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Joint Universality of Certain Dirichlet Series

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Abstract—In this paper, we define the Dirichlet series $\zeta_{u_T j}(s)$, $j = 1, \dots, r$, absolutely converging in the half-plane $\operatorname{Re} s > 1/2$ and prove that the set of shifts $(\zeta_{u_T 1}(s + ia_1\tau), \dots, \zeta_{u_T r}(s + ia_r\tau))$ approximating a given set of analytic functions has a positive density on the interval $[T, T+H]$, $H = o(T)$ as $T \to \infty$. Here $a_1, \dots, a_r \in \mathbb{R}$ are algebraic numbers linearly independent over $\mathbb{Q}$ and $u_T \to \infty$ as $T \to \infty$.

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1. INTRODUCTION

Continuing the research of Bohr and Courant on the density of the set of values of the Riemann zeta function $\zeta(s)$, $s = \sigma + it$, S. M. Voronin discovered the universality property of this function in [1]. Let $0 < r < 1/4$. He proved that if the function $f(s)$ is analytic in a disk $|s| < r$, and it is continuous and has no zeros up to the boundary of this disk, then, for every $\varepsilon > 0$, there exists a $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s| \le r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Since the space of analytic functions is infinite-dimensional, Voronin's theorem is an infinite-dimensional generalization of the result due to Bohr and Courant [2] that, for $1/2 < \sigma \le 1$, states that the set

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\}$$

is everywhere dense in $\mathbb{C}$.

Voronin's theorem has a more general form. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, let $\mathcal{K}$ be the class of compact subsets of the strip D having connected complements, and let $H_0(K)$, $K \in \mathcal{K}$, be the class of continuous functions without zeros in K and analytic inside K. In that case (see, for example, [3]) if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} > 0. \tag{1.1}$$

Here $\operatorname{meas} A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The last inequality shows that there exists infinitely many shifts $\zeta(s + i\tau)$ approximating this function from the class $H_0(K)$.

Several modifications of Voronin's theorem are known. Universality theorems on the approximation of analytic functions by generalized shifts $\zeta(s + i\gamma(\tau))$ with some function $\gamma(\tau)$ are known [4], [5], and proofs were given for discrete universality theorems with shifts $\zeta(s + ikh)$, $h > 0$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, [6], and $\zeta(s + i\gamma(k))$ [4], [7]-[9], or even with $\zeta(s + i\gamma_k)$ [10] and $\zeta(s + it_k)$ [11], where $\{\gamma_k > 0\}$ is the sequence of imaginary parts of nontrivial zeros of the function $\zeta(s)$, and $\{t_k\}$ is the sequence of Gram

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points. Also known are universality theorems with weights [12], [13]. In [14], "lim inf" in (1.1) was replaced by "lim".

More complicated are joint universality theorems when several analytic functions are approached simultaneously by shifts of the same zeta function. The first joint universality theorem was also obtained by Voronin for Dirichlet L-functions [15].

It is clear that, in joint universality theorems, the approximating shifts must be independent in some sense. So, in [15], shifts with pairwise nonequivalent Dirichlet characters were used. In [16], the following theorem was obtained.

Theorem 1. *Suppose that $a_1, \dots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$ and*

$$\widehat{a}(Ta)^{1/3}(\log Ta)^{26/15} \le H \le T,$$

where

$$a = \max_{1\le j\le r} |a_j|^{-1} \qquad \text{and} \qquad \widehat{a} = \max_{1\le j\le r} |a_j|.$$

Let $K_j \in \mathcal{K}$, and let $f_j(s) \in H_0(K_j)$, $j = 1, \dots, r$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{H} \operatorname{meas}\Big\{\tau \in [T, T+H] : \sup_{1\le j\le r} \sup_{s\in K_j} |\zeta(s + ia_j\tau) - f_j(s)| < \varepsilon\Big\} > 0.$$

Also, "lim inf" *can be replaced by* "lim" *with the exception of at most a countable set of values $\varepsilon > 0$.*

The purpose of this paper is to replace the function $\zeta(s)$ in Theorem 1 by an absolutely convergent Dirichlet series.

