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Characterizations of σ-Solvable Finite Groups

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Abstract—All the groups considered in this paper are finite, and G always denotes a finite group; σ is a partition of the set $\mathbb{P}$ of all primes, i.e., $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$. A group G is said to be *σ-primary* if G is a σ_i-group for some $i = i(G)$, and *σ-solvable* if every chief factor of G is σ-primary. A set of subgroups $\mathcal{H}$ of a group G is called a *complete Hall σ-set* of G if every element $\neq 1$ of the set $\mathcal{H}$ is a Hall σ_i-subgroup G for some i, and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of the group G for all i such that $\sigma_i \cap \pi(G) \neq \varnothing$. A subgroup A of a group G is said to be K-$\mathfrak{S}_\sigma$-subnormal in G if G contains a series of subgroups $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or the group $A_i/(A_{i-1})_{A_i}$ is σ-solvable for all $i = 1, \ldots, t$.

We say that a subgroup A of a group G is *weakly K-$\mathfrak{S}_\sigma$-subnormal* in G if G contains K-$\mathfrak{S}_\sigma$-subnormal subgroups S and T such that $G = AT$ and $A \cap T \leq S \leq A$. In the present paper, we study conditions under which a group is σ-solvable. In particular, we prove that a group G is σ-solvable if and only if at least one of the following two conditions is satisfied: (i) G has a complete Hall σ-set $\mathcal{H}$ all of whose elements are weakly K-$\mathfrak{S}_\sigma$-subnormal in G; (ii) in every maximal chain of subgroups $\cdots < M_3 < M_2 < M_1 < M_0 = G$ of the groups G, at least one of the subgroups M_3, M_2, or M_1 is weakly K-$\mathfrak{S}_\sigma$-subnormal in G.

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1. INTRODUCTION

All the groups considered in the paper are finite, and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, and σ is some partition of $\mathbb{P}$, i.e.,

$$\sigma = \{\sigma_i \mid i \in I\},$$

where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$. The symbol $\pi(G)$ denotes the set of all prime divisors of $|G|$, and $\sigma(G) = \{\sigma_i \mid \sigma_i \cap \pi(G) \neq \varnothing\}$. We say that a chain of subgroups

$$\cdots < M_3 < M_2 < M_1 < M_0 = G$$

of G is a *maximal chain* in G if M_i is a maximal subgroup in M_{i-1} for all i. The groups A and B are called *groups of equal order* if $|A| = |B|$. If A is a subgroup of a group G, then A_G is the largest normal subgroup of G contained in A.

Let $\mathfrak{F}$ be a class of groups. Then a subgroup A of a group G is said to be $\mathfrak{F}$-*subnormal in the sense of Kegel* [1] or K-$\mathfrak{F}$-*subnormal in G* [2] if G contains a series of subgroups

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

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such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i} \in \mathfrak{F}$ for all $i = 1, \dots, t$.

Recall some notions of the theory of σ-properties of a group [3]–[6].

A group G is said to be *σ-primary* if G is a σ_i-group for some $i = i(G)$; *σ-solvable* if every chief factor of G is σ-primary. We use the symbols $\mathfrak{S}_\sigma$ and $\mathfrak{S}$ to denote the classes of all σ-solvable and all solvable groups, respectively; $G^{\mathfrak{S}_\sigma}$ stands for the intersection of all normal subgroups N of a group G with $G/N \in \mathfrak{S}_\sigma$.

A set of subgroups $\mathcal{H}$ of a group G is called a *complete Hall σ-set* of G if every element $\neq 1$ of the set $\mathcal{H}$ is a Hall σ_i-subgroup of G for some i and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$.

Definition 1. We say that a subgroup A of a group G is *weakly K-$\mathfrak{S}_\sigma$-subnormal* in G if there are K-$\mathfrak{S}_\sigma$-subnormal subgroups T and S such that $G = AT$ and $A \cap T \leq S \leq A$.

Remark 1. (i) Each K-$\mathfrak{S}_\sigma$-subnormal subgroup $S = S \cap G$ is weakly K-$\mathfrak{S}_\sigma$-subnormal in the group G, since G is a K-$\mathfrak{S}_\sigma$-subnormal subgroup of G.

(ii) A subgroup A of a group G is said to be *σ-subnormal* in G [3] if there is a series of subgroups $A = A_0 \leq A_1 \leq \cdots A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or the quotient $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \dots, n$. Every subnormal subgroup is σ-subnormal, and every σ-subnormal subgroup is K-$\mathfrak{S}_\sigma$-subnormal in the group.

(iii) A subgroup S of a group G is said to be *σ-permutable* in G [3] if G has a complete Hall σ-set $\mathcal{H}$ such that $SH^x = H^xS$ for all $H \in \mathcal{H}$ and all $x \in G$.

By Theorem B of [3], every σ-permutable subgroup is σ-subnormal and, therefore, K-$\mathfrak{S}_\sigma$-subnormal in the group.

Now consider the following example.

