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Asymptotics of Sums of Cosine Series with Fractional Monotonicity Coefficients

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Abstract—The paper examines the following question: Under what orders of monotonicity are the upper and lower bounds of the sum of a cosine series near zero valid if they are obtained using the function $\sum_{n=0}^{[\pi/x]}(n+1)\Delta(\mathbf{a})_n$?

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1. INTRODUCTION

One of the classical problems of the theory of trigonometric series is to obtain asymptotic estimates near zero of sums of trigonometric series with monotone coefficients. The first of the works in this direction, apparently, is the paper [1] of Salem; see also [2, pp. 668–676]. The research was later continued in the works of Telyakovsky [3], Popov and Solodov [4], Popov[5] and many other mathematicians.

However, in this problem, the properties of sine and cosine series differ significantly. If sums of sine series with monotone decreasing coefficients

$$\sum_{n=1}^{\infty} a_n \sin nx,$$

where $x \in (0, \pi)$, are usually estimated using the expression

$$h(x) \equiv x \sum_{n=1}^{[\pi/x]} n a_n,$$

then, for a cosine series with monotone decreasing coefficients

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{1}$$

a similar role is played by

$$q(x) \equiv \sum_{n=0}^{[\pi/x]} (n+1)\Delta a_n,$$

where $\Delta a_n = a_n - a_{n+1}$, $n = 0, 1, \dots$.

In this note, we will focus our attention on cosine series.

The classical version for an upper bound is as follows.

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Theorem A. *Let the coefficients of the series* (1) *satisfy the conditions* $a_n \to 0$ *as* $n \to \infty$ *and*

$$\Delta^2(a)_n \equiv \Delta(\Delta a)_n \equiv a_n - 2a_{n+1} + a_{n+2} \geq 0$$

for $n = 0, 1, \dots$ *. Then if* $f(x)$ *is the sum of the series* (1)*, then, for* $x \in (0, \pi)$*, the following estimates hold*:

$$0 \leq f(x) \leq 5q(x). \tag{2}$$

Of course, the constant 5 is not optimal, but questions about best constants in inequalities are not discussed in this paper.

For lower bounds, a greater 'degree" of monotonicity is usually required. The following statement is known.

Theorem B. *Let the coefficients of the series* (1) *satisfy the conditions* $a_n \to 0$ *as* $n \to \infty$ *and*

$$\Delta^3(a)_n \equiv \Delta(\Delta(\Delta(a)))_n \equiv a_n - 3a_{n+1} + 3a_{n+2} - a_{n+3} \geq 0$$

for $n = 0, 1, \dots$ *. Then, for some constant* $C > 0$*, if* $f(x)$ *is the sum of the series* (1)*, then, for* $x \in (0, 1]$*, the following estimate holds*:

$$f(x) \geq Cq(x). \tag{3}$$

Unfortunately, the authorship of Theorems A and B is, apparently, unknown, but, for many years, these theorems have been included in special courses for students. It is also known that estimates (2) (even with another constant on the right-hand side) is, generally speaking, no longer valid if only the monotonicity of the coefficients a_n is required, while estimate (3) does not hold under the conditions of their convexity. In this connection, it is of interest to consider the problem on classes of fractional monotonicity, which were previously introduced by the author in [6].

Let us give the corresponding definitions.

Definition 1. Let $-\infty < \alpha < \infty$. By *Cesaro numbers* $\{A_n^\alpha\}_{n=0}^\infty$ we mean the coefficients of the expansion

$$(1 - x)^{-\alpha - 1} = \sum_{n=0}^{\infty} A_n^\alpha x^n$$

for $x \in (0, 1)$.