Let $\theta_j > 1/2$ be a fixed number, let $u > 0$, and let

$$v_{uj}(m) = \exp\bigg\{-\bigg(\frac{m}{u}\bigg)^{\theta_j}\bigg\}, \qquad m \in \mathbb{N}, \quad j = 1, \dots, r.$$

We define

$$\zeta_{uj}(s) = \sum_{m=1}^{\infty} \frac{v_{uj}(m)}{m^s}, \qquad j = 1, \dots, r.$$

Then these series absolutely converge in the plane $\sigma > \sigma_0$ for any fixed σ_0. Inequality $\theta_j > 1/2$ is related to representation (3.2) and its application to the proof of Lemma 3, because the inequality $\operatorname{Re}(s+z) > 1$ must hold for $s \in K$. Let us define another probability space. Let $\mathcal{B}(\mathbb{X})$ denote the Borel σ-field of the space $\mathbb{X}$. We put

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\mathbb{P}$ is the set of all primes and $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the set Ω is a compact topological Abelian group by Tikhonov's theorem. Therefore, Haar probability measure can be defined on $(\Omega, \mathcal{B}(\Omega))$. Further, let

$$\underline{\Omega} = \Omega_1 \times \dots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then $\underline{\Omega}$ is again a compact topological group and there is a Haar measure m_H on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$. We obtain the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$. Let $\omega(p)$ be the pth component of an element $\omega_j \in \Omega_j$, $j = 1, \dots, r$, $p \in \mathbb{P}$. Let $H(D)$ denote the space of analytic functions in the strip D, functions equipped with the topology of uniform convergence on compact sets. Let

$$H^r(D) = \underbrace{H(D) \times \dots \times H(D)}_{r}, \qquad \underline{\omega} = (\omega_1, \dots, \omega_r) \in \underline{\Omega},$$

and, on the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$, we define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \underline{\omega}) = (\zeta(s, \omega_1), \dots, \zeta(s, \omega_r)),$$

where

$$\zeta(s, \omega_j) = \prod_{p \in \mathbb{P}} \biggl(1 - \frac{\omega_j(p)}{p^s}\biggr)^{-1}, \qquad j = 1, \dots, r.$$

For almost all ω_j, this product converges uniformly on compact subsets of the strip D. Then the following statement holds.

Theorem 2. *Let $a_1, \dots, a_r$ and H be the same as in Theorem 1; let $u_T \to \infty$ and $u_T \ll \exp\{o(T)\}$ as $T \to \infty$. Let $K_j \in \mathcal{K}$; let $f_j(s) \in H_0(K_j)$, $j = 1, \dots, r$. Then, with the exception of at most a countable set of values $\varepsilon_1 > 0, \dots, \varepsilon_r > 0$, the following limit exists*:

$$\lim_{T \to \infty} \frac{1}{T} \operatorname{meas}\Bigl\{\tau \in [T, T+H] : \sup_{s \in K_1} |\zeta_{u_T 1}(s + ia_1\tau) - f_1(s)| < \varepsilon_1, \dots, \sup_{s \in K_r} |\zeta_{u_T r}(s + ia_r\tau) - f_r(s)| < \varepsilon_r\Bigr\}$$
$$= m_H\Bigl\{\underline{\omega} \in \underline{\Omega} : \sup_{s \in K_1} |\zeta(s, \omega_1) - f_1(s)| < \varepsilon_1, \dots, \sup_{s \in K_r} |\zeta(s, \omega_r) - f_r(s)| < \varepsilon_r\Bigr\} > 0.$$

Theorem 2 is derived by a slight modification of Theorem 1 and from the proximity of functions $\zeta(s)$ and $\zeta_{uj}(s)$ by using the method of characteristic functions. For absolutely convergent Dirichlet series, one-dimensional universality theorems were obtained in [17]-[20].