Example 1. (i) A subgroup A of a group G is said to be *weakly σ-permutable* in G ([7], [8]) if G contains a σ-permutable subgroup S and a σ-subnormal subgroup T such that $G = AT$ and $A \cap T \leq S \leq A$. By Remark 1 (ii, iii), every weakly σ-permutable subgroup is weakly K-$\mathfrak{S}_\sigma$-subnormal in the group.

(ii) In the general case, the class of all weakly K-$\mathfrak{S}_\sigma$-subnormal subgroups is wider than the class of all weakly σ-permutable subgroups. Let $\sigma = \{\{2\}, \{3\}, \dots\}$. In this case, every σ-subnormal subgroup is subnormal, and any σ-permutable subgroup H is S-permutable in the group ([9], [10]), i.e., $HP = PH$ for all Sylow subgroups P of the group.

Now let A be a non-Abelian group of order p^3 of a simple odd exponent p. Let $G = A \wr C_2 = B \rtimes C_2$, where B is the base of the regular wreath product G. Let $L = \Phi(A_1)$, where A_1 is the first copy of the group A in B. Then $|L| = p$, $L \leq \Phi(B)$, and L is subnormal, and hence weakly K-$\mathfrak{S}$-subnormal in G, considering Remark 1.2 (ii). It is also clear that L is a nonnormal subgroup of G. Now let us show that L is not weakly S-permutable in G. Suppose that G contains a subnormal subgroup T and an S-permutable subgroup S such that $G = LT$ and $L \cap T \leq S \leq L$. Then $B = L(T \cap B)$, and thus $T \cap B = B$, since $L \leq \Phi(B)$. Hence $L \cap T = L = S$ is S-permutable in G. Then L is normal in G by [9, Lemma 1.2.16]. This contradiction shows that L is not weakly S-permutable in G.

(iii) A subgroup A of a group G is said to be *c-normal* in G [11] if $AT = G$ and $A \cap T \leq A_G$ for some normal subgroup T of G. Consequently, any c-normal subgroup is weakly K-$\mathfrak{S}_\sigma$-subnormal in the group for every partition σ of $\mathbb{P}$.

(iv) A subgroup M of a group G is said to be

(a) *modular* in G [12] if M is a modular element (in the sense of Kurosh [12, p. 43]) of the lattice of all subgroups $\mathcal{L}(G)$ of G, i.e., the following conditions hold:

(1) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leq G$ and $Z \leq G$ such that $X \leq Z$;

(2) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G$ and $Z \leq G$ such that $M \leq Z$;

(b) *submodular* in G ([13], [14]) if G contains a series of subgroups $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that A_{i-1} is a modular subgroup of A_i for all $i = 1, \dots, t$.

It follows from the main result of the theory of modular subgroups [12, Theorem 5.1.14] that every submodular subgroup is K-$\mathfrak{S}$-subnormal and, therefore, weakly K-$\mathfrak{S}$-subnormal in the group.

Although the concept of σ-solvable group was first introduced in a recent paper [3], this concept proved to be very useful in the analysis of many open questions (see, for example, [3]–[8], [15]–[26]), and thus the problem of finding conditions under which a group is σ-solvable is very interesting and relevant. In this paper, we prove the following result.

Theorem 1. *The following conditions are equivalent*:

(i) *a group G is σ-solvable*;

(ii) *every subgroup of G is K-$\mathfrak{S}_\sigma$-subnormal*;

(iii) *G has a complete Hall σ-set $\mathcal{H}$ all of whose elements are weakly K-$\mathfrak{S}_\sigma$-subnormal in G*;

(iv) *in every maximal chain $\cdots < M_3 < M_2 < M_1 < M_0 = G$ of G, at least one of the subgroups M_3, M_2 or M_1 is weakly K-$\mathfrak{S}_\sigma$-subnormal in G*;

(v) *$i_{K,\mathfrak{S}_\sigma}(G) \leq 2|\sigma(G)|$, where $i_{K,\mathfrak{S}_\sigma}(G)$ is the number of classes of non-K-$\mathfrak{S}_\sigma$-subnormal subgroups of G of equal order.*

Theorem 1 covers many known results. In particular, taking into account Remark 1 and Example 1, we see that the following well-known results are special cases of this theorem.

Corollary 1 (Guo, Skiba [19]). *If, in every maximal chain $\cdots < M_3 < M_2 < M_1 < M_0 = G$ of a group G, at least one of the subgroups M_3, M_2, or M_1 is σ-subnormal in G, then G is σ-solvable.*

Corollary 2 (Zhang, Wu, Guo [7]). *If G has a complete Hall σ-set $\mathcal{H}$ all of whose elements are weakly σ-permutable in G, then G is σ-solvable.*

Corollary 3 (Zimmermann [14]). *If, in every maximal chain $\cdots < M_3 < M_2 < M_1 < M_0 = G$ of a group G, at least one of the subgroups M_3, M_2, or M_1 is submodular in G, then G is solvable.*

Corollary 4 (Spencer [27]). *If, in every maximal chain $\cdots < M_3 < M_2 < M_1 < M_0 = G$ of a group G, at least one of the subgroups M_3, M_2, or M_1 is subnormal in G, then G is solvable.*

