ISSN 0001-4346, Mathematical Notes, 2021, Vol. 110, No. 5, pp. 732–737. © Pleiades Publishing, Ltd., 2021.
Russian Text © The Author(s), 2021, published in Matematicheskie Zametki, 2021, Vol. 110, No. 5, pp. 744–750.

Automorphisms of Surfaces of Markov Type

A. Yu. Perepechko[1,2,3*]

[1] *Institute for Information Transmission Problems of Russian Academy of Sciences (Kharkevich Institute), Moscow, 127051 Russia*

[2] *Moscow Institute of Physics and Technology (National Research University), Dolgoprudnyi, Moscow Oblast, 141701 Russia*

[3] *National Research University Higher School of Economics, Moscow, 101978 Russia*

Received March 12, 2020; in final form, June 6, 2021; accepted June 25, 2021

Abstract—Affine algebraic surfaces of Markov type of the form

$$x^2 + y^2 + z^2 - xyz = c$$

are studied. Their automorphism groups are found.

DOI: 10.1134/S0001434621110109

Keywords: *affine surface, automorphism group, Thompson group, Markov numbers, birational transformations, boundary divisor.*

1. INTRODUCTION

By analogy with Markov surfaces, we define more general surfaces of Markov type and find their automorphism groups. Section 2 contains preliminary information about the completions of affine surfaces, their birational transformations, and dual graphs of boundary divisors. Section 3 contains a description of a completion, the boundary divisor, relationship with the Thompson group, and the automorphism group for each of the considered surfaces.

2. PRELIMINARIES ON COMPLETIONS AND DUAL GRAPHS

2.1. Completions of Affine Surfaces

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, and let Y be an irreducible affine algebraic surface over $\mathbb{K}$. By a *completion* of Y we mean an open embedding $Y \hookrightarrow X$ into an irreducible projective surface X. For example, if $Y \subset \mathbb{A}^n \subset \mathbb{P}^n$, then can take $X = \overline{Y}$, the Zariski closure of Y in the projective space $\mathbb{P}^n$.

We assume that the *boundary divisor* $D = X \setminus Y$ of a completion is *simple normal crossing*, i.e., each singular point of the divisor is the transversal intersection of a pair of its different irreducible components (i.e., curves). Under these conditions, the pair (X, D) is called an *NC-pair*. The divisor D is associated with the weighted *dual graph* $\Gamma(D)$ defined as follows:

- the vertices of $\Gamma(D)$ are the irreducible components of D;
- the weights of vertices are the self-intersection numbers of the components;
- the edges are the intersection points of the components of D.

Note that the dual graph may have multiple edges.

The automorphisms of an affine surface Y extend to birational transformations of the completion X and can be described in terms of blow-ups and blow-downs of the boundary divisor D. The group of birational transformations of X regular on Y is denoted by $\mathrm{Bir}(X, D)$. It is isomorphic to $\mathrm{Aut}(Y)$. The induced *birational* transformations of the dual graph $\Gamma(D)$ were described in [1].

*E-mail: a@perep.ru

2.2. Weighted Graphs

We recall the necessary information and results of [1] concerning dual graphs.

Definition 2.1 [1, Definition 2.13]. By $((w_1,\dots,w_n))$, where $w_1,\dots,w_n\in\mathbb{N}$, we denote the weighted graph on n vertices such that this graph is a cycle and the weights of vertices (numbered in the order in which they occur on traveling along the cycle) are $w_1,\dots,w_n$. Such a weighted graph is said to be *circular*.

A circular graph $((w_1,\dots,w_n))$ is *standard* if one of the following conditions holds:

- $w_1=\dots=w_{2k}=0$ and $w_{2k+1},\dots,w_n\le -2$ for some $k\ge 0$;
- $w_1=\dots=w_{n-1}=0$ and $w_n\le 0$;
- $w_1=\dots=w_{n-2}=0$, $w_{n-1}=w_n=-1$, and n is even.

Definition 2.2 [1, Definition 2.3]. An *inner blow-up* at an edge e is a transformation of a weighted graph Γ which consists in subdividing the edge e by a new vertex of weight -1 and decreasing the weights of the endvertices of e by 1. If the edge e is a loop, then the weight of its terminal vertex is decreased by 2.

An *outer blow-up* at a vertex v is the transformation of a weighted graph Γ which consists in adding a leaf vertex of weight -1 and an edge joining it with v followed by decreasing the weight of v by 1.

An outer (inner) *blow-down* is a transformation of Γ inverse to an outer (respectively, inner) blow-up.