2. JOINT UNIVERSALITY OF THE FUNCTION $\zeta(s)$

For brevity, let

$$\underline{\zeta}(s + i\underline{a}\tau) = (\zeta(s + ia_1, \tau), \dots, \zeta(s + ia_r\tau)), \qquad \underline{a} = a_1, \dots, a_r,$$

and, for sets $A \in \mathcal{B}(H^r(D))$, let

$$P_{T,H}(A) = \frac{1}{H} \operatorname{meas}\{\tau \in [T, T+H] : \underline{\zeta}(s + i\underline{a}\tau) \in A\}.$$

Lemma 1 [16]. *Suppose that $a_1, \dots, a_r$, and let H be the same as in Theorem 1. Then, as $T \to \infty$, $P_{T,H}$ converges weakly to the distribution of the random element $\underline{\zeta}(s, \underline{\omega})$, i.e., to the measure*

$$P_{\underline{\zeta}}(A) \overset{\text{def}}{=} m_H\{\underline{\omega} \in \underline{\Omega} : \underline{\zeta}(s, \underline{\omega}) \in A\}, \qquad A \in \mathcal{B}(H^r(D)).$$

In addition, the set

$$(\{g \in H(D) : g(s) \neq 0 \quad \text{or} \quad g(s) \equiv 0\})^r$$

is the support of the measure $P_{\underline{\zeta}}$.

Lemma 2. *Suppose that $a_1, \dots, a_r$ and H are the same as in Theorem 1. Let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$, $j = 1, \dots, r$. Then, with the exception of at most a countable set of values $\varepsilon_1 > 0, \dots, \varepsilon_r > 0$, the following limit exists*:

$$\lim_{T \to \infty} \frac{1}{T} \operatorname{meas}\Bigl\{\tau \in [T, T+H] : \sup_{s \in K_1} |\zeta(s + ia_1\tau) - f_1(s)| < \varepsilon_1, \dots, \sup_{s \in K_r} |\zeta(s + ia_r\tau) - f_r(s)| < \varepsilon_r\Bigr\}$$
$$= m_H\Bigl\{\underline{\omega} \in \underline{\Omega} : \sup_{s \in K_1} |\zeta(s, \omega_1) - f_1(s)| < \varepsilon_1, \dots, \sup_{s \in K_r} |\zeta(s, \omega_r) - f_r(s)| < \varepsilon_r\Bigr\} > 0.$$

Proof. Let the mapping $h\colon H^r(D) \to \mathbb{R}^r$ be defined by the formula

$$h(g_1,\dots,g_r) = \Big(\sup_{s\in K_1} |g_1(s)-f_1(s)|,\dots,\sup_{s\in K_r}|g_r(s)-f_r(s)|\Big), \qquad (g_1,\dots,g_r)\in H^r(D).$$

Then h is continuous. Therefore, from the properties of the weak convergence of probability measures and from Lemma 1, we find that, as $T\to\infty$, $P_{T,H}h^{-1}$ converges weakly to $P_{\underline{\zeta}}h^{-1}$, where

$$P_{T,H}h^{-1}(A) = P_{T,H}(h^{-1}A), \quad P_{\underline{\zeta}}h^{-1}(A) = P_{\underline{\zeta}}(h^{-1}A) \qquad \text{for all} \quad A\in\mathcal{B}(\mathbb{R}^r).$$

It is well known that the weak convergence of probability measures on $(\mathbb{R}^r,\mathcal{B}(\mathbb{R}^r))$ is equivalent to the weak convergence of distribution functions. Let us define the distribution functions

$$F_{T,H}(\varepsilon_1,\dots,\varepsilon_r) = \frac{1}{H}\operatorname{meas}\Big\{\tau\in[T,T+H] : \sup_{s\in K_1}|\zeta(s+ia_1\tau)-f_1(s)|<\varepsilon_1,\dots, \sup_{s\in K_r}|\zeta(s+ia_r\tau)-f_r(s)|<\varepsilon_r\Big\},$$

$$F(\varepsilon_1,\dots,\varepsilon_r) = m_H\Big\{\underline{\omega}\in\underline{\Omega} : \sup_{s\in K_1}|\zeta(s,\omega_1)-f_1(s)|<\varepsilon_1,\dots,\sup_{s\in K_r}|\zeta(s,\omega_r)-f_r(s)|<\varepsilon_r\Big\}.$$

Then we see that, for $T\to\infty$, $F_{T,H}(\varepsilon_1,\dots,\varepsilon_r)$ converges weakly to $F(\varepsilon_1,\dots,\varepsilon_r)$, i.e., converges for $\varepsilon_1,\dots,\varepsilon_r$, such that the one-dimensional distribution functions $F(\varepsilon_1,\infty,\dots,\infty),\dots,F(\infty,\dots,\infty,\varepsilon_r)$ are continuous. Since the distribution function has at most a countable set of points of discontinuity, the lemma will be proved if we show the positivity of $F(\varepsilon_1,\dots,\varepsilon_r)$.