The following properties of these numbers are known (see [7]):

1) $A_n^0 = 1$ for $n = 0, 1, \dots$ and $A_0^\alpha = 1$ for any α;

2) if $\alpha \neq -1, -2, \dots$, then there are constants $C_1 > 0$ and $C_2 > 0$ depending only on α such that

$$C_2 n^\alpha \leq |A_n^\alpha| \leq C_1 n^\alpha$$

for all $n > 0$;

3) for $\alpha > -1$ and any n, $A_n^\alpha > 0$; for $\alpha > 0$, $A_n^\alpha \uparrow \infty$ as $n \to \infty$; and, for $-1 < \alpha < 0$, $A_n^\alpha \downarrow 0$ as $n \to \infty$;

4) the following equality holds:

$$\sum_{k=0}^{n} a_{n-k}^\alpha A_k^\beta = A_n^{\alpha+\beta+1}$$

for all α and β, and $n = 0, 1, \dots$. In particular, $A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$.

If a number sequence $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ and a real α are given, then we denote

$$\Delta^{\alpha}(\mathbf{a})_n = \sum_{k=0}^{\infty} A_k^{-\alpha-1} a_{n+k}$$

for $n = 0, 1, \dots$ in the case where such a sum exists, for example, if $\alpha > 0$ and the sequence $\mathbf{a}$ is bounded.

Definition 2. Let $\alpha > 0$, and let $\mathbf{a}$ be a sequence of real numbers. Then we say that $\mathbf{a} \in M_\alpha$ if $\lim_{n\to\infty} a_n = 0$ and $\Delta^{\alpha}(\mathbf{a})_n \geq 0$ for $n = 0, 1, \dots$.

It follows from Definition 2 that the class M_0 coincides with the class of zero-tending sequences of nonnegative numbers, and M_1 is the class of monotone nonincreasing sequences tending to zero, etc. In addition, the author found that, for $\alpha > \beta \geq 0$, the inclusion $M_\alpha \subset M_\beta$ is valid (see [6, Lemma 1, item b)]).

It should be noted that many important auxiliary results needed for the study of monotonicity of fractional order were established by Andersen [8].

The purpose of this paper is to obtain additions to Theorem A and to strengthen Theorem B in terms of fractional monotonicity. More precisely, the following statements will be established.

Theorem 1. *For any $\alpha \in (1, 2)$, there exists a sequence $\mathbf{a} \in M_\alpha$ and a monotone zero-tending sequence $\{t_l\}_{l=1}^{\infty}$ such that*

$$\frac{q(t_l)}{f(t_l)} \to 0 \text{ as } l \to \infty,$$

where $f(x)$ and $q(x)$ were defined above.

Theorem 2. *Let $\alpha > 2$. Then there exists a constant $C = C(\alpha) > 0$ such that if the sequence $\mathbf{a}$ is contained in M_α, then, for $x \in (0, \pi/6)$, the sum of the series* (1) *satisfies the inequality $f(x) \geq C(\alpha) q(x)$.*

It should be noted that the interval $(0, \pi/6)$ in Theorem 2 is not definitive, and the question of how much it can be extended remains open.

In the section "Additions", some related problems will be discussed and also, for completeness, we will prove that, in Theorem 2, we cannot take $\alpha = 2$.

2. AUXILIARY RESULTS

The following results were established by the author in [6].

Lemma 1. *Let, for the numbers α, γ and the sequence* $\mathbf{a}$, *one of the following conditions holds*:

a) $\alpha < 0$, $\gamma < 0$, *and* $\mathbf{a} \in M_0$;

b) $\alpha > 0$, $\gamma < 0$, *and* $\mathbf{a} \in M_\alpha$;

c) $\gamma > 0$, $\alpha = -\gamma$, $\mathbf{a} \in M_0$, *and there exists a bounded sequence* $\{\Delta^{\alpha}(\mathbf{a})_n\}_{n=0}^{\infty}$.

Then

$$0 \leq \Delta^{\gamma}(\Delta^{\alpha}(\mathbf{a}))_n = \Delta^{\gamma+\alpha}(\mathbf{a})_n \text{ for } n = 0, 1, \dots .$$

Note that, in items a) and b) of Lemma 1, infinite values are not excluded.

For $\alpha > 0$, denote

$$K_n^{\alpha}(x) = \frac{A_n^{\alpha-1}}{2} + \sum_{k=1}^{n} A_{n-k}^{\alpha-1} \cos kx$$

for $n = 0, 1, \dots$.

