ISSN 0001-4346, Mathematical Notes, 2021, Vol. 110, No. 5, pp. 767–772. © Pleiades Publishing, Ltd., 2021.
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On Fourier Series on the Torus and Fourier Transforms

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Received June 2, 2021; in final form, June 30, 2021; accepted July 1, 2021

Abstract—The question of the representability of a continuous function on $\mathbb{R}^d$ in the form of the Fourier integral of a finite Borel complex-valued measure on $\mathbb{R}^d$ is reduced in this article to the same question for a simple function. This simple function is determined by the values of the given function on the integer lattice $\mathbb{R}^d$. For $d = 1$, this result is already known: it is an inscribed polygonal line. The article also describes applications of the obtained theorems to multiple trigonometric Fourier series.

DOI: 10.1134/S0001434621110134

Keywords: *Fourier series of a measure on the torus $\mathbb{T}^d$ and functions from $L_1(\mathbb{T}^d)$, variation of a measure, Wiener Banach algebras, positive definite functions, exponential entire functions, $(C,1)$-means of Fourier series, Vitali variation, Banach–Alaoglu theorem.*

We will write the Fourier series of a function $f \in L_1(\mathbb{T}^d)$, where $\mathbb{T}^d = [-\pi,\pi)^d$ is the torus, in the form ($x = (x_1,\dots,x_d)$, $(x,y) = \sum_{i=1}^d x_j y_j$, $|x| = \sqrt{(x,x)}$)

$$f \sim \sum_{k\in\mathbb{Z}^d} \widehat{f}_k e_k, \qquad \widehat{f}_k = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x)e^{-i(k,x)}\,dx, \qquad e_k = e^{i(k,x)}.$$

If μ is a finite Borel (complex-valued) measure on $\mathbb{T}^d$, then we write its Fourier series in the form (see [1, Chap. 3])

$$\sum_{k\in\mathbb{Z}^d} \widehat{f}_k e_k, \qquad \widehat{f}_k = \int_{\mathbb{T}^d} e^{-i(k,x)}\,d\mu(x).$$

If μ is a finite Borel measure on $\mathbb{R}^d$ and $|\mu|$ is its variation (see, for example, [2, Chap. XI]), then Wiener Banach algebras are defined as follows:

$$W = W(\mathbb{R}^d) = \left\{ f : f(x) = \int_{\mathbb{R}^d} e^{-i(x,y)}\,d\mu(y),\ \|f\|_W = |\mu|(\mathbb{R}^d)\right\},$$
$$W_0 = W_0(\mathbb{R}^d) = \left\{ f : f(x) = \int_{\mathbb{R}^d} g(y)e^{-i(x,y)}\,dy,\ \|f\|_{W_0} = \|g\|_{L_1(\mathbb{R}^d)}\right\};$$

see [3, Chap. 6] and, most importantly, the survey [4], in which the list of references contains 175 titles.

The set of continuous positive definite functions on $\mathbb{R}^d$ will be denoted by $W^+(\mathbb{R}^d)$. These are functions from $W(\mathbb{R}^d)$ defined by the condition $\|f\|_W = f(0)$.

Let $d = 1$. Denote by l_f a piecewise linear continuous function defined by the conditions $l_f(k) = f(k)$, $k \in \mathbb{Z}$ (a polygonal line). Further, it was noted in the book [5, Chaps. XIX, 16] that, together with f, also l_f belongs to $W^+(\mathbb{R}^1)$. Therefore,

$$\|l_f\|_{W^+} = l_f(0) = f(0) = \|f\|_{W^+}. \tag{1}$$

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It follows that always

$$\|l_f\|_W \le 6\|f\|_W. \tag{2}$$

Indeed, if the measure μ in the representation of f is real, then

$$f(x) = \int_{\mathbb{R}} e^{-ixy}\, d\mu(y) = \int_{\mathbb{R}} e^{-ixy}\, d|\mu| - \int_{\mathbb{R}} e^{-ixy}\, d(|\mu| - \mu) = f_1(x) - f_2(x)$$

($f_1, f_2 \in W^+(\mathbb{R})$). Obviously, $l_{f_1} - l_{f_2} = l_f$ and $\|l_f\|_W \le 3\|f\|_W$. In the general case, $\|l_f\|_W \le 6\|f\|_W$.

