ISSN 0001-4346, Mathematical Notes, 2021, Vol. 110, No. 4, pp. 543–559. © Pleiades Publishing, Ltd., 2021.
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Regularity of the Solution of the Prandtl Equation

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Received May 5, 2021; in final form, May 5, 2021; accepted May 25, 2021

Abstract—Solvability and regularity of the solution of the Dirichlet problem for the Prandtl equation

$$\frac{u(x)}{p(x)} - \frac{1}{2\pi}\int_{-1}^{1}\frac{u'(t)}{t-x}\,dt = f(x)$$

is studied. Here $p(x)$ is a positive function on $(-1,1)$ such that $\sup(1-x^2)/p(x) < \infty$. We introduce the scale of spaces $\widetilde{H}^s(-1,1)$ in terms of the special integral transformation on the interval $(-1,1)$. We obtain theorems about the existence and uniqueness of the solution in the classes $\widetilde{H}^s(-1,1)$ with $0 \le s \le 1$. In particular, for $s = 1$ the result is as follows: if $r^{1/2}f \in L_2$, then $r^{-1/2}u, r^{1/2}u' \in L_2$, where $r(x) = 1 - x^2$.

DOI: 10.1134/S0001434621090248

Keywords: *Prandtl equation, weak solution, Fourier integral transformation, integral transformation on the interval.*

INTRODUCTION

0.1. The Prandtl Equation: Physical Motivation

The Prandtl equation

$$u(x) - p(x)\frac{1}{2\pi}\int_{-1}^{1}\frac{u'(t)}{t-x}\,dt = p(x)f(x), \qquad u(-1) = u(1) = 0, \tag{0.1}$$

is one of the universal equations of mathematical physics. It is used in almost all cases when thin plates (shells with boundary) are studied. In aerodynamics and hydrodynamics [1], [2], this equation describes the circulation (the load averaged along the chord $c(x)$) on a three-dimensional thin wing with span L in a stream running at the angle of attack $\alpha_0(x)$ with the speed U_0. In this case, $p(x) = a_0 c(x)/L$ and $f(x) = \alpha_0(x)U_0$, where a_0 is some constant coefficient. In magnetostatics [3], equation (0.1) describes the surface (averaged along the thickness $\delta(x)$) magnetization induced in a thin plate of width L of a ferromagnetic material with susceptibility $\varkappa$ by a transverse external magnetic field with a tangential component $f(x)$. In this case, $p(x) = \varkappa\delta(x)/L$.

In two-dimensional mechanics [4], equation (0.1) is the main tool in the study of contact problems and calculations of stiffeners. Equation (0.1) can be considered as the potential theory equation for the following boundary-value problem:

$$\begin{cases} \Delta\Phi(x,y) = 0, & (x,y) \in \mathbb{R}^2 \setminus [-1,1], \\ \partial_y\Phi^+(x,0) = \partial_y\Phi^-(x,0), & x \in [-1,1], \\ \Phi^+(x,0) - \Phi^-(x,0) + p(x)\,\partial_y\Phi^+(x,0) = p(x)f(x), & x \in [-1,1], \end{cases} \tag{0.2}$$

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if the solution is found as a double layer potential with density $u(t)$:

$$\Phi(x,y) = \frac{1}{2\pi}\int_{-1}^{1} u(t)\,\frac{\partial}{\partial\tau}\ln\frac{1}{\rho}\bigg|_{\tau=0} dt, \qquad \rho = \sqrt{(x-t)^2+(y-\tau)^2}\,,$$

or for the adjoint problem (in the sense of the Cauchy–Riemann conditions)

$$\begin{cases} \Delta\Psi(x,y) = 0, & (x,y)\in\mathbb{R}^2\setminus[-1,1], \\ \Psi^+(x,0) = \Psi^-(x,0), & x\in[-1,1], \\ \partial_y\Psi^+(x,0) - \partial_y\Psi^-(x,0) - \partial_x(p(x)\,\partial_x\Psi(x,0)) = \partial_x(p(x)f(x)), & x\in[-1,1], \end{cases} \tag{0.3}$$

if the solution is found as a single layer potential with density $u'(t)$:

$$\Psi(x,y) = \frac{1}{2\pi}\int_{-1}^{1} u'(t)\ln\frac{1}{\rho}\,dt, \qquad \rho = \sqrt{(x-t)^2+y^2}\,.$$

Boundary-value problems of the type (0.2), (0.3) arise from general three-dimensional problems, when one of the parameters of the domain (or of the surface) becomes “thin”.