Lemma 2. *Let $1 < \alpha < 2$, and let $\mathbf{a} \in M_\alpha$. Then, for $n = 0, 1, \dots$ and $x \in (0, \pi)$,*

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx = \sum_{k=0}^{n} \Delta^\alpha(\mathbf{a})_k K_n^\alpha(x) + o(1)$$

as $n \to \infty$.

Corollary 1. *Let $1 < \alpha < 2$, and let $\mathbf{a} \in M_\alpha$. Then, for $x \in (0, \pi)$,*

$$f(x) = \sum_{k=0}^{\infty} \Delta^\alpha(\mathbf{a})_k K_n^\alpha(x).$$

We will need another auxiliary statement.

Lemma 3. *Let $\alpha \in (0, 1)$. Then there exists a constant $C_3 = C_3(\alpha) > 0$ such that, for any sequence $\mathbf{a} \in M_\alpha$ and for any natural numbers $k_1 < k_2 < k_3$ for which $k_2 - k_1 > (k_3 - k_1)/4$, the following inequality holds*:

$$\sum_{n=k_1}^{k_2} a_n \ge C_3 \sum_{n=k_1}^{k_3} a_n.$$

Proof. Let us define the sequence $\mathbf{b} = \{b_n\}_{n=0}^\infty$, where $b_n = \Delta^\alpha(\mathbf{a})_n$ for $n = 0, 1, \dots$. By assumption, this is a sequence of nonnegative numbers and, by Lemma 1, b), we have

$$a_n = \Delta^{-\alpha}(\mathbf{b})_n = \sum_{r=0}^{\infty} A_r^{\alpha-1} b_{n+r} = \sum_{\nu=n}^{\infty} A_{\nu-n}^{\alpha-1} b_\nu.$$

But then

$$\sum_{n=k_1}^{k_2} a_n = \sum_{n=k_1}^{k_2} \sum_{\nu=n}^{\infty} A_{\nu-n}^{\alpha-1} b_\nu = \sum_{\nu=k_1}^{k_2} b_\nu \sum_{n=k_1}^{\nu} A_{\nu-n}^{\alpha-1} + \sum_{\nu=k_2+1}^{\infty} b_\nu \sum_{n=k_1}^{k_2} A_{\nu-n}^{\alpha-1} \equiv \sum_{\nu=k_1}^{\infty} b_\nu f_\nu. \tag{4}$$

Similarly,

$$\sum_{n=k_1}^{k_3} a_n = \sum_{\nu=k_1}^{k_3} b_\nu \sum_{n=k_1}^{\nu} A_{\nu-n}^{\alpha-1} + \sum_{\nu=k_3+1}^{\infty} b_\nu \sum_{n=k_1}^{k_3} A_{\nu-n}^{\alpha-1} \equiv \sum_{\nu=k_1}^{\infty} b_\nu f'_\nu. \tag{5}$$

Note that, for $k_1 \le \nu \le k_2$,

$$f_\nu = f'_\nu. \tag{6}$$

If $k_2 + 1 \le \nu \le k_3$, then

$$f_\nu \ge C_2(\nu - k_1)^{\alpha-1}(k_2 - k_1 + 1) > C_2(k_3 - k_1)^{\alpha-1}\frac{1}{4}(k_3 - k_1) = \frac{C_2}{4}(k_3 - k_1)^\alpha,$$

where the constants C_1 and C_2 given below are taken from property 2) of the Cesaro numbers, while

$$f'_\nu = \sum_{n=k_1}^{\nu} A_{\nu-n}^{\alpha-1} = \sum_{r=0}^{\nu-k_1} A_r^{\alpha-1} \le C_1 \sum_{r=0}^{\nu-k_1} (r+1)^{\alpha-1} \le C_4(\alpha)(k_3 - k_1)^\alpha.$$

Thus, for $k_2 + 1 \le \nu \le k_3$, we have

$$f_\nu \ge C_5(\alpha) f'_\nu, \tag{7}$$

where the constant $C_5(\alpha) > 0$ depends only on α.