In the recent paper [6], this inequality with coefficient 1 (instead of 6) was proved and different applications were given (from Wiener algebras to Fourier series and from Fourier series to algebras). These applications are completely new and cannot be obtained without theorems of this kind. Incidently, similar arguments were given in [7], but without applications, because an important theorem was lacking (see Theorem 1 below).

The purpose of this paper is to prove the following two theorems and their application to Fourier series in d variables.

Theorem 1. *For the trigonometric series $\sum_{k\in\mathbb{Z}^d} c_k e_k$ to be the Fourier series of a Borel measure μ on $\mathbb{T}^d$ (of a function $f \in L_1(\mathbb{T}^d)$), it is necessary and sufficient that there exist a function $\varphi \in W(\mathbb{R}^d)$ ($\varphi \in W_0(\mathbb{R}^d)$) with the condition $\varphi(k) = c_k$, $k \in \mathbb{Z}^d$. In addition,*

$$|\mu|(\mathbb{T}^d) = \min_{\varphi} \|\varphi\|_W$$

(*the minimum over such functions*) *and this minimum is attained at*

$$\varphi_0(x) = \int_{\mathbb{T}^d} e^{-i(x,y)}\, d\mu(y).$$

In the class of entire functions of exponential type at most π in each variable, there is only one such function in W_0. Further, the measure $\mu \ge 0$ if and only if such a function $\varphi \in W^+(\mathbb{R}^d)$ exists.

Theorem 2. 1) *Consider the cube*

$$\Pi_k = \{x \in \mathbb{R}^d : k_j \le x_j \le k_j + 1,\ 1 \le j \le d\}.$$

Any function $\mathbb{R}^d \to \mathbb{C}$ which is linear in each variable $x_1, \dots, x_d$ on each such cube is completely determined by the values at the vertices of such cubes ($k \in \mathbb{Z}^d$) and is continuous on $\mathbb{R}^d$.

2) *If $f \in W$, and l_f is a function from* 1) *defined by the condition $l_f(k) = f(k)$, $k \in \mathbb{Z}^d$, then*

$$\|l_f\|_W \le \|f\|_W, \qquad \|l_f\|_{W_0} \le \|f\|_{W_0}, \qquad \|l_f\|_{W^+} = \|f\|_{W^+}.$$

Proof of Theorem 1. In the case of the Fourier series of a measure μ, we have

$$\min_{\varphi} \|\varphi\|_W \le \|\varphi_0\|_W = |\mu|(\mathbb{T}^d). \tag{3}$$

On the other hand, if

$$\varphi(x) = \int_{\mathbb{R}^d} e^{-i(x,y)}\, d\mu(y), \qquad \|\varphi\|_W = |\mu|(\mathbb{R}^d),$$

then, for $k \in \mathbb{Z}^d$, by virtue of the periodicity and Fubini's theorem,

$$\begin{aligned} c_k = \varphi(k) &= \int_{\mathbb{R}^d} e^{-i(k,y)}\, d\mu(y) = \sum_{m\in\mathbb{Z}^d} \int_{\mathbb{T}^d + 2\pi m} e^{-i(k,y)}\, d\mu(y) \\ &= \sum_{m\in\mathbb{Z}^d} \int_{\mathbb{T}^d} e^{-i(k,y)}\, d\mu(y + 2\pi m) = \int_{\mathbb{T}^d} e^{-i(k,y)} \sum_{m\in\mathbb{Z}^d} d\mu(y + 2\pi m), \end{aligned}$$

i.e.,

$$c_k = \int_{\mathbb{T}^d} e^{-i(k,y)}\, d\mu_1(y), \qquad |\mu_1|(\mathbb{T}^d) \le |\mu|(\mathbb{R}^d),$$

and this series is the Fourier series of the measure μ_1.