A *birational transformation* of a graph Γ is a composition of blow-downs and blow-ups.

Remark 2.3 [1, Sec. 3.5]. Let (X,D) be an NC-pair. A blow-up on the boundary divisor D at an intersection point of two components corresponds to an inner blow-up of the dual graph $\Gamma(D)$, and that at a smooth point, to an outer one. Birational maps of an NC-pair (X,D) regular on $Y=X\setminus D$ generate birational transformations of the graph $\Gamma(D)$, and vice versa (up to the choice of a blow-up point in the case of an outer blow-up).

Definition 2.4 [1, Definition 2.10]. Suppose given a weighted graph Γ and its vertex v of degree 2 and weight 0. An *elementary* transformation of the graph Γ consists in the inner blow-up at an edge incident to v and the inner blow-down at v followed by replacing the vertex v by a new vertex of weight 0 with increasing the weight of one of the vertices neighboring v by 1 and decreasing the weight of the other by 1.

Theorem 2.5 [1, Corollary 3.19]. *Any birational transformation of a standard circular weighted graph Γ can be represented as a composition of elementary transformations.*

3. AUTOMORPHISMS OF SURFACES

3.1. Markov Surface

The positive integer solutions of the Diophantine equation

$$x^2+y^2+z^2=3xyz \tag{3.1}$$

are called the *Markov triples*, and the elements of Markov triples are called the *Markov numbers*. They were described in 1879–1880 by Markov [2], [3]. Their most remarkable property is that all Markov triples are obtained from $(1,1,1)$ by permuting coordinates and applying the involution

$$x\mapsto 3yz-x,\qquad y\mapsto y,\qquad z\mapsto z.$$

The celebrated *unicity conjecture* is that *each Markov number is the largest element of precisely one Markov triple* (*ordered by decreasing magnitude*). Various interpretations of the Markov numbers, as well as their context and history, were described in [4].

The equation $x^2+y^2+z^2=xyz$ is equivalent to Eq. (3.1): its solutions are the tripled solutions of (3.1) [4, Proposition 2.2]. The algebraic hypersurfaces determined by these equations in $\mathbb{A}^3 = \operatorname{Spec}\mathbb{K}[x,y,z]$ are isomorphic; the isomorphism is given by the map $(x,y,z) \mapsto (3x,3y,3z)$. Each of these two surfaces is called the *Markov surface* [5].

We deal with the surfaces

$$M_c = \{x^2 + y^2 + z^2 - xyz = c\} \subset \mathbb{A}^3,$$

where $c \in \mathbb{Z}$, and call them *surfaces of Markov type*. They are also known as generalized Markov surfaces, or tetrahedral Goursat surfaces.

We also use the two-dimensional algebraic torus

$$\mathbb{T} = (\mathbb{K}^\times)^2 = \operatorname{Spec}\mathbb{K}[x^{\pm 1}, y^{\pm 1}].$$

3.2. Completions

Consider the completion

$$\mathbb{T} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 = \{((x_1 : x_2), (y_1 : y_2))\}$$

of the two-dimensional algebraic torus defined by

$$(x,y) \mapsto ((x:1),(y:1)).$$

For surfaces M_c, $c \in \mathbb{K}$, we will consider completions

$$M_c \subset \overline{M}_c = \{t(x^2+y^2+z^2) - xyz = ct^3\} \subset \mathbb{P}^3 = \{(x:y:z:t)\}.$$

3.3. Boundary Divisors

The boundary divisor $D_{\mathbb{T}} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathbb{T}$ consists of the following four straight lines (below ∞ denotes $(1:0)$):

$$\begin{aligned} L_1 &= \{x_1 = 0\} = \{(0,t) \mid t \in \mathbb{K} \cup \{\infty\}\}, \\ L_2 &= \{x_2 = 0\} = \{(\infty,t) \mid t \in \mathbb{K} \cup \{\infty\}\}, \\ L_3 &= \{y_1 = 0\} = \{(t,0) \mid t \in \mathbb{K} \cup \{\infty\}\}, \\ L_4 &= \{y_2 = 0\} = \{(t,\infty) \mid t \in \mathbb{K} \cup \{\infty\}\}. \end{aligned}$$

The self-intersection numbers of these lines are zero. The dual graph of the divisor is the cycle on the four vertices corresponding to the given lines, i.e., the cycle $((0,0,0,0))$ in the notation of [1, Sec. 2.1].