As a rule, we have $p(x)\ge 0$. This function can vanish only at the ends of the interval. In general, the potential theory equations for boundary-value problems (0.2), (0.3) contain hypersingular operators. In problem (0.2), this is the normal derivative of the double layer potential, and in problem (0.3), this is the second tangential derivative of the single layer potential. However, the degeneracy at the ends of the coefficient $p(x)$ facing the integral operator smooths out such a singularity. In particular, if $p(x) = (1-x^2)p_0(x)$ and $p_0(\pm1)\ne 0$, then the operator that corresponds to the second term in (0.1) is simply singular.

Usually, it is not possible to solve the Prandtl equation exactly[1]. That is why the main literature concerning applications is devoted to the search for convenient numerical schemes. One of the most popular schemes is the Multopp method. A significant part of the monograph [4] is devoted to the justification of this method. However, the conditions under which the Multopp method is justified are too restrictive (for example, the function $\sqrt{1-x^2}/p(x)$ is required to be Hölder).

Thus, a rigorous functional study of the Prandtl equation under natural (from the point of view of physical applications) conditions on the coefficient $p(x)$ is very relevant.

0.2. Relation to the Theory of the Schrödinger Equation with Fractional Laplacian

The integral operator in equation (0.1) can also be represented as

$$-\frac{1}{2\pi}\int_{-1}^{1}\frac{u'(t)}{t-x}\,dt = \sqrt{-\frac{d^2}{dx^2}}\,[u](x).$$

The number of papers devoted to the study of equations with fractional Laplacian is huge. In particular, generalized (or weak) solutions of boundary-value problems for equations of the form

$$(-\Delta)^\sigma u(x) + V(x)u(x) = f(x), \qquad x\in\Omega\subset\mathbb{R}^n, \quad 0<\sigma<1,$$

are constructed. A survey, some new results and extensive literature on this topic can be found in [5] and [6]. Note that the one-dimensional case is usually not distinguished, although it has an important feature: the boundary of the domain is disconnected (it consists of two points ±1).

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[1] The only widely known case is the elliptic wing. If $p(x) = p_0\sqrt{1-x^2}$ and $f(x)=1$, then $u(x) = 2p_0(p_0+2)^{-1}\sqrt{1-x^2}$.

0.3. Main Results

We study problem (0.1) assuming that the coefficient $p(x)$ is a positive function on $(-1,1)$, it may vanish for $x=\pm 1$, but the order of zeros is not higher than the first degree below (see condition (1.2) on $V(x)=p(x)^{-1}$). This case includes interesting from the practical point of view examples of triangular wing of an airplane and a composite wing with a chord break of first kind.

In Sec. 1, using the Fourier transform approach, we define the weak solution of problem (0.1) in the class $H^{1/2}_{00}(-1,1)$ (consisting of functions u on the interval $(-1,1)$ that can be extended by zero to functions of the Sobolev class $H^{1/2}(\mathbb{R})$). The equation is replaced by appropriate integral identity, and it is assumed that $f(x)$ belongs to the class dual to $H^{1/2}_{00}(-1,1)$ (with respect to the pairing in $L_2(-1,1)$). We prove theorem about the existence and uniqueness of the solution. (Of course, analogs of these results are known; see [5].)

However, the obtained solution $u\in H^{1/2}_{00}(-1,1)$ does not even have to be continuous. And from the physical point of view, one is interested in continuous solutions. Therefore, we need to investigate the question of additional regularity of the solution (under the previous assumptions on the coefficient $p(x)$ by strengthening the assumptions on the function $f(x)$).

In the authors opinion, the Fourier transformation on the axis is not the most convenient tool for studying the problem on a finite interval. Therefore, investigating regularity of the solution, we use a special integral transformation $\mathscr{P}$ on the interval, which was introduced and studied in [7]. Earlier, application of the transform $\mathscr{P}$ made it possible to solve many problems on the interval, which were previously either solved in a much more laborious way [8] or remained open [9], [10]. The definition and the main properties of $\mathscr{P}$ are described in Sec. 2. We introduce the interpolational scale of spaces $\widetilde{H}^s(-1,1)$, $s\ge 0$, in terms of the transform $\mathscr{P}$.

