbf{R}^N)} + \|bf\|_{L^{\kappa,q}(\mathbf{R}^N)} \right)$$

for all $f \in L^1_{loc}(\mathbf{R}^N)$.

When $\tau = \kappa = 0$, we refer the reader to [11] and [12].

Proof of Theorem 3. In view of Theorem 1, it suffices to show that

$$\|bMf\|_{L^{\kappa,q}(\mathbf{R}^N)} \le C \tag{4.1}$$

when f is a nonnegative measurable function f on $\mathbf{R}^N$ with $\|f\|_{L^{\tau,p}(\mathbf{R}^N)} + \|bf\|_{L^{\kappa,q}(\mathbf{R}^N)} \le 1$. Note that

$$\begin{aligned} & b(x)\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \\ &= \frac{1}{|B(x,r)|}\int_{B(x,r)} \{b(x)-b(y)\}f(y)\,dy + \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)f(y)\,dy \\ &\le C\frac{1}{|B(x,r)|}\int_{B(x,r)} |x-y|^{\theta} f(y)\,dy + \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)f(y)\,dy \end{aligned}$$

for $x \in \mathbf{R}^N$ and $r > 0$. Hence

$$b(x)Mf(x) \le CI_\theta f(x) + M[bf](x)$$

for $x \in \mathbf{R}^N$. In view of Theorem 2, we have

$$\|I_\theta f\|_{L^{\kappa,q}(\mathbf{R}^N)} \le C\|f\|_{L^{\tau,p}(\mathbf{R}^N)}.$$

Moreover, we obtain, by Theorem 1,

$$\|M[bf]\|_{L^{\kappa,q}(\mathbf{R}^N)} \le C\|bf\|_{L^{\kappa,q}(\mathbf{R}^N)},$$

which gives (4.1). □

Corollary 3. *Let $s, p, t, q \in \mathcal{P}_1(\mathbf{R}^N)$. Suppose $1/t(x) = 1/s(x) - \theta/N$ and $1/q(x) = 1/p(x) - \theta/N$. Then there is a constant $C > 0$ such that*

$$\|Mf\|_{\mathcal{L}^{s,p}(\mathbf{R}^N)} + \|bMf\|_{\mathcal{L}^{t,q}(\mathbf{R}^N)} \le C\Big(\|f\|_{\mathcal{L}^{s,p}(\mathbf{R}^N)} + \|bf\|_{\mathcal{L}^{t,q}(\mathbf{R}^N)}\Big)$$

for all $f \in L^1_{loc}(\mathbf{R}^N)$.

5. SOBOLEV'S INEQUALITY IN DOUBLE-PHASE LORENTZ SPACES OF VARIABLE EXPONENTS

By Theorem 2, we obtain the following Sobolev inequality.

Theorem 4. *Let $p, q \in \mathcal{P}_1(\mathbf{R}^N)$ and $\kappa, \tau \in \mathcal{P}(\mathbf{R}^N)$. Suppose $p^+ < N/(\alpha+\theta)$. Moreover, suppose*

(D1) $1/q(x) = 1/p(x) - \theta/N$;

(D2) $\kappa(x) = \tau(x)q(x)/p(x)$;

(T5) $(\alpha+\theta)p(0) - N < \tau(0) < N(p(0)-1)$ *and* $(\alpha+\theta)p(\infty) - N < \tau(\infty) < N(p(\infty)-1)$;

(T6) $\alpha q(0) - N < \kappa(0) < N(q(0)-1)$ *and* $\alpha q(\infty) - N < \kappa(\infty) < N(q(\infty)-1)$.

Then there is a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{\tau p^\sharp/p,p^\sharp}(\mathbf{R}^N)} + \|bI_\alpha f\|_{L^{\kappa q^\sharp/q,q^\sharp}(\mathbf{R}^N)} \le C\Big(\|f\|_{L^{\tau,p}(\mathbf{R}^N)} + \|bf\|_{L^{\kappa,q}(\mathbf{R}^N)}\Big)$$

for all $f \in L^1_{loc}(\mathbf{R}^N)$.

Proof. By Theorem 2, it suffices to show

$$\|bI_\alpha f\|_{L^{\kappa q^\sharp/q,q^\sharp}(\mathbf{R}^N)} \le C \tag{5.1}$$

when f is a nonnegative measurable function f on $\mathbf{R}^N$ with $\|f\|_{L^{\tau,p}(\mathbf{R}^N)} + \|bf\|_{L^{\kappa,q}(\mathbf{R}^N)} \le 1$. Note that

$$\begin{aligned} b(x)I_\alpha f(x) &= \int_{\mathbf{R}^N} |x-y|^{\alpha-N}\{b(x)-b(y)\}f(y)\,dy + \int_{\mathbf{R}^N} |x-y|^{\alpha-N}b(y)f(y)\,dy \\ &\le C\int_{\mathbf{R}^N} |x-y|^{\alpha+\theta-N}f(y)\,dy + \int_{\mathbf{R}^N} |x-y|^{\alpha-N}b(y)f(y)\,dy \end{aligned}$$

for $x \in \mathbf{R}^N$. Therefore,

$$b(x)I_\alpha f(x) \le CI_{\alpha+\theta}f(x) + I_\alpha[bf](x)$$

for $x \in \mathbf{R}^N$. In view of Theorem 2, we have

$$\|I_{\alpha+\theta}f\|_{L^{\kappa q^\sharp/q,q^\sharp}(\mathbf{R}^N)} \le C\|f\|_{L^{\tau,p}(\mathbf{R}^N)}$$

and

$$\|I_\alpha[bf]\|_{L^{\kappa q^\sharp/q,q^\sharp}(\mathbf{R}^N)} \le C\|bf\|_{L^{\kappa,q}(\mathbf{R}^N)},$$

which proves (5.1). □

Corollary 4. *Let $s,p,t,q \in \mathcal{P}_1(\mathbf{R}^N)$, $1/t(x) = 1/s(x) - \theta/N$ and $1/q(x) = 1/p(x) - \theta/N$. Suppose $s^+ < N/(\alpha+\theta)$ and $p^+ < N/(\alpha+\theta)$. Then there is a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{L}^{s^\sharp,p^\sharp}(\mathbf{R}^N)} + \|bI_\alpha f\|_{\mathcal{L}^{t^\sharp,q^\sharp}(\mathbf{R}^N)} \le C\left(\|f\|_{\mathcal{L}^{s,p}(\mathbf{R}^N)} + \|bf\|_{\mathcal{L}^{t,q}(\mathbf{R}^N)}\right)$$

for all $f \in L^1_{loc}(\mathbf{R}^N)$.

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