It remains to take into account that, for any extension of φ from $\mathbb{Z}^d$ to $\mathbb{R}^d$ (see also (3)),

$$\|\varphi_0\|_W = |\mu_1|(\mathbb{T}^d) \le |\mu|(\mathbb{R}^d) = \|\varphi\|_W.$$

But if φ is an entire function of type at most π in $z_1,\dots,z_d$, then the uniqueness of φ_0 follows from the fact that any such function vanishing for $z=k$, $k\in\mathbb{Z}^d$, after division by $\prod_{j=1}^d \sin\pi z_j$, is entire and bounded on $\mathbb{C}^d$ (see, for example, [3, 3.4.4]) and, therefore, it is a constant. But then, for some $\lambda\in\mathbb{C}$,

$$\varphi(x) = \varphi_0(x) + \lambda\prod_{j=1}^d \sin\pi x_j,$$

and if the limits of φ and φ_0 exist as $|x|\to\infty$, then we have $\lambda=0$.

The same argument applies to Fourier series of functions $f\in L_1(\mathbb{T}^d)$. □

Proof of Theorem 2. 1) The boundary of Π_k consists of cubes of dimension from 1 to $d-1$. On the edges of Π_k (all the coordinates, except one, of the points are fixed), such a linear function is uniquely defined by the values at the endpoints (these are the vertices of Π_k). In the case of squares (all the coordinates, except two, are fixed), we draw a segment parallel to the coordinate axis and again use linearity, etc.

The coefficients of such a polynomial of degree d are easy to find.

For example, for $d=2$, we can express this polynomial as

$$\begin{aligned} a_1(x_1-k_1-1)(x_2-k_2-1) &+ a_2(x_1-k_1)(x_2-k_2-1) \\ + a_3(x_1-k_1-1)(x_2-k_2) &+ a_4(x_1-k_1)(x_2-k_2) \end{aligned}$$

and, substituting the vertices (k_1,k_2), (k_1+1,k_2), (k_1,k_2+1), and (k_1+1,k_2+1), we successively obtain a_1, a_2, a_3, and a_4. Further, the constant mixed derivative of the polynomial a_1 is equal to the mixed difference over 2^d vertices of Π_k.

2) For $x\in\mathbb{R}^d$ ($h\in\mathbb{R}$, $h_+=\max\{h,0\}$), we assume

$$l_f(x) = \sum_{k\in\mathbb{Z}^d} f(k)\prod_{j=1}^d (1-|x_j-k_j|)_+. \tag{4}$$

Let

$$\Pi_k \subset \widetilde{\Pi}_k := \{x : |x_j-k_j|\le 1,\ 1\le j\le d\}.$$

If $m\in\mathbb{Z}^d$, and $x\notin\widetilde{\pi}_m$, then there exists a j_0 such that $|x_{j_0}-k_{j_0}|>1$, and hence $l_f(x)=0$. Therefore, for $x\in\widetilde{\Pi}_m$,

$$l_f(x) = f(m)\prod_{j=1}^d (1-|x_j-m_j|), \qquad l_f(m)=f(m)$$

and, for $x\in\Pi_m$,

$$l_f(x) = f(m)\prod_{j=1}^d (m_j+1-x_j).$$

It follows from the same linearity property that $l_f\in C(\mathbb{R}^d)$.

Let us now prove that if $f \in W(\mathbb{R}^d)$, then $\|l_f\|_W \le \|f\|_W$. Let us first assume that the function f is compactly supported. Then the sum in the definition of l_f (see (4)) is finite.