Corollary 5 (Schmid [28]). *A group G is solvable if each of its* 3*-maximal subgroups is modular.*

Corollary 6 (Kovaleva [15]). *If $i_\sigma(G) \leq 2|\sigma(G)|$, where $i_\sigma(G)$ stands for the number of classes of non-σ-subnormal subgroups of equal order of a group G, then G is σ-solvable.*

Corollary 7 (Lu, Meng [29]). *If the number of conjugacy classes of nonsubnormal subgroups of a group G does not exceed $2|\pi(G)|$, then G is solvable.*

2. PROOF OF THEOREM 1

Lemma 1. (1) *The class $\mathfrak{S}_\sigma$ is closed with respect to taking products of normal $\mathfrak{S}_\sigma$-subgroups, homomorphic images, and subgroups. Moreover, any extension of a σ-solvable group with the help of a σ-solvable group is σ-solvable.*

(2) *A group is σ-solvable if and only if all its maximal subgroups are K-$\mathfrak{S}_\sigma$-subnormal.*

Proof. (1) This assertion is obvious.

(2) It is clear that, in every σ-solvable group, all its maximal subgroups are K-$\mathfrak{S}_\sigma$-subnormal.

Suppose now that all maximal subgroups of a group G are K-$\mathfrak{S}_\sigma$-subnormal in G. Then $G/M_G \in \mathfrak{S}_\sigma$ for every maximal subgroup M of G. Therefore, $G/\Phi(G) \in \mathfrak{S}_\sigma$ and, consequently, $G \in \mathfrak{S}_\sigma$, by part (1).

This completes the proof of the lemma. ☐

From Lemma 1 and from the main result of [1], we obtain the following assertion.

Lemma 2. *The set of all K-$\mathfrak{S}_\sigma$-subnormal subgroups of a group G forms a sublattice of the lattice of all subgroups of G.*

Lemma 3. *Let A, H, and N be subgroups of G, where A is K-$\mathfrak{S}_\sigma$-subnormal and N is normal in G. Then*

(1) *$A \cap H$ is K-$\mathfrak{S}_\sigma$-subnormal in H;*

(2) *AN/N is K-$\mathfrak{S}_\sigma$-subnormal in G/N;*

(3) *if $N \le H$ and H/N is K-$\mathfrak{S}_\sigma$-subnormal in G/N, then H is K-$\mathfrak{S}_\sigma$-subnormal in G;*

(4) *the subgroup $A^{\mathfrak{S}_\sigma}$ is subnormal in G;*

(5) *if A is σ-solvable and N is a non-σ-primary minimal normal subgroup of G, then $A \le C_G(N)$.*

Proof. (1)–(4). These assertions are corollaries of Lemmas 6.1.6, 6.1.7, and 6.1.9 in [2].

(5) Assume that this assertion fails to hold; let G be a counterexample of minimal order. Then $A \ne G$. By assumption, there is a series of subgroups

$$A = A_0 \le A_1 \le \cdots \le A_r = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-solvable for all $i = 1, \dots, r$. Let $M = A_{r-1}$. Without loss of generality, we can assume that $M \ne G$.

Let $E = NA$. Let us first assume that $E < G$. It is clear that $N = N_1 \times \cdots \times N_n$ for some minimal normal subgroups $N_1, \dots, N_n$ of E and N_i is not σ-primary for all i. By Lemma 3(1), A is K-$\mathfrak{S}_\sigma$-subnormal in E and, therefore, due to the choice of G, this means that $A \le C_E(N_i)$ for all i and, therefore, $A \le C_E(N)$. Hence $NA = E = G$. Then $N \nleq M$, and hence G/M_G is not σ-solvable, since $N \simeq NM_G/M_G$ is not σ-primary. This implies that M is normal in G, and hence $N \cap M = 1$. This implies that $[N, M] = 1$, and thus $A \le C_G(N)$.

This completes the proof of the lemma. □

Lemma 4. *Let A, H, and N be subgroups of a group G, where A is weakly K-$\mathfrak{S}_\sigma$-subnormal and N is normal in G.*

(1) *If either $N \le A$ or $(|N|, |A|) = 1$, then AN/N is weakly K-$\mathfrak{S}_\sigma$-subnormal in G/N.*

(2) *If $N \le H$ and H/N is weakly K-$\mathfrak{S}_\sigma$-subnormal in G/N, then H is weakly K-$\mathfrak{S}_\sigma$-subnormal in G.*

(3) *If $A \le E \le G$, then A is weakly K-$\mathfrak{S}_\sigma$-subnormal in E.*

Proof. Let T and S be K-$\mathfrak{S}_\sigma$-subnormal subgroups of G such that $G = AT$ and $A \cap T \le S \le A$.