Further, let $k_3 + 1 \le \nu \le 2k_3 - k_1$. Then, since $\nu - k_1 \le 2(k_3 - k_1)$, we obtain

$$f_\nu \ge C_2(\nu - k_1)^{\alpha-1}(k_2 - k_1 + 1) > 2^{\alpha-1} C_2(k_3 - k_1)^{\alpha-1}\frac{1}{4}(k_3 - k_1)$$

$$= 2^{\alpha-3}C_2(k_3 - k_1)^\alpha \geq C_2 2^{-3}(\nu - k_1)^\alpha,$$

and, further,

$$f'_\nu \leq C_1 \sum_{r=0}^{\nu-k_1}(r+1)^{\alpha-1} \leq C_4(\alpha)(\nu - k_1)^\alpha.$$

Therefore, also for $k_3 + 1 \leq \nu \leq 2k_3 - k_1$, we have

$$f_\nu \geq C_6(\alpha) f'_\nu, \tag{8}$$

where the constant $C_6(\alpha) > 0$ depends only on α.

Finally, let $\nu > 2k_3 - k_1$. Then

$$f_\nu \geq C_2(\nu - k_1)^{\alpha-1}(k_2 - k_1 + 1),$$

and, further,

$$f'_\nu \leq C_1(\nu - k_3)^{\alpha-1}(k_3 - k_1 + 1).$$

Note that $k_2 - k_1 + 1 > (k_3 - k_1 + 1)/4$ and $\nu - k_1 = \nu - k_3 + k_3 - k_1 < 2(\nu - k_3)$, whence

$$(\nu - k_1)^{\alpha-1} > 2^{\alpha-1}(\nu - k_3)^{\alpha-1}.$$

Therefore, in this case, we have

$$f_\nu \geq C_7(\alpha) f'_\nu, \tag{9}$$

where the constant $C_7(\alpha) > 0$ depends only on α. Let us put $C_3 = \min(1, C_5, C_6, C_7)$. Now the result of Lemma 3 follows from (4)–(9) and the nonnegativity of the numbers b_ν. □

3. MAIN RESULTS

Proof of Theorem 1. Because of the embedding of the classes M_α, we assume without loss of generality that $\alpha \in (3/2, 2)$. Let $\{m_l\}_{l=1}^\infty$ be an increasing sequence of natural numbers that satisfies the following conditions:

1) all the m_l are fourth powers of natural numbers;

2) $m_{l+1} > m_l^4$ for $l = 1, 2, \ldots$;

3) $m_{l+1}^{1-\alpha/2} > 100^l m_l$ for $l = 1, 2, \ldots$;

4) $m_1 > 100$.

Let us define the sequence $\mathbf{b}$ as follows:

$$b_r = \begin{cases} \dfrac{10^{-l}}{A_{m_l}^{\alpha-1}} & \text{for } r = m_l, \quad l = 1, 2, \ldots, \\ 0 & \text{for the other } r. \end{cases}$$

Note that, for any $n \geq 0$,

$$\sum_{k=0}^\infty A_k^{\alpha-1} b_{n+k} = \sum_{m_l \geq n} A_{m_l - n}^{\alpha-1} \frac{10^{-l}}{A_{m_l}^{\alpha-1}} \leq \sum_{m_l \geq n} 10^{-l}.$$

Thus, there exists a sequence tending to zero, $\{\Delta^{-\alpha}(\mathbf{b})_n\}_{n=0}^\infty$. Let us put $a_n = \Delta^{-\alpha}(\mathbf{b})_n$ for $n = 0, 1, \ldots$ and consider the series

$$\frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx. \tag{10}$$

By Lemma 1 c), for all n we have $\Delta^\alpha(\mathbf{a})_n = b_n \geq 0$, i.e., $\mathbf{a} \in M_\alpha \subset M_1$. Therefore, the series (10) converges for $x \in (0,\pi)$ to some function $f(x)$. By Corollary 1, for $x \in (0,\pi)$ we have

$$f(x) = \sum_{k=0}^{\infty} \Delta^\alpha(\mathbf{a})_k K_n^\alpha(x) = \sum_{l=1}^{\infty} \frac{10^{-l}}{A_{m_l}^{\alpha-1}} K_{m_l}^\alpha(x). \tag{11}$$