We have

$$\int_{\mathbb{R}^d} l_f(x)e^{i(x,y)}\,dx = \sum_k f(k)\int_{\mathbb{R}^d}(1-|x_j-k_j|)_+e^{i(x,y)}\,dx$$
$$= \sum_k f(k)\prod_{j=1}^d\int_{\mathbb{R}^1}(1-|x_j-k_j|)_+e^{ix_jy_j}\,dx_j.$$

Since, for k and $y \in \mathbb{R}^1$,

$$\int_{\mathbb{R}^1}(1-|x-k|)_+e^{ixy}\,dx = e^{iky}\int_{\mathbb{R}^1}(1-|x|)_+e^{ixy}\,dx = e^{iky}\biggl(\frac{2\sin(y/2)}{y}\biggr)^2,$$

it follows that

$$\int_{\mathbb{R}^d} l_f(x)e^{i(x,y)}\,dx = \sum_{k\in\mathbb{Z}^d} f(k)e^{i(k,y)}\prod_{j=1}^d\biggl(\frac{2\sin(y_j/2)}{y_j}\biggr)^2$$

and, by the inverse formula for the Fourier transform,

$$\|l_f\|_W = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\biggl|\sum_k f(k)e^{i(k,y)}\prod_{j=1}^d\biggl(\frac{2\sin(y_j/2)}{y_j}\biggr)^2\biggr|\,dy.$$

In the general case, we apply this equality to $f_n(x) = f(x)\prod_{j=1}^d(1-|x_j|/n)_+$, obtaining

$$(2\pi)^d\|l_{f_n}\|_W = \int_{\mathbb{R}^d}\biggl|\sum_k f(k)\prod_{j=1}^d\biggl(1-\frac{|k_j|}{n}\biggr)_+e^{i(k,y)}\prod_{j=1}^d\biggl(\frac{2\sin(y_j/2)}{y_j}\biggr)^2\biggr|\,dy.$$

As before, $\mathbb{R}^d = \bigcup_m(\mathbb{T}^d + 2\pi m)$ and, due to periodicity,

$$(2\pi)^d\|l_{f_n}\|_W = \int_{\mathbb{T}^d}\biggl|\sum_k f(k)\prod_{j=1}^d\biggl(1-\frac{|k_j|}{n}\biggr)_+e^{i(k,y)}\sum_m\prod_{j=1}^d\biggl(\frac{2\sin(y_j/2)}{y_j+2\pi m_j}\biggr)^2\biggr|\,dy.$$

But always

$$\sum_m\prod_j|a_{m_j}| \le \prod_j\sum_m|a_{m_j}|$$

and, for $y \in \mathbb{R}$,

$$\sum_{m=-\infty}^{\infty}\biggl(\frac{2\sin(y/2)}{y+2\pi m}\biggr)^2 \equiv 1$$

(this well-known equality can be obtained, for example, from the partial fraction expansion of the meromorphic function $1/\sin^2(z/2)$). Thus,

$$(2\pi)^d\|l_{f_n}\|_W \le \int_{\mathbb{T}^d}\biggl|\sum_k f(k)\prod_{j=1}^d\biggl(1-\frac{|k_j|}{n}\biggr)_+e^{i(k,y)}\biggr|\,dy.$$

By virtue of Theorem 1 (under the condition $f \in W(\mathbb{R}^d)$), the series $\sum_k f(k)e^{i(k,y)}$ is the Fourier series of a measure μ on $\mathbb{T}^d$. But then, under the modulus sign, we have the $(C,1)$-means of σ_n of this series and, therefore,

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}|\sigma_n(y)|\,dy \le |\mu|(\mathbb{T}^d) \le \|f\|_W.$$

Let us pass to the limit as $n \to \infty$, taking into account the fact that $l_{f_n} \to l_f$ everywhere and that $l_f \in C(\mathbb{R}^d)$ (see [8, Theorem 2]). Then we see that $\|l_f\|_W \le \|f\|_W$, and also that f and l_f belong to $W^+(\mathbb{R}^d)$ (see (1)), and now, by virtue of (2), also that f and l_f belong to $W_0(\mathbb{R}^d)$. □

Also note that the condition $l_f \in W$ needs to be checked only near ∞, because W is an algebra with a local property and, for $|x| \le N$, the function l_f has a bounded mixed derivative if the function itself is bounded (see [4, 7.2, 7.3]).