The boundary divisor $\overline{D} = \overline{M}_c \setminus M_c$ consists of three lines with self-intersection number -1:

$$\begin{aligned} L_x &= \{x = t = 0\}, \\ L_y &= \{y = t = 0\}, \\ L_z &= \{z = t = 0\}. \end{aligned}$$

Blowing down one of them, say L_z, we obtain a completion $\widehat{M_c}$ with boundary divisor $\widehat{D} = \widehat{L}_x \cup \widehat{L}_y$, where $\widehat{L}_x$ and $\widehat{L}_y$ are the images of L_x and L_y, respectively. The dual graph $\Gamma(\widehat{D})$ is the cycle $((0,0))$ on two vertices of weight 0.

Remark 3.1. The projection $\phi\colon \mathbb{P}^3 \to \mathbb{P}^2$, $(x:y:z:t) \mapsto (x:y:z)$, birationally maps the (completed) Markov surface $\overline{M}_0$ to $\mathbb{P}^2$. The inverse map is given by

$$\phi^{-1} : (x:y:z) \mapsto (xQ : yQ : zQ : xyz),$$

where $Q = x^2 + y^2 + z^2$. The conic $\{Q = 0\}$ intersects each of the coordinate lines $x = 0$, $y = 0$, and $z = 0$ at two points, and these points are not the intersection points of coordinate lines.

The map $\phi^{-1}\colon \mathbb{P}^2 \to \overline{M}_0$ is the composition of the blow-ups of the six intersection points of Q with the coordinate lines and the blow-down of the (-2)-curve Q, which gives a quadratic singular point, i.e., a Du Val singularity of type A_1, at $(0:0:0:1) \in \mathbb{P}^3$. The images of the coordinate lines in $\mathbb{P}^2$ form a boundary divisor $\overline{M}_0 \setminus M_0$, and their self-intersection numbers equal -1.

3.4. Automorphism Groups

It is well known (see, e.g., [6, Example 1]) that

$$\begin{aligned}\mathrm{Aut}(\mathbb{T}) &= \left\{(x,y)\mapsto(t_1x^ay^b,t_2x^cy^d)\mid(t_1,t_2)\in\mathbb{T},\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}(2,\mathbb{Z})\right\}\\&\cong\mathbb{T}\rtimes\mathrm{GL}(2,\mathbb{Z}).\end{aligned}$$

This follows directly from the fact that automorphisms preserve the set $\{tx^ay^b\mid a,\,b\in\mathbb{Z},\,t\in\mathbb{K}\}\subset\mathbb{K}[\mathbb{T}]$ of invertible functions. The connected component $\mathrm{Aut}^\circ(\mathbb{T})=\mathbb{T}$ acts trivially on the dual graph $\Gamma(\mathbb{P}^1\times\mathbb{P}^1,D_{\mathbb{T}})$, and the discrete part $\mathrm{Aut}(\mathbb{T})/\mathbb{T}$ is isomorphic to $\mathrm{GL}(2,\mathbb{Z})$.

Let us introduce notation for some elements of $\mathrm{GL}(2,\mathbb{Z})$:

$$C_2=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}:(x,y)\mapsto\left(\frac{1}{x},\frac{1}{y}\right),$$

$$C_4=\begin{pmatrix}0&1\\-1&0\end{pmatrix}:(x,y)\mapsto\left(y,\frac{1}{x}\right),$$

$$C_6=\begin{pmatrix}1&1\\-1&0\end{pmatrix}:(x,y)\mapsto\left(xy,\frac{1}{x}\right).$$

We have $C_4^2=C_6^3=C_2$, $C_2^2=\mathrm{id}$; moreover, as is known [7, Proposition 2.1],

$$\mathrm{SL}(2,\mathbb{Z})=\langle C_4\rangle *_{\langle C_2\rangle}\langle C_6\rangle$$

is the amalgamated product of the finite cyclic subgroups generated by C_4 and C_6, respectively.

The automorphism C_4 cyclically permutes the components as $L_1\to L_4\to L_2\to L_3\to L_1$, and C_6 is the composition of two elementary transformations at opposite vertices; namely, it consists in blowing up the points $p_1=(0,\infty)$ and $p_2=(\infty,0)$ and blowing down the images of the lines L_1 and L_2. Blowing up the points p_1 and p_2, whose exceptional curves we denote by E_1 and E_2, respectively, we obtain a completion $T\hookrightarrow Bl_{p_1,p_2}(\mathbb{P}^1\times\mathbb{P}^1)$ with boundary divisor $D'_{\mathbb{T}}$, whose components are cyclically permuted by the automorphism C_6 as $L_1\to E_1\to L_4\to L_2\to E_2\to L_3\to L_1$.