By Mergelyan's theorem on the approximation of analytic functions [21], here exist polynomials $p_1(s),\dots,p_r(s)$ such that

$$\sup_{s\in K_1}|f_1(s)-\mathrm{e}^{p_1(s)}|<\frac{\varepsilon_1}{2}, \quad \dots, \quad \sup_{s\in K_r}|f_r(s)-\mathrm{e}^{p_r(s)}|<\frac{\varepsilon_r}{2}. \tag{2.1}$$

By Lemma 1, the collection $(\mathrm{e}^{p_1(s)},\dots,\mathrm{e}^{p_r(s)})$ is an element of the support of the measure $P_{\underline{\zeta}}$; hence

$$P_{\underline{\zeta}}(G_{\varepsilon_1,\dots,\varepsilon_r})>0, \tag{2.2}$$

where

$$G_{\varepsilon_1,\dots,\varepsilon_r} = \Big\{(g_1,\dots,g_r)\in H^r(D) : \sup_{s\in K_1}|g_1(s)-\mathrm{e}^{p_1(s)}|<\frac{\varepsilon_1}{2},\dots,\sup_{s\in K_r}|g_r(s)-\mathrm{e}^{p_r(s)}|<\frac{\varepsilon_r}{2}\Big\}.$$

However, it follows from inequalities (2.1) that the set $G_{\varepsilon_1,\dots,\varepsilon_r}$ lies in the set

$$\Big\{(g_1,\dots,g_r)\in H^r(D) : \sup_{s\in K_1}|g_1(s)-f_1(s)|<\varepsilon_1,\dots,\sup_{s\in K_r}|g_r(s)-f_r(s)|<\varepsilon_r\Big\},$$

which, together with (2.2), proves that $F(\varepsilon_1,\dots,\varepsilon_r)>0$. □

3. PROXIMITY OF $\zeta(s)$ AND $\zeta_{u_T}(s)$

Let $\theta>1/2$, $u>0$,

$$v_u(m) = \exp\Big\{-\Big(\frac{m}{u}\Big)^\theta\Big\}, \qquad m\in\mathbb{N},$$

$$\zeta_u(s) = \sum_{m=1}^\infty \frac{v_u(m)}{m^s}.$$

Let us prove the following statement.

Lemma 3. *Suppose that $a_1, \dots, a_r$ and H are the same as in Theorem 1. Let $u_T \to \infty$ and $u_T \ll \exp\{o(T)\}$ as $T \to \infty$. Then, for every $a \in \mathbb{R} \setminus \{0\}$ and any compact set $K \subset D$,*

$$\lim_{T\to\infty} \frac{1}{H} \int_T^{T+H} \sup_{s\in K} |\zeta(s+ia\tau) - \zeta_{u_T}(s+ia\tau)|\, d\tau = 0.$$

Proof. It was found in [16] that, for a fixed $\sigma \in (1/2, 1)$ and for every $\tau \in \mathbb{R}$,

$$\frac{1}{H} \int_T^{T+H} |\zeta(\sigma + ia\tau + iv)|^2 \, d\tau \ll_{\sigma,a} 1 + |v|. \tag{3.1}$$

For $\zeta_{u_T}(s)$ in the strip D, we have the following representation [3]:

$$\zeta_{u_T}(s) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z) l_{u_T}(z) \frac{dz}{z}, \qquad \text{where} \quad l_{u_T}(z) = \frac{z}{\theta} \Gamma\left(\frac{z}{\theta}\right) u_T^z. \tag{3.2}$$