In Zygmund's book [7], the following estimates were established:

$$\left|\frac{K_r^\alpha(x)}{A_r^{\alpha-1}}\right| \leq r+1 \tag{12}$$

for $r = 0,1,\dots$ and all x,

$$\left|\frac{K_r^\alpha(x)}{A_r^{\alpha-1}}\right| \leq C_8(\alpha) r^{-\alpha+1} x^{-\alpha} \tag{13}$$

for $r = 1,2,\dots$ and $x \in (0,\pi)$, where the constant $C_8(\alpha)$ depends only on α;

$$\frac{K_r^\alpha(x)}{A_r^{\alpha-1}} = \frac{1}{A_r^{\alpha-1}} \cdot \frac{\sin((r+\alpha/2)x - \pi(\alpha-1)/2)}{(2\sin(x/2))^\alpha} + \frac{2\theta(\alpha-1)}{r(2\sin(x/2))^2} \tag{14}$$

for $x \in (0,\pi)$, where $|\theta| \leq 1$.

For all l, we put $n_l = \sqrt{m_l}$. Obviously, there exists a $t_l \in (\pi/(2n_l), 2\pi/n_l)$ such that

$$\sin\biggl(\biggl(m_l + \frac{\alpha}{2}\biggr)t_l - \frac{\pi(\alpha-1)}{2}\biggr) = 1.$$

Hence, using (14), we obtain

$$\frac{K_{m_l}^\alpha(t_l)}{A_{m_l}^{\alpha-1}} \geq \frac{C_9(\alpha) n_l^\alpha}{m_l^{\alpha-1}} - \frac{C_{10}(\alpha) n_l^2}{m_l} = C_9(\alpha) m_l^{1-\alpha/2} - C_{10}(\alpha),$$

where the positive constants $C_9(\alpha)$ and $C_{10}(\alpha)$ depend only on α. Combining this with formulas (11)–(13), we see that, for any l,

$$\begin{aligned}
f(t_l) &\geq 10^{-l}(C_9(\alpha) m_l^{1-\alpha/2} - C_{10}(\alpha)) - \sum_{r=1}^{l-1} \frac{10^{-r}}{A_{m_r}^{\alpha-1}} |K_{m_r}^\alpha(t_l)| - \sum_{r=l+1}^{\infty} \frac{10^{-r}}{A_{m_r}^{\alpha-1}} |K_{m_r}^\alpha(t_l)| \\
&\geq 10^{-l}(C_9(\alpha) m_l^{1-\alpha/2} - C_{10}(\alpha)) - \sum_{r=1}^{l-1} 10^{-r}(m_r+1) - \sum_{r=l+1}^{\infty} 10^{-r} C_8(\alpha) {m_r}^{-\alpha+1} n_l^\alpha \\
&\geq 10^{-l}(C_9(\alpha) m_l^{1-\alpha/2} - C_{10}(\alpha)) - 2m_{l-1} - \sum_{r=l+1}^{\infty} 10^{-r} C_8(\alpha) \\
&> \frac{C_9(\alpha)}{2} 10^{-l} m_l^{1-\alpha/2},
\end{aligned} \tag{15}$$

where l is sufficiently large.

Let us now estimate $q(t_l)$. Obviously, if $r \geq l \geq 1$ and $k \leq 2n_l$, then $m_r - k \geq m_r/2$, and hence

$$\frac{A_{m_r-k}^{\alpha-2}}{A_{m_r}^{\alpha-1}} \leq \frac{C_1(\alpha)(m_r-k)^{\alpha-2}}{C_2(\alpha) m_r^{\alpha-1}} \leq \frac{2^{2-\alpha} C_1(\alpha)}{C_2(\alpha)} \cdot \frac{1}{m_r} \equiv \frac{C_{11}(\alpha)}{m_r} \leq \frac{C_{11}(\alpha)}{m_l}.$$