In Sec. 3, we give an independent definition of the weak solution

$$u\in \widetilde{H}^{1/2}(-1,1)=H^{1/2}_{00}(-1,1)$$

of problem (0.1) in terms of the transform $\mathscr{P}$, and then establish a theorem about additional regularity of the solution. Namely, under the assumption that $\int_{-1}^{1}(1-x^2)|f(x)|^2\,dx<\infty$ it is proved that the solution belongs to the class $\widetilde{H}^1(-1,1)$ (which is distinguished by condition (2.6)) and satisfies the estimate

$$\int_{-1}^{1}\biggl(\frac{|u(x)|^2}{1-x^2}+(1-x^2)|u'(x)|^2\biggr)\,dx\le C\int_{-1}^{1}(1-x^2)|f(x)|^2\,dx.$$

We obtain the solvability of problem (0.1) in $\widetilde{H}^s(-1,1)$, where $1/2\le s\le 1$, by interpolation (under the assumption that f belongs to the class dual to $\widetilde{H}^{1-s}(-1,1)$). For $s>1/2$, the solution is continuous on the closed interval $[-1,1]$ and satisfies the boundary conditions $u(\pm 1)=0$. Finally, by duality arguments, we prove solvability of problem (0.1) in $\widetilde{H}^s(-1,1)$, where $0\le s<1/2$ (under the assumption that f belongs to the class dual to $\widetilde{H}^{1-s}(-1,1)$). In particular, for $s=0$ we obtain existence and uniqueness of the so called "very weak" solution from the class $\widetilde{H}^0(-1,1)$ (which is distinguished by the condition $\int_{-1}^{1}(1-x^2)^{-1}|u(x)|^2\,dx<\infty$).

Thus, the application of an adequate apparatus allowed us to introduce a suitable scale of spaces and prove additional regularity of the solution under wide assumptions on the coefficient $p(x)$ and the function $f(x)$ on the right-hand side.

1. WEAK SOLUTION OF THE PRANDTL EQUATION

1.1. Statement of the Problem

It is convenient to denote $V(x):=p(x)^{-1}$ and rewrite the Prandtl equation (0.1) with the Dirichlet conditions as

$$V(x)u(x)-\frac{1}{2\pi}\int_{-1}^{1}\frac{u'(t)}{t-x}\,dt=f(x),\qquad u(-1)=u(1)=0. \tag{1.1}$$

Here and below, singular integrals are understood in the mean value sense. We assume that $V(x)$ is a measurable function on $(-1,1)$ satisfying the following conditions:

$$V(x) \geq 0 \quad \text{for a.e. } x \in (-1,1), \qquad (1-x^2)V(x) \leq M < \infty. \tag{1.2}$$

The assumptions on the right-hand side f will be formulated later.

Our first goal is to define the weak solution of equation (1.1) and to prove theorem about existence and uniqueness of the solution.

1.2. Definition of Weak Solutions. Approach via the Fourier Transform

As usual in the theory of weak solutions of boundary-value problems, we will first present formal considerations that will tell us how to define the solution correctly. Due to the boundary conditions, it is natural to consider the function $u(x)$ on the real axis extending it by zero. Let $g(x)$ be a function supported on $[-1,1]$, but also defined on the whole axis. Multiply equation (1.1) by $\overline{g(x)}$ and integrate:

$$\int_{-1}^{1} V(x)u(x)\overline{g(x)}\,dx + \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{g(x)}\int_{-\infty}^{\infty}\frac{u'(t)}{x-t}\,dt\,dx = \int_{-1}^{1} f(x)\overline{g(x)}\,dx. \tag{1.3}$$

Next, we use the Fourier transform, which is taken in the form

$$\widehat{u}(\zeta) = \int_{-\infty}^{\infty} u(x)e^{-ix\zeta}\,dx, \qquad u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{u}(\zeta)e^{ix\zeta}\,d\zeta.$$

Let us transform the second term in the left-hand side of (1.3) with the help of the Parseval identity

$$\int_{-\infty}^{\infty} v(x)\overline{g(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{v}(\zeta)\overline{\widehat{g}(\zeta)}\,d\zeta. \tag{1.4}$$