Let $K \subset D$ be a fixed compact set. Then there will be an $\varepsilon > 0$ such that, for every $s = \sigma + it \in K$, the inequalities $1/2 + 2\varepsilon \le \sigma \le 1 - \varepsilon$ will hold. For such σ we have $\widehat{\theta} = 1/2 + \varepsilon - \sigma < 0$. Therefore, representation (3.2) and the residue theorem lead to the equality

$$\zeta_{u_T}(s) - \zeta(s) = \frac{1}{2\pi i} \int_{\widehat{\theta} - i\infty}^{\widehat{\theta} + i\infty} \zeta(s+z) l_{u_T}(z) \, \frac{dz}{z} + \frac{l_{u_T}(1-s)}{1-s}.$$

After elementary transformations, this formula can be written as

$$\begin{aligned} &\frac{1}{H} \int_T^{T+H} \sup_{s\in K} |\zeta(s+ia\tau) - \zeta_{u_T}(s+ia\tau)| \, d\tau \\ &\quad \ll \int_{-\infty}^{\infty} \left(\frac{1}{H} \int_T^{T+H} \left| \zeta\left(\frac{1}{2} + \varepsilon + ia\tau + iv\right) \right| d\tau \right) \sup_{s\in K} \left| \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} \right| dv \\ &\qquad + \frac{1}{H} \int_T^{T+H} \sup_{s\in K} \left| \frac{l_{u_T}(1-s-ia\tau)}{1-s-ia\tau} \right| d\tau \\ &\quad \stackrel{\text{def}}{=} I_1(T) + I_2(T). \end{aligned} \tag{3.3}$$

The gamma function $\Gamma(\sigma + it)$ satisfies the estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \qquad c > 0,$$

which is uniform in any strip $\sigma_1 < \sigma < \sigma_2$. Hence

$$\frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} \ll_\theta u_T^{1/2+\varepsilon-\sigma} \exp\left\{ -\frac{c}{\theta} |v - t| \right\} \ll_{\theta,K} u_T^{-\varepsilon} \exp\{-c_1|v|\}, \qquad c_1 > 0.$$

From this and estimate (3.1), we obtain

$$I_1(T) \ll_{\varepsilon,a,\theta,K} u_T^{-\varepsilon} \int_{-\infty}^{\infty} (1+|v|)^{1/2} \exp\{-c_1|v|\} \, dv \ll_{\varepsilon,a,\theta,K} u_T^{-\varepsilon}. \tag{3.4}$$

Similarly, we find that, for all $s \in K$,

$$\frac{l_{u_T}(1-s-ia\tau)}{1-s-ia\tau} \ll_\theta u_T^{1-\sigma} \exp\left\{ -\frac{c}{\theta} |t + a\tau| \right\} \ll_{\theta,K} u_T^{1/2-2\varepsilon} \exp\{-c_2|a|\,|\tau|\}, \qquad c_2 > 0.$$

Therefore,

$$I_2(T) \ll_{\theta,K} u_T^{1/2-2\varepsilon} \exp\{-c_2|a|T\}.$$

Since $u_T \ll \exp\{\varepsilon_T T\}$ as $\varepsilon_T \to 0$, in view of (3.4) and (3.3), we obtain the assertion of the lemma. □

4. PROOF OF THEOREM 2

Theorem 2 readily follows from Lemmas 2 and 3.

Proof of Theorem 2. It suffices to show that the distribution function

$$\widehat{F}_{T,H}(\varepsilon_1,\dots,\varepsilon_r)=\frac{1}{H}\,\mathrm{meas}\Big\{\tau\in[T,T+H]:\sup_{s\in K_1}|\zeta_{u_T}(s+ia_1\tau)-f_1(s)|<\varepsilon_1,\dots,\\ \sup_{s\in K_r}|\zeta_{u_T}(s+ia_r\tau)-f_r(s)|<\varepsilon_r\Big\}$$