Therefore,

$$q(t_l) \leq \sum_{k=0}^{2n_l} (k+1)\Delta(\mathbf{a})_k = \sum_{k=0}^{m_{l-1}} (k+1)\Delta(\mathbf{a})_k + \sum_{k=m_{l-1}+1}^{2n_l} (k+1)\Delta(\Delta^{-\alpha}(\mathbf{b}))_k$$

$$\le (m_{l-1}+1)a_0 + \sum_{k=m_{l-1}+1}^{2n_l} (k+1)\Delta(\Delta^{-1}(\Delta^{-\alpha+1}(\mathbf{b})))_k$$

$$\le (m_{l-1}+1)a_0 + \sum_{k=m_{l-1}+1}^{2n_l} (k+1)\Delta^{-\alpha+1}(\mathbf{b})_k$$

$$\le (m_{l-1}+1)a_0 + (2n_l+1)\sum_{k=m_{l-1}+1}^{2n_l}\sum_{r=l}^{\infty} A_{m_r-k}^{\alpha-2}\cdot\frac{10^{-r}}{A_{m_r}^{\alpha-1}}$$

$$\le (m_{l-1}+1)a_0 + (2n_l+1)2n_l\frac{C_{11}(\alpha)}{m_l}\sum_{r=l}^{\infty}10^{-r} \le (m_{l-1}+1)a_0 + C_{12}(\alpha). \tag{16}$$

Formulas (9), (10) and Condition 3) imposed on the sequence $\{m_l\}_{l=1}^{\infty}$ imply the result of Theorem 1. □

Proof of Theorem 2. Without loss of generality, we can assume that $\alpha \in (2,3)$. Since, in particular, $\mathbf{a} \in M_2$, for $x \in (0,\pi)$, it follows that

$$f(x) = \sum_{n=0}^{\infty}\Delta^2(\mathbf{a})_n K_n(x), \qquad \text{where} \quad K_n(x) = \frac{1-\cos(n+1)x}{4\sin^2(x/2)}.$$

For brevity, we denote $b_n = \Delta^2(\mathbf{a})_n$ for $n = 0,1,\dots$. Then we have the sequence $\mathbf{b} = \{b_n\}_{n=0}^{\infty} \in M_{\alpha-2}$.

Further,

$$f(x) = \sum_{n=0}^{[\pi/x]} b_n K_n(x) + \sum_{n=[\pi/x]+1}^{\infty} b_n K_n(x) \equiv S_1 + S_2.$$

If $0 \le n \le [\pi/x]$, then

$$K_n(x) \ge \frac{2\sin^2((n+1)x/2)}{x^2} \ge \frac{1}{\pi^2}(n+1)^2.$$

Hence, applying the Abel transformation, we obtain

$$S_1 \ge \frac{1}{10}\sum_{n=0}^{[\pi/x]}(n+1)^2\Delta^2(\mathbf{a})_n = \frac{1}{10}\sum_{n=0}^{[\pi/x]}((n+1)^2-n^2)\sum_{r=n}^{[\pi/x]}\Delta^2(\mathbf{a})_r$$

$$\ge \frac{1}{10}\,q(x) - \frac{1}{10}\left[\frac{\pi}{x}\right]^2\Delta^1(\mathbf{a})_{[\pi/x]+1}. \tag{17}$$

Note that, for any $x \in (0,\pi/6)$ and for any natural k, the interval $\big[[(2k-1)\pi/x]+1, [(2k-1/3)\pi/x]\big]$ will contain at least a quarter of integer points from the segment $\big[[(2k-1)\pi/x]+1, [(2k+1)\pi/x]\big]$. Hence, taking into account the nonnegativity of the Fejér kernels and Lemma 3, we obtain

$$S_2 = \sum_{k=1}^{\infty}\sum_{n=[(2k-1)\pi/x]+1}^{[(2k+1)\pi/x]} b_n K_n(x) \ge \sum_{k=1}^{\infty}\sum_{n=[(2k-1)\pi/x]+1}^{[(2k-1/3)\pi/x]} b_n K_n(x)$$