(1) First, note that

$$(AN/N)(TN/N) = ATN/N = G/N, \qquad SN/N \le AN/N,$$

where the subgroups TN/N and SN/N are K-$\mathfrak{S}_\sigma$-subnormal in G/N by Lemma 3(2). Thus, it remains only to show that

$$(AN/N) \cap (TN/N) \le SN/N.$$

If $N \le A$, then

$$(AN/N) \cap (TN/N) = (A \cap TN)/N = N(A \cap T)/N \le SN/N.$$

Suppose now that $(|N|, |A|) = 1$. Since $G = AT$, it follows that $|G : T|$ divides $|A|$. Hence $|NT : T|$ divides $|A|$. However, $|NT : T| = |N : N \cap T|$ divides $|N|$. Thus, $N \leq T$, and hence

$$(AN/N) \cap (TN/N) = (AN/N) \cap (T/N) = (AN \cap T)/N = N(A \cap T)/N \leq SN/N.$$

Therefore, the subgroup AN/N is weakly K-$\mathfrak{S}_\sigma$-subnormal in G/N.

(2) By assumption, G/N contains K-$\mathfrak{S}_\sigma$-subnormal subgroups Z/N and D/N such that

$$G/N = (H/N)(Z/N), \qquad (H \cap Z)/N = (H/N) \cap (Z/N) \leq D/N \leq H/N.$$

Then $G = HZ$ and $H \cap Z \leq D \leq H$, where Z and D are K-$\mathfrak{S}_\sigma$-subnormal subgroups of G by Lemma 3 (3) and, therefore, H is weakly K-$\mathfrak{S}_\sigma$-subnormal in G.

(3) Note that

$$E = E \cap AT = A(E \cap T), \qquad A \cap (E \cap T) = A \cap T \leq S \leq A,$$

where $E \cap T$ and S are weakly K-$\mathfrak{S}_\sigma$-subnormal in E by Lemma 3 (1). Hence A is weakly K-$\mathfrak{S}_\sigma$-subnormal in E.

This completes the proof of the lemma. □

Proof of Theorem 1. (i) ⇒ (ii) Let $H \leq M < G$, where M is a maximal subgroup of G. Then M is K-$\mathfrak{S}_\sigma$-subnormal in G by Lemma 1 (2). On the other hand, H is K-$\mathfrak{S}_\sigma$-subnormal in M by induction. Hence H is K-$\mathfrak{S}_\sigma$-subnormal in G.

The implication (ii) ⇒ (i) follows from Lemma 1 (2).

Since a σ-solvable group is σ_i-separable, and hence has a Hall σ_i-subgroup for all i, it follows from the implication (ii) ⇒ (i) and from Remark 1 (i) that (ii) ⇒ (iii), (iv), (v).

(iii) ⇒ (i) Suppose that this assertion fails to hold; let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \dots, H_t\}$. Then $t > 1$. Without loss of generality, we can assume that H_i is a nonidentity σ_i-group for all $i = 1, \dots, t$. According to the condition, for any i, the group G has K-$\mathfrak{S}_\sigma$-subnormal subgroups T_i and S_i such that $G = H_iT_i$ and $H_i \cap T_i \leq S_i \leq H_i$.

Let us first show that G/L is σ-solvable for every nontrivial σ-primary normal subgroup L of G and, therefore, there is no σ-primary normal subgroup in G. Indeed, suppose that G contains a minimal normal subgroup L which is a σ_i-group for some i. It can readily be seen that $\{H_1L/L, \dots, H_tL/L\}$ is a complete Hall σ-set of G/L. Moreover, $L \leq H_i$ and $(|L|, |H_j|) = 1$ for all $j \neq i$. However, then H_kL/L is weakly K-$\mathfrak{S}_\sigma$-subnormal in G/L for all $k = 1, \dots, t$ by Lemma 4 (1). Then condition (iii) holds for G/L, and hence G/L is σ-solvable due to the choice of G. Thus, G is σ-solvable, which contradicts the choice of G. Hence every minimal normal subgroup G is not σ-primary.

This implies that $S^{\mathfrak{S}_\sigma} \neq 1$ for any nontrivial K-$\mathfrak{S}_\sigma$-subnormal subgroup S of G by Lemma 3 (5) and, therefore, $S_i = 1$ for all i, and T_1 is not a σ_2-group. Hence $t > 2$, and T_i is a supplement to H_i to G. Therefore, T_i is a Hall σ_i'-subgroup in G. Then $T_2 \cap \dots \cap T_t$ is a nonidentity Hall σ_1-subgroup of G according to [31, A, 1.6 (b)], and this subgroup is K-$\mathfrak{S}_\sigma$-subnormal in G by Lemma 2, a contradiction. Therefore, the implication (iii) ⇒ (i) holds.

(iv) ⇒ (i) Assume that this assertion fails to hold; let G be a counterexample of minimal order.

(1) *If N is a minimal normal subgroup of G, then G/N is a σ-solvable group. Thus, N is not a σ-primary group. Moreover, N is a unique minimal normal subgroup of G, $C_G(N) = 1$, and $L \leq N$ for any minimal subnormal subgroup L of G.*

By Lemma 4 (1), the hypothesis is true for G/N, which means that G/N is σ-solvable by the choice of G. Consequently, N is not σ-primary.