as $T\to\infty$ converges weakly to the distribution function $F(\varepsilon_1,\dots,\varepsilon_r)$. To do this, we use the method of characteristic functions. Let $\varphi_{T,H}(v_1,\dots,v_r)$, $\widehat{\varphi}_{T,H}(v_1,\dots,v_r)$, and $\varphi(v_1,\dots,v_r)$, $v_1,\dots,v_r\in\mathbb{R}$, be the characteristic functions, respectively, of the distribution functions $F_{T,H}(\varepsilon_1,\dots,\varepsilon_r)$, $\widehat{F}_{T,H}(\varepsilon_1,\dots,\varepsilon_r)$, and $F(\varepsilon_1,\dots,\varepsilon_r)$. It is well known that the weak convergence of distribution functions implies the convergence of the corresponding characteristic functions. Therefore, from the proof of Lemma 2, we obtain

$$\varphi_{T,H}(v_1,\dots,v_r)=\varphi(v_1,\dots,v_r)+o(1) \tag{4.1}$$

uniformly over $|v_j|\le C_j$ for any $0<C_j<\infty$, $j=1,\dots,r$. From the definition of the characteristic function, we obtain

$$\begin{aligned}\widehat{\varphi}_{T,H}(v_1,\dots,v_r)&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{i(\varepsilon_1v_1+\dots+\varepsilon_rv_r)\}\,d\widehat{F}_{T,H}(\varepsilon_1,\dots,\varepsilon_r)\\
&=\frac{1}{H}\int_T^{T+H}\exp\Big\{i\Big(v_1\sup_{s\in K_1}|\zeta_{u_T1}(s+ia_1\tau)-f_1(s)|+\dots\\
&\qquad+v_r\sup_{s\in K_r}|\zeta_{u_Tr}(s+ia_r\tau)-f_r(s)|\Big)\Big\}\,d\tau\\
&=\varphi_{T,H}(v_1,\dots,v_r)+\frac{1}{H}\int_T^{T+H}\Big(\exp\Big\{i\sum_{j=1}^r v_j\sup_{s\in K_j}|\zeta_{u_Tj}(s+ia_j\tau)-f_j(s)|\Big\}\\
&\qquad-\exp\Big\{i\sum_{j=1}^r v_j\sup_{s\in K_j}|\zeta(s+ia_j\tau)-f_j(s)|\Big\}\Big)\,d\tau\\
&=\varphi_{T,H}(v_1,\dots,v_r)+\frac{1}{H}\int_T^{T+H}\exp\Big\{i\sum_{j=1}^r v_j\sup_{s\in K_j}|\zeta(s+ia_j\tau)-f_j(s)|\Big\}\\
&\qquad\times\Big(\exp\Big\{i\sum_{j=1}^r\Big(v_j\sup_{s\in K_j}|\zeta_{u_Tj}(s+ia_j\tau)-f_j(s)|\\
&\qquad-\sup_{s\in K_j}|\zeta(s+ia_j\tau)-f_j(s)|\Big)\Big\}-1\Big)\,d\tau.\end{aligned}$$

Thus, by virtue of the inequality $|e^{iv}-1|\le|v|$, $v\in\mathbb{R}$, and the triangle inequality, we have

$$\begin{aligned}&|\widehat{\varphi}_{T,H}(v_1,\dots,v_r)-\varphi_{T,H}(v_1,\dots,v_r)|\\
&\qquad\le\frac{1}{H}\int_T^{T+H}\sum_{j=1}^r|v_j|\Big|\sup_{s\in K_j}|\zeta(s+ia_j\tau)-f_j(s)|-\sup_{s\in K_j}|\zeta_{u_Tj}(s+ia_j\tau)-f_j(s)|\Big|\,d\tau\\
&\qquad\le\frac{1}{H}\sum_{j=1}^r|v_j|\int_T^{T+H}\sup_{s\in K_j}|\zeta(s+ia_j\tau)-\zeta_{u_tj}(s+ia_j\tau)|\,d\tau.\end{aligned}$$

Hence, from Lemma 3 and (4.1), we obtain

$$\widehat{\varphi}_{T,H}(v_1,\dots,v_r)=\varphi(v_1,\dots,v_r)+o(1)$$

uniformly over $|v_j| \leq C_j$, for any $C_j < \infty$, $j = 1, \dots, r$. Therefore, $\widehat{F}_{T,H}(\varepsilon_1, \dots, \varepsilon_r)$ converges weakly to $F(\varepsilon_1, \dots, \varepsilon_r)$, and the theorem is proved. □

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