$$\ge \frac{1}{x^2}\sum_{k=1}^{\infty}\sum_{n=[(2k-1)\pi/x]+1}^{[(2k-1/3)\pi/x]} b_n(1-\cos(n+1)x) \ge \frac{1}{8x^2}\sum_{k=1}^{\infty}\sum_{n=[(2k-1)\pi/x]+1}^{[(2k-1/3)\pi/x]} b_n$$

$$\ge \frac{C_3}{8x^2}\sum_{k=1}^{\infty}\sum_{n=[(2k-1)\pi/x]+1}^{[(2k+1)\pi/x]} b_n = \frac{C_3}{8x^2}\,\Delta^1(\mathbf{a})_{[\pi/x]+1}. \tag{18}$$

Now if

$$\left[\frac{\pi}{x}\right]^2 \Delta^1(\mathbf{a})_{[\pi/x]+1} < \frac{1}{2}\, q(x),$$

then the result of Theorem 2 follows from (17) and, otherwise, from formula (18). □

4. ADDITIONS

1. Let us give an example showing that the condition $\mathbf{a} \in M_2$ does not guarantee the validity of the lower bound in terms of $q(x)$. Let $n_k = 2^{2^k} - 1$, $k = 1, 2, \dots$, and let

$$b_r = \begin{cases} k^{-2} 2^{-2^k} & \text{for } r = n_k,\ k = 1, 2, \dots, \\ 0 & \text{for the other } r. \end{cases}$$

Let

$$c_n = \sum_{r=n}^{\infty} b_r$$

for $n = 0, 1, \dots$. Note that, for any $k \ge 1$, for $n_k < n \le n_{k+1}$, we have $c_n \le 2b_{n_{k+1}}$, whence it is clear that the numbers

$$a_n = \sum_{l=n}^{\infty} c_l \le \sum_{k:\ n_k \ge n} n_k 2 b_{n_k} \le \sum_{k:\ n_k \ge n} 2k^{-2}$$

are defined and tend to zero as $n \to \infty$. It is also obvious that

$$\Delta^2(\mathbf{a})_n = b_n$$

for all n. Thus, $\mathbf{a} \in M_2$. Let $f(x)$ be the sum of the series (1).

Obviously,

$$f(x) = \sum_{n=0}^{\infty} b_n \frac{1 - \cos(n+1)x}{4\sin^2(x/2)} = \sum_{m=1}^{\infty} m^{-2} 2^{-2^m} \frac{1 - \cos 2^{2^m} x}{4\sin^2(x/2)}\,.$$

Let $t_k = 2\pi/2^{2^k}$ for $k = 1, 2, \dots$. Then

$$f(t_k) = \sum_{m=1}^{k-1} m^{-2} 2^{-2^m} \frac{1 - \cos 2^{2^m} t_k}{4\sin^2(t_k/2)} \le \sum_{m=1}^{k-1} m^{-2} 2^{-2^m} 2^{2^{m+1}} = \sum_{m=1}^{k-1} m^{-2} 2^{2^m} \le 2(k-1)^{-2} 2^{2^{k-1}}. \quad (19)$$

At the same time,

$$q(t_k) = \sum_{n=0}^{[\pi/t_k]} (n+1)\Delta(\mathbf{a})_n = \sum_{n=0}^{2^{2^k-1}} (n+1)\Delta(\mathbf{a})_n \ge \sum_{n=2^{2^k-2}}^{2^{2^k-1}} (n+1) k^{-2} 2^{-2^k} \ge \frac{1}{k^2} \cdot 2^{2^k-4}. \quad (20)$$

It follows from (19)–(20) that

$$\frac{q(t_k)}{f(t_k)} \to \infty$$

as $k \to \infty$, which was required to verify.

2. If $\mathbf{a} \in M_1$, then an elementary estimate involving the Abel transformation shows that

$$|f(x)| \le q(x) + \frac{\pi}{x}\, a_{[\pi/x]+1}$$

for $x \in (0, \pi)$.

3. It would be of interest to obtain results similar to Theorems 1 and 2 for sine series.

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