Observe that the internal integral in (1.3) is the Fourier-convolution[2] $u'(t) * t^{-1}$. The Fourier-image of $u'(t)$ is given by $i\zeta\widehat{u}(\zeta)$, and the Fourier-image of the function t^{-1} equals $-i\pi\,\operatorname{sign}\zeta$. Thus, equation (1.3) can be written as

$$\int_{-1}^{1} V(x)u(x)\,\overline{g(x)}\,dx + \frac{1}{4\pi}\int_{-\infty}^{\infty}|\zeta|\widehat{u}(\zeta)\overline{\widehat{g}(\zeta)}\,d\zeta = \int_{-1}^{1} f(x)\overline{g(x)}\,dx. \tag{1.5}$$

Denote the sesquilinear form in the left-hand side of (1.5) by

$$[u,g] := \int_{-1}^{1} V(x)u(x)\overline{g(x)}\,dx + \frac{1}{4\pi}\int_{-\infty}^{\infty}|\zeta|\widehat{u}(\zeta)\overline{\widehat{g}(\zeta)}\,d\zeta. \tag{1.6}$$

A natural class to look for the weak solution is the class[3] $H^{1/2}_{00} := H^{1/2}_{00}(-1,1)$, defined as the subspace in the Sobolev space $H^{1/2}(\mathbb{R})$ consisting of functions equal to zero almost everywhere outside the interval $[-1,1]$. Concerning the properties of this class of functions, see [11, Chap. 1, Sec. 11.5]. The norm in $H^{1/2}_{00}$ is defined as the standard norm in $H^{1/2}(\mathbb{R})$:

$$\|u\|^2_{H^{1/2}_{00}} = \|u\|^2_{H^{1/2}(\mathbb{R})} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1+|\zeta|^2)^{1/2}|\widehat{u}(\zeta)|^2\,d\zeta, \qquad u \in H^{1/2}_{00}.$$

Note that the set $C_0^\infty(-1,1)$ is dense in $H^{1/2}_{00}$.

We need the following property of the functions from $H^{1/2}_{00}$ (for the sake of completeness, we provide the proof).

[2] In what follows, we will also consider convolution for another integral transform.

[3] In the literature, various notations are used for this class and its dual; we accept the notation from [11].

Lemma 1. *Any function* $u \in H_{00}^{1/2}(-1,1)$ *satisfies*

$$\int_{-1}^{1} \frac{|u(x)|^2}{1-x^2}\,dx \le \frac{1}{2}\int_{-\infty}^{\infty} |\zeta||\widehat{u}(\zeta)|^2\,d\zeta. \tag{1.7}$$

Proof. We extend $u \in H_{00}^{1/2}$ by zero, keeping the same notation. Let us use the following identity valid on the class $H^{1/2}(\mathbb{R})$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy = \int_{-\infty}^{\infty} |\zeta||\widehat{u}(\zeta)|^2\,d\zeta. \tag{1.8}$$

To check (1.8), denote the left-hand side by $J[u]$ and substitute $y = x+z$:

$$\begin{aligned} J[u] &= \int_{-\infty}^{\infty} \frac{dz}{|z|^2}\int_{-\infty}^{\infty} |u(x+z)-u(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{|z|^2}\int_{-\infty}^{\infty} |e^{iz\zeta}-1|^2|\widehat{u}(\zeta)|^2\,d\zeta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{u}(\zeta)|^2\,d\zeta \int_{-\infty}^{\infty} \frac{|e^{iz\zeta}-1|^2}{|z|^2}\,dz. \end{aligned}$$

Here we have used the Parseval identity (1.4) and the Fubini theorem. The internal integral is equal to $2\pi|\zeta|$. This implies (1.8).

Since $u(y) = 0$ for $y \in \mathbb{R}\setminus[-1,1]$, from (1.8) it follows that

$$\begin{aligned} \int_{-\infty}^{\infty} |\zeta||\widehat{u}(\zeta)|^2\,d\zeta &\ge \int_{-1}^{1} |u(x)|^2\,dx \int_{-\infty}^{-1} \frac{dy}{|x-y|^2} + \int_{-1}^{1} |u(x)|^2\,dx\int_{1}^{\infty}\frac{dy}{|x-y|^2} \\ &= 2\int_{-1}^{1}\frac{|u(x)|^2}{1-x^2}\,dx, \end{aligned}$$

which proves (1.7). □

Let us check that the relation $u \in H_{00}^{1/2}$ is equivalent to the condition $[u,u] < \infty$.