If G contains a minimal normal subgroup $R \neq N$, then G/N and G/R are σ-solvable groups, and hence it follows from the isomorphisms $R \simeq R/1 = R/(R \cap N) \simeq RN/N$ that R is a σ-primary group, which contradicts the fact proved above. Hence N is a unique minimal normal subgroup of G and $C_G(N) = 1$, since the subgroup N is non-Abelian.

It follows from [31, A, 13.4] that $N \leq N_G(L)$. Moreover, $L \nleq C_G(N) = 1$ and, therefore, $L \leq N$. Hence assertion (1) holds.

(2) *If p is an arbitrary odd prime dividing $|N|$ and N_p is the Sylow p-subgroup of N, then $N_p = N \cap G_p$, $G_p \leq N_G(N_p) \leq M$, and $G = NM$ for some Sylow p-subgroup G_p and some maximal subgroup M of G. Hence p does not divide $|G : M|$, and $M_G = 1$.*

By Frattini's argument, $G = NN_G(N_p)$. Since N is not an Abelian group, it follows that $N_G(N_p) \neq G$. Then $N \nleq M$ for a maximal subgroup M of G containing $N_G(N_p)$. Hence $G = NM$ and $M_G = 1$. Moreover, if $N_p \leq G_p$, where G_p is a Sylow p-subgroup of G, then $N_p = N \cap G_p$, and hence $G_p \leq N_G(N_p) \leq M$.

(3) *The intersection $D := M \cap N$ is not nilpotent. In particular, $D \nleq \Phi(M)$.*

Assume that D is nilpotent; let N_p be the Sylow p-subgroup of D. Then N_p is normal in M, because N_p is a characteristic subgroup of D, and D is normal in M. Therefore, the subgroup $Z(J(N_p))$ is normal in M. Since $M_G = 1$, it follows that $N_G(Z(J(N_p))) = M$. Then $N_N(Z(J(N_p))) = D$ is a nilpotent group. This implies that N has a normal p-complement by the Glauberman–Thompson theorem, since p is odd. However, then N is an p-group, which contradicts assertion (1). Therefore, (3) holds.

(4) *$V^{\mathfrak{S}_\sigma} \neq 1$ for every nontrivial K-$\mathfrak{S}_\sigma$-subnormal subgroup V of G.*

Indeed, $V^{\mathfrak{S}_\sigma} = 1$ implies that V is σ-solvable and, therefore, $1 < V \leq C_G(N) = 1$ by Lemma 3(5). Hence (4) holds.

(5) *If H is K-$\mathfrak{S}_\sigma$-subnormal in G and is contained in M, then $H = 1$.* (5)

Let $W = H^{\mathfrak{S}_\sigma}$ be the σ-solvable residual of H. Then W is subnormal in G by Lemma 3(4) and, therefore,

$$W^G = W^{NM} = W^M \leq M_G = 1$$

by [31, A, 14.3]. Hence $W = 1$. Thus, H is σ-solvable and, therefore, $H = 1$ by part (4).

(6) *If H is a weakly K-$\mathfrak{S}_\sigma$-subnormal subgroup of G and is contained in M, then $G = HT$ and $H \cap T = 1$ for some K-$\mathfrak{S}_\sigma$-subnormal subgroup T of G* (this follows from part (5) and from the definition of weakly K-$\mathfrak{S}_\sigma$-subnormal subgroup).

(7) *M is not weakly K-$\mathfrak{S}_\sigma$-subnormal in G. In particular, M is σ-solvable.*

Suppose that M is weakly K-$\mathfrak{S}_\sigma$-subnormal in G. Then $G = MT$ and $M \cap T = 1$ for some K-$\mathfrak{S}_\sigma$-subnormal subgroup T of G by assertion (6). Moreover, $W := T^{\mathfrak{S}_\sigma} \neq 1$ according to part (4), and W is a subnormal subgroup of G according to Lemma 3(4).

Let now L be a minimal subnormal subgroup of G contained in W. Then L is a minimal normal subgroup of N by part (1). Hence p divides $|L|$, where $|L|$ divides $|T| = |G : M|$. However, p does not divide $|G : M|$ by part (2). This contradiction completes the proof of the first part of (7). Then, in any maximal chain

$$\cdots < M_3 < M_2 < M_1 = M < G$$

of G, one of the subgroups M_3 and M_2 is weakly K-$\mathfrak{S}_\sigma$-subnormal in G by assumption. Hence this subgroup is weakly K-$\mathfrak{S}_\sigma$-subnormal in M by Lemma 4(3). Thus, the conjecture holds for M, and thus M is σ-solvable by the choice of G.

(8) *The relation $N < G$ holds.*

Suppose that $N = G$ is a non-Abelian simple group; let H be an arbitrary proper K-$\mathfrak{S}_\sigma$-subnormal subgroup of G. Suppose that $H \neq 1$.