Consider the following automorphisms of the affine surface M_c:

- the sign changes $(x,y,z)\mapsto(\varepsilon_xx,\varepsilon_yy,\varepsilon_zz)$, where $\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{-1,1\}$ and $\varepsilon_x\varepsilon_y\varepsilon_z=1$, form the Klein group $K\cong(\mathbb{Z}/2\mathbb{Z})^2$;
- the symmetric group S_3 acts on M_c by permutations of coordinates, and its extension to $\overline{M}_c$ acts on the boundary divisor by permutations of the components L_x, L_y, and L_z;
- the map

  $$\sigma:\qquad x\mapsto y,\quad y\mapsto x,\quad z\mapsto xy-z$$

  is an involution on M_c, and its extension to $\overline{M}_c$ transposes the lines L_x and L_y, blows down the line L_z to a point, and blows up the intersection point of L_x and L_y.

Proposition 3.2. *The subgroup of those automorphisms of M_c which have regular extensions to $\overline{M}_c$ coincides with the image of $S_3\ltimes K$ under the embedding specified above.*

Proof. Let $g\in\mathrm{Aut}(M_c)$ be an automorphism admitting a regular extension to $\overline{M}_c$ and not lying in S_3. Up to multiplication by an element of S_3, we can assume that g takes each of the lines L_x, L_y, and L_z to itself. It follows that $g^*(x)=ax+b$ for some $a,b\in\mathbb{K}^\times$, and $g^*(y)$ and $g^*(z)$ have similar expressions. It is easy to check that $g^*(x)=\pm x$, $g^*(y)=\pm y$, $g^*(z)=\pm z$, and $g\in K$. This contradiction completes the proof. $\square$

Theorem 3.3. *The following isomorphisms hold*: $\mathrm{Aut}(M_c)=\langle S_3,K,\sigma\rangle\cong(\mathbb{Z}/2\mathbb{Z})^2\rtimes\mathrm{PGL}(2,\mathbb{Z})$.

Proof. Consider the following maps of boundary divisors, which are unramified double coverings:

$$\psi\colon D_{\mathbb{T}} \to \widehat{D}, \qquad L_1, L_2 \xrightarrow{\cong} \widehat{L}_y, \qquad L_3, L_4 \xrightarrow{\cong} \widehat{L}_x,$$
$$\psi'\colon D'_{\mathbb{T}} \to \overline{D}, \qquad L_1, L_2 \xrightarrow{\cong} L_y, \qquad L_3, L_4 \xrightarrow{\cong} L_x, \qquad E_1, E_2 \xrightarrow{\cong} L_z.$$

The blow-down of E_1, E_2, and L_z transforms ψ' into ψ.

Note that C_4 and C_6 induce birational transformations of the dual graph $\Gamma(D_{\mathbb{T}})$ which are equivariant with respect to ψ. Moreover, their images generate all elementary transformations of the dual graph $\Gamma(\overline{D})$. On the other hand, the action of C_4 on $\Gamma(D_{\mathbb{T}})$ coincides with that of σ, and the action of C_6 coincides with that of the permutation $x \mapsto y \mapsto z \mapsto x$ of coordinates.

Therefore, by Theorem 2.5 and Proposition 3.2, the group $\mathrm{Aut}(M_c)$ is generated by $S_3 \ltimes K$ and σ. Since S_3 and σ are lifted to transformations of $\Gamma(D_{\mathbb{T}})$ and K acts trivially on $\Gamma(D_{\mathbb{T}})$, it follows that

$$\mathrm{Aut}(M_c) \cong K \rtimes \mathrm{GL}(2,\mathbb{Z})/C_2 \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathrm{PGL}(2,\mathbb{Z}). \quad \square$$

Corollary 3.4. *The automorphisms of surfaces of Markov type preserve the subset of integer points and are generated by involutions.*

Remark 3.5. Although the group $\mathrm{Aut}(M_c)$ of algebraic automorphisms is discrete, the group of holomorphic automorphisms has an open orbit [8, Theorem 1.4].

Example 3.6. Consider the surface $Y = \mathbb{T}/C_2$. Its function algebra equals

$$\mathbb{K}[Y] = \mathbb{K}[\mathbb{T}]^{C_2} = \mathbb{K}[a,b,c]/(a^2+b^2+c^2-abc-4),$$

where $a = x + 1/x$, $b = y + 1/y$, and $c = xy + 1/(xy)$. Therefore, $Y = M_4$.