Turning to the applications, we denote by l_c the function from Theorem 2 with the conditions

$$l_c(k) = c_k, \qquad k \in \mathbb{Z}^d.$$

Proposition 1. *For the series $\sum_{k\in\mathbb{Z}^d} c_k e_k$ to be the Fourier series of a function (measure), it is necessary and sufficient that $l_c \in W_0(\mathbb{R}^d)$ ($l_c \in W(\mathbb{R}^d)$).*

Proof. It follows from Theorems 1 and 2. □

Proposition 2. *For the series $\sum_{k\in\mathbb{Z}^d} c_k e_k$ to be the Fourier series of a function of Vitali bounded variation on $\mathbb{T}^d$, it is necessary and sufficient that $l_c(x) \in W_0(\mathbb{R}^d)$ and $l_c(x)\prod_{j=1}^d x_j \in W(\mathbb{R}^d)$.*

Proof. By definition (see, for example, [4, 4.2]), the Vitali variation is

$$V_{\text{vit}}(f) = \sup \sum |\Delta_h f(x)|,$$

where (e_j is the unit vector on the axis ox_j, $|h_j| > 0$, and the supremum is taken over all admissible x and h)

$$\Delta_h f(x) = \left(\prod_{j=1}^d \Delta_{h_j}\right) f(x), \qquad \Delta_{h_j} f(x) = f(x + h_j e_j) - f(x).$$

For example, for smooth functions,

$$V_{\text{vit}}(f) = \int \left| \frac{\partial^d f(x)}{\partial x_1 \cdots \partial x_d} \right| dx.$$

It is only necessary to take into account that, for periodic functions f,

$$f \in V_{\text{vit}}(\mathbb{T}^d)$$
$$\longleftrightarrow \quad \sup_n \int_{\mathbb{T}^d} \left| \frac{\partial^d \sigma_n(f)}{\partial x_1 \cdots \partial x_d} \right| dx = \sup_n \int_{\mathbb{T}^d} \left| \sum_k f(k) \prod_{j=1}^d k_j \left(1 - \frac{|k_j|}{n}\right)_+ e^{i(k,x)} \right| dx < \infty.$$

The need for a condition involving σ_n is obvious if we proceed from the definition of the variation V_{vit} (the case $d = 1$ was considered in [9, Chaps. 1, 60]). To prove sufficiency, we apply either the Banach–Alaoglu theorem or simply Banach's theorem, because the space $C(\mathbb{T}^d)$ is separable. The condition in question means that the norms of σ_n in the space conjugate to $C(\mathbb{T}^d)$ are, for example, bounded by the number M. But the ball in such a space is weakly compact, i.e., there is a subsequence σ_n weakly converging to a function from $C(\mathbb{T}^d)$ (converging pointwise everywhere). But then, also for the limit function, we have $V_{\text{vit}} \le M$. □

Proposition 3. *If $\sum c_k e_k$ is the Fourier series of a measure, $\lim_{|k|\to\infty} c_k = 0$ (which is also necessary), and*

$$\int_{\mathbb{R}^d} \left| \frac{\partial^d l_c(x)}{\partial x_1 \cdots \partial x_d} \right| dx < \infty$$

or, which is the same, $l_c(x)\prod_{j=1}^d x_j \in W(\mathbb{R}^d)$, then $\sum c_k e_k$ is the Fourier series of a function from $L_1(\mathbb{T}^d)$.

Proof. Let us use Proposition 2, Theorem 2 from [8] (if $f \in W(\mathbb{R}^d)$: $f(\infty) = 0$ and, outside of some cube, f is a function of Vitali bounded variation, then $f \in W_0(\mathbb{R}^d)$), and Theorem 1.

Just as in Propositions 1 and 2 (which are criteria), it is possible to formulate a boundedness criterion for Fourier partial sums in $L_1(\mathbb{T}^d)$ and a convergence criterion for Fourier series in $L_1(\mathbb{T}^d)$. In the first case, the sequence of norms W_0 will be bounded, while, in the second case, the zero limit will appear. □

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