Then G has a proper nonidentity subgroup V such what $H \leq V$ and either V is normal in G or G/V_G is a σ-solvable group. However, the first condition is impossible, since the group G is simple. Hence $V_G = 1$, and then $G/V_G = G/1 \simeq G$ is a σ-solvable group, which contradicts the choice of G. Thus, any proper K-$\mathfrak{S}_\sigma$-subnormal subgroup of G is trivial.

Let Q be a Sylow q-subgroup of G, where q is the least prime dividing $|G|$, and let L be a maximal subgroup of G containing Q. Then, taking into account [32, IV, 2.8], we obtain $|Q| > q$. Let V be a maximal subgroup of Q and S be a maximal subgroup of V. Then there is a 3-maximal subgroup $1 < W$ of G such that $W \leq Q$. Indeed, if $S \neq 1$, then this is obvious. On the other hand, if $S = 1$, then Q is an

Abelian group and, therefore, $Q < L$ by [32, IV, 7.4]. Therefore, in this case, there exists a 3-maximal subgroup W of G such that $V \le W \le Q$.

Due to condition (iv), G has a weakly K-$\mathfrak{S}_\sigma$-subnormal subgroup U such what $W \le U < G$. Hence G contains K-$\mathfrak{S}_\sigma$-subnormal subgroups T and R such that $G = UT$ and $U \cap T \le R \le U$. Then $T = G$, and thus $U = U \cap T \le R \le U$.

Hence $U = R$ is K-$\mathfrak{S}_\sigma$-subnormal in G. Then $U = 1$, a contradiction. Thus, assertion (8) holds.

(9) *If $G_p \le V \le M$, then V is not weakly K-$\mathfrak{S}_\sigma$-subnormal in G.*

Assume that V is weakly K-$\mathfrak{S}_\sigma$-subnormal in G. Then $G = VT$ and $V \cap T = 1$ for some K-$\mathfrak{S}_\sigma$-subnormal subgroup T of G by assertion (6). Then $1 < T < G$, and the subgroup $T^{\mathfrak{S}_\sigma}$ is nontrivial by part (4). Moreover, this subgroup is subnormal in G by Lemma 3(4). Hence $Z \le T$ for a minimal subnormal subgroup Z of G contained in $T^{\mathfrak{S}_\sigma}$. Then Z is a p'-group, because $V \cap T = 1$ and $G_p \le V$. On the other hand, Z is a minimal normal subgroup of N by part (1) and, therefore, p divides $|Z|$. This contradiction completes the proof of (9).

(10) The *group $M = D \rtimes L$* is *solvable, where L is a group of prime order*.

By [32, IV, 7.4], $G_p \le L$ for some maximal subgroup L of M, since $p > 2$ and G is unsolvable. Let us first show that $G_p = L$. Indeed, suppose that $G_p \le V$ for some maximal subgroup V of L. Then, in every maximal chain

$$\cdots < V = M_i < \cdots < M_3 < M_2 < M_1 = M < M_0 = G$$

of G that contains V below M_3, all subgroups M_3, M_2, and M_1 are not weakly K-$\mathfrak{S}_\sigma$-subnormal in G by part (9), which contradicts the condition. Hence $G_p = L$ is a maximal subgroup of M and, therefore, M is solvable according to [32, IV, 7.4]. If $G_p \le D$, then $D = M \cap N = M$, since $G_p \ne D$ by part (2), whence it follows that $N = G$, which contradicts part (8). Hence $G_p \not\le D$, and thus $DL = M$. Therefore, $D \cap L = 1$ in the case of $|G_p| = p$, whence it follows that $M = D \rtimes L$, and thus assertion (10) holds because M is solvable.

Finally, suppose that $|G_p| > p$. First of all, note what L is not weakly σ-subnormal in G by part (9). However, on the other hand, M is also not weakly K-$\mathfrak{S}_\sigma$-subnormal in G by (7). Therefore, every maximal subgroup of $L = G_p$ is weakly K-$\mathfrak{S}_\sigma$-subnormal in G.

It follows from $G_p \not\le D$ that $N_p = N \cap G_p \le V$ for some maximal subgroups V of G_p. Since V is weakly K-$\mathfrak{S}_\sigma$-subnormal in G, it follows that V has a K-$\mathfrak{S}_\sigma$-subnormal complement T in G by part (7). Hence $T \cap N_p = 1$. Lemma 3(4) and part (4) imply that $Z \le T^{\mathfrak{S}_\sigma} \le T$ for some minimal subnormal subgroup Z of G. Then $Z \cap N_p = 1$ and, therefore, $Z \not\le N$, which contradicts part (1). Therefore, assertion (10) holds.

A concluding contradiction for (iv) $\Rightarrow$ (i). It follows from part (10) that $|D|$ is a power of a prime, which contradicts part (2). Therefore, the implication (iv) $\Rightarrow$ (i) holds.

(v) $\Rightarrow$ (i) Suppose that this assertion fails to hold; let G be a counterexample of minimal order.