Question 3.7. Is it true that the set of all points with positive integer coordinates on M_4 decomposes into orbits of the form $\mathrm{Aut}(M_4)\cdot(a,a,2)$, and these orbits are different for different a?

Remark 3.8. In [9], surfaces determined by equations of the form $xyz = P(x,y,z)$, where $P(x,y,z)$ is any second-degree polynomial with zero constant term, were studied; they were called *surfaces of Markov type* in [9]. There surfaces have the same boundary divisor formed by the three straight lines L_x, L_y, and L_z.

The argument concerning transformations of dual graphs in the proof of Theorem 3.3 apply to these surfaces and give an alternative proof of Theorem 2 of [9]. In particular, the involutions t_1, t_2, and t_3 in [9] are obtained by conjugating the involution σ by elements of S_3.

Remark 3.9. Consider all vertices of the weighted graph $((0,0,0,0))$ which can obtained by applying compositions of inner blow-ups. There is a natural one-to-one correspondence between the set of vertices thus obtained and the set of dyadic-rational points on the unit circle $[0,1]/(0\sim 1)$. Namely, the initial vertices are identified with 0, 1/4, 1/2, and 3/4, respectively, and each vertex obtained by blowing up an intersection point is identified with the midpoint of the segment joining the points corresponding to the intersecting curves.

The resulting mapping is equivariant and gives an embedding of $\mathrm{SL}(2,\mathbb{Z})$ in the Thompson group T of piecewise linear transformations of the unit circle with dyadic-rational break points; see [10]–[12] for details. For the weighted graph $((0,0))$, there is a similar correspondence, which gives an embedding of $\mathrm{PSL}(2,\mathbb{Z})$ in T.

ACKNOWLEDGMENTS

The author thanks Vlad Sergiescu for a comprehensible introduction to the Thompson group. The author is particularly grateful to Mikhail Zaidenberg for immersing him in automorphisms of affine surfaces and for many discussions, comments, and suggestions concerning the paper. Finally, the author is grateful to Yurii Prokhorov, whose comments have enriched the paper.

FUNDING

This work was supported by the Russian Science Foundation under grant 18-71-00153.

REFERENCES

1. H. Flenner, S. Kaliman, and M. Zaidenberg, "Birational transformations of weighted graphs," in *Affine Algebraic Geometry* (Osaka Univ. Press, Osaka, 2007), pp. 107–147; "Corrigendum," in *Affine Algebraic Geometry*, 54 (Amer. Math. Soc. , Providence, RI, 2011), pp. 35–38.
2. A. Markoff, "Sur les formes quadratiques binaires indéfinies," Math. Ann. **15**, 381–406 (1879).
3. A. A. Markoff, "Sur les formes quadratiques binaires indéfinies II," Math. Ann. **17**, 379–399 (1880).
4. M. Aigner, *Markov's Theorem and 100 Years of the Uniqueness Conjecture* (Springer, Cham, 2013).
5. U. Rehmann and E. Vinberg, "On a phenomenon discovered by Heinz Helling," Transform. Groups **22**, 259–265 (2017).
6. I. Arzhantsev and S. Gaifullin, "The automorphism group of a rigid affine variety," Math. Nachr. **290** (5-6), 662–671 (2017).
7. A. Bjorner, F. Cohen, C. De Concini, C. Procesi, and M. Salvetti, *Configuration Spaces. Geometry, Combinatorics and Topology* (Edizioni della Normale, Pisa, 2012).
8. S. Kaliman, F. Kutzschebauch, and M. Leuenberger, "Complete algebraic vector fields on affine surfaces," Internat. J. Math. **31** (3) (2020), Paper no. 2050018.
9. M. H. Èl'-Huti, "Cubic surfaces of Markov type," Math. USSR-Sb. **22** (3), 333–348 (1974).
10. É. Ghys and V. Sergiescu, "Sur un groupe remarquable de difféomorphismes du cercle," Comment. Math. Helv. **62** (2), 185–239 (1987).
11. A. Fossas, "PSL$(2,\mathbb{Z})$ as a non-distorted Subgroup of Thompson's group T," Indiana Univ. Math. J. **60** (6), 1905–1925 (2011).
12. J. Diller and J.-L. Lin, "Rational surface maps with invariant meromorphic two-forms," Math. Ann. **364** (1-2), 313–352 (2016).