First, note that, if $H \le A \le G$, where H is not K-$\mathfrak{S}_\sigma$-subnormal in A, then H is not K-$\mathfrak{S}_\sigma$-subnormal in G by Lemma 3(1). Hence $i_{K,\mathfrak{S}_\sigma}(A) \le i_{K,\mathfrak{S}_\sigma}(G)$ for every subgroup A of G. Moreover, if $H/N \le G/N$, where H/N is not K-$\mathfrak{S}_\sigma$-subnormal in G/N, then H is not K-$\mathfrak{S}_\sigma$-subnormal in G by Lemma 3(2). Hence $i_{K,\mathfrak{S}_\sigma}(G/N) \le i_{K,\mathfrak{S}_\sigma}(G)$.

(1) *If N is a nonidentity normal subgroup of G such that $|\sigma(G/N)| = |\sigma(G)|$, then G/N is σ-solvable. Moreover, if L is a proper subgroup of G such that $|\sigma(L)| = |\sigma(G)|$, then L is σ-solvable.*

Since

$$i_{K,\mathfrak{S}_\sigma}(G/N) \le i_{K,\mathfrak{S}_\sigma}(G) \le 2|\sigma(G)| = 2|\sigma(G/N)|,$$

it follows that the conjecture holds for G/N and, therefore, G/N is σ-solvable by the choice of G. Similarly, it follows from $i_{K,\mathfrak{S}_\sigma}(L) \le i_{K,\mathfrak{S}_\sigma}(G)$ that L is σ-solvable.

(2) *The equality $O_{\sigma_i}(G) = 1$ holds for every $\sigma_i \in \sigma(G)$.*

Suppose that $O := O_{\sigma_i}(G) \neq 1$ for some i. Let us first show that $O \nleq \Phi(G)$. Indeed, if $O \leq \Phi(G)$, then $\pi(G/O) = \pi(G)$ and, therefore, $\sigma(G/O) = \sigma(G)$, and thus G/O is σ-solvable by part (1). Consequently, G is σ-solvable; a contradiction. Hence $O \nleq \Phi(G)$, and thus there is a maximal subgroup M of G such that $G = OM$. Then M is not a σ-solvable group and, therefore, $|\sigma(M)| \neq |\sigma(G)|$ by part (1). On the other hand, $|G : M| = |O : Q \cap M|$ is a σ_i-number and, therefore, $|\sigma(M)| = |\sigma(G)| - 1$. This implies that $M \cap O = 1$ and, therefore, O is a Hall σ_i-subgroup of G.

The inequality $i_\sigma(M) \geq 2|\sigma(M)| + 1$ holds by the choice of G. Let $i_{K,\mathfrak{S}_\sigma}(M) = k$. If $k = 0$, then all subgroups of M are K-$\mathfrak{S}_\sigma$-subnormal in M and, therefore, M is σ-solvable by implication (ii) $\Rightarrow$ (i). Thus, $k \neq 0$. Let $T_1, \dots, T_k$ be representatives of the classes of non-K-$\mathfrak{S}_\sigma$-subnormal subgroups of M of equal order. Since $T_j = T_j(O \cap M) = OT_j \cap M$ is not a K-$\mathfrak{S}_\sigma$-subnormal subgroup of M, it follows that OT_j is not K-$\mathfrak{S}_\sigma$-subnormal in G by Lemma 3 (1). Moreover, since $M \cap O = 1$ and $|T_{j_1}| \neq |T_{j_2}|$ for $j_1 \neq j_2$, it follows that $|OT_{j_1}| \neq |OT_{j_2}|$ for all $j_1 \neq j_2$.

Thus, taking into account the inequality $i_{K,\mathfrak{S}_\sigma}(G) \geq i_{K,\mathfrak{S}_\sigma}(M)$ and counting the subgroups OT_i, we obtain

$$i_{K,\mathfrak{S}_\sigma}(G) \geq 2i_{K,\mathfrak{S}_\sigma}(M) \geq 2(2|\sigma(M)| + 1).$$

However, since $2 \leq |\sigma(M)| = |\sigma(G)| - 1$, it follows that

$$i_{K,\mathfrak{S}_\sigma}(G) \geq 2(|\sigma(G)| - 1) + 2|\sigma(M)| + 2 \geq 2|\sigma(G)| + 4.$$

This contradiction completes the proof of assertion (2).

(3) *If H is a nonidentity σ-solvable subgroup of G, then H is not K-$\mathfrak{S}_\sigma$-subnormal in G* (this follows from part (2) and Lemma 3 (5)).

(4) *The equality $O^{\sigma_i}(G) = G$ holds for every $\sigma_i \in \sigma(G)$.*

Suppose that $O = O^{\sigma_i}(G) < G$ for some i. Since G/O is a σ_i-group, it follows that G/O is σ-solvable. Hence

$$i_{K,\mathfrak{S}_\sigma}(O) \geq 2|\sigma(O)| + 1$$

by the choice of G.

By assertion (1), $|\sigma(O)| = |\sigma(G)| - 1$ and, therefore, O is a Hall σ_i'-subgroup of G. Therefore, by the Schur–Zassenhaus theorem, there is a complement H for O in G, H is a Hall σ_i-subgroup of G, and every Hall σ_i-subgroup of G is conjugate to H.

Note that H is simple. Indeed, suppose what $N \neq 1$ is a proper normal subgroup of H. Then ON is a proper normal subgroup of G, and $|\sigma(ON)| = |\sigma(G)|$. Hence ON is σ-solvable by assertion (1), which implies that G is σ-solvable, since G/ON is a σ_i-group, a contradiction. Thus, H is a simple group.

Since O and H are Hall subgroups of G, it follows that either O or H is solvable by the Feit–Thompson theorem on the solvability of groups of odd order. However, if O is solvable, then O is σ-solvable and, therefore, G is σ-solvable. Hence H is solvable, and thus $|H| = q$ for some $q \in \pi(G)$. Consequently, H is not K-$\mathfrak{S}_\sigma$-subnormal in G by part (3). Moreover, every nonidentity Sylow subgroup of O is not K-$\mathfrak{S}_\sigma$-subnormal in O. Let P be some Sylow p-subgroup of O, $p \in \pi(O)$. According to Frattini's argument, $G = ON_G(P)$. Since q does not divide $|O|$, it follows that q divides $|N_G(P)|$. Hence there is an $x \in G$ such that $H^x \leq N_G(P)$ and, therefore, H^xP is a subgroup of G. Since

$$P = P(H^x \cap O) = H^xP \cap O$$

is not K-$\mathfrak{S}_\sigma$-subnormal in O, it follows that H^xP is not K-$\mathfrak{S}_\sigma$-subnormal in G by Lemma 3 (1). We also note that $|\pi(O)| \geq 2$ by the p^aq^b-Burnside theorem. Therefore, the number of subgroups of the form H^xP is not less than 2. Thus, taking into account that

$$i_{K,\mathfrak{S}_\sigma}(G) \geq i_{K,\mathfrak{S}_\sigma}(O) \geq 2|\sigma(O)| + 1 = 2(|\sigma(G)| - 1) + 1 = 2(|\sigma(G)| - 1$$

and taking into account the subgroups H^xP and H, we obtain

$$i_{K,\mathfrak{S}_\sigma}(G) \geq i_{K,\mathfrak{S}_\sigma}(O) + 2 + 1 \geq 2|\sigma(G)| + 2.$$

This contradiction completes the proof of assertion (4).

A concluding contradiction for (v) $\Rightarrow$ (i). Since G is unsolvable, it follows that G has a noncyclic Sylow subgroup P by [32, VI, 10.3]. Then a maximal subgroup V of P is not identity and, therefore, V is not K-$\mathfrak{S}_\sigma$-subnormal in G by part (3). Moreover, every nonidentity Sylow subgroup of G is not K-$\mathfrak{S}_\sigma$-subnormal in G. Therefore, $m \geq |\pi(G)| + 1$ for the number m of all classes of non-K-$\mathfrak{S}_\sigma$-subnormal primary subgroups of equal order.

By assertion (4), G is not p-nilpotent for any $p \in \pi(G)$. Therefore, taking into account [32, III, 5.2], we see that there is a p-closed Schmidt subgroup E of G with $p \in \pi(E)$ for every $p \in \pi(G)$.

Now let $p > q$ be distinct prime divisors of $|G|$. Let V be a p-closed Schmidt subgroup with $p \in \pi(V)$, and let W be a q-closed Schmidt subgroup with $q \in \pi(W)$. By [32, IV, 5.4], $V = V_p \rtimes V_r$ and $W = W_q \rtimes W_t$, where V_p is a Sylow p-subgroup and V_r is a cyclic Sylow r-subgroup of V ($r \neq p$); W_q is a Sylow q-subgroup and W_t is a cyclic Sylow t-subgroup of W ($t \neq q$). By part (4), V and W are not K-$\mathfrak{S}_\sigma$-subnormal in G. If $|V| \neq |W|$, then V and W are representatives of two distinct classes of non-K-$\mathfrak{S}_\sigma$-subnormal subgroups in G of equal order.

Now suppose that $|V| = |W|$. In this case, $r = q$ and $t = p$. Let T be a maximal subgroup of V such that $V_p \leq T$. If $V_p = T$, then $|V_q| = q$, and thus, taking into account that $|V| = |W|$, we see that $|W_q| = q$. Hence W is supersolvable. As is well known, every supersolvable group is Ore dispersive. Therefore, W is Ore dispersive, which implies that $q > p$, a contradiction. Thus, $V_p < T$. Moreover, T is not K-$\mathfrak{S}_\sigma$-subnormal in G by part (4), and T is not a primary group. Thus, for every $p \in \pi(G)$, there is a class of non-K-$\mathfrak{S}_\sigma$-subnormal subgroups in G of equal order, and each representative of any of these classes is not a primary group. It is also clear that $r \geq \pi(G)|$ for the number r of such classes. Hence

$$i_{K,\mathfrak{S}_\sigma}(G) \geq |\pi(G)| + 1 + |\pi(G)| \geq 2|\sigma(G)| + 1.$$

This contradiction completes the proof of the implication (v) $\Rightarrow$ (i).

This completes the proof of the theorem. □

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