Suppose that $[u,u] < \infty$. Then by (1.7)

$$\int_{-1}^{1} |u(x)|^2\,dx \le \int_{-1}^{1}\frac{|u(x)|^2}{1-x^2}\,dx \le \frac{1}{2}\int_{-\infty}^{\infty} |\zeta||\widehat{u}(\zeta)|^2\,d\zeta. \tag{1.9}$$

From (1.6) and (1.9), by the Parseval identity (1.4), it follows that

$$\|u\|^2_{H_{00}^{1/2}} \le \int_{-1}^{1} |u(x)|^2\,dx + \frac{1}{2\pi}\int_{-\infty}^{\infty} |\zeta||\widehat{u}(\zeta)|^2\,d\zeta \le (2\pi+2)[u,u]. \tag{1.10}$$

On the other hand, by (1.2) and (1.7),

$$\int_{-1}^{1} V(x)|u(x)|^2\,dx \le M\int_{-1}^{1}\frac{|u(x)|^2}{1-x^2}\,dx \le M\pi\|u\|^2_{H_{00}^{1/2}}.$$

Together with (1.6) this implies that

$$[u,u] \le \left(M\pi + \frac{1}{2}\right)\|u\|^2_{H_{00}^{1/2}}, \qquad u \in H_{00}^{1/2}. \tag{1.11}$$

From (1.10) and (1.11) it follows that the form $[u,u]^{1/2}$ determines the norm in $H_{00}^{1/2}$ equivalent to the standard one. Then the sesquilinear form $[u,g]$ given by (1.6) can be taken as the inner product in this space.

Let us write identity (1.5) in the form

$$[u,g] = (f,g), \tag{1.12}$$

where $(f,g) := (f,g)_{L_2(-1,1)}$. The natural class for f is the space $(H_{00}^{1/2})^*$ dual to $H_{00}^{1/2}$ with respect to the pairing in $L_2(-1,1)$. In other words, a distribution f that is an anti-linear continuous functional over $C_0^\infty(-1,1)$ belongs to $(H_{00}^{1/2})^*$ if

$$\sup_{0\neq u\in C_0^\infty(-1,1)} \frac{|(f,u)|}{\|u\|_{H_{00}^{1/2}}} < \infty. \tag{1.13}$$

Then the pairing (f,u) in $L_2(-1,1)$ extends to the pairs $f \in (H_{00}^{1/2})^*$, $u \in H_{00}^{1/2}$, and the left-hand side of (1.13) is taken as the norm of f in $(H_{00}^{1/2})^*$. Note that

$$\|f\|_{(H_{00}^{1/2})^*} = \sup_{0\neq u\in H_{00}^{1/2}} \frac{|(f,u)|}{\|u\|_{H_{00}^{1/2}}}. \tag{1.14}$$

Now, we give a definition of weak solution of problem (1.1).

Definition 1. Let $f \in (H_{00}^{1/2})^*$. An element $u \in H_{00}^{1/2}$ satisfying integral identity (1.5) for any $g \in H_{00}^{1/2}$ is called the *weak solution of problem* (1.1).

Theorem 1. *Suppose that $V(x)$ satisfies* (1.2). *Then for any $f \in (H_{00}^{1/2})^*$ there exists a unique weak solution $u \in H_{00}^{1/2}$ of problem* (1.1). *The solution satisfies the estimate*

$$\|u\|_{H_{00}^{1/2}} \le (2\pi+2)\|f\|_{(H_{00}^{1/2})^*}. \tag{1.15}$$

Proof. We write identity (1.5) in the form (1.12). By (1.14), the right-hand side $l_f(g) = (f,g)$ is an anti-linear continuous functional over $g \in H_{00}^{1/2}$ satisfying the estimate

$$|l_f(g)| \le \|f\|_{(H_{00}^{1/2})^*}\|g\|_{H_{00}^{1/2}}. \tag{1.16}$$

We consider $H_{00}^{1/2}$ as the Hilbert space with the inner product $[u,g]$. By the Riesz theorem about the general form of an anti-linear continuous functional in a Hilbert space, there exists a unique element $u \in H_{00}^{1/2}$ such that $l_f(g) = [u,g]$ for all $g \in H_{00}^{1/2}$, i.e., identity (1.12) holds. This proves the existence and uniqueness of the solution.

The estimate (1.15) follows from the identity $[u,u] = l_f(u)$ and relations (1.10) and (1.16). □

Example. Denote $r(x) := 1 - x^2$ and assume that $f \in L_{2,r}(-1,1) =: L_{2,r}$, i.e.,

$$\|f\|_{L_{2,r}}^2 = \int_{-1}^{1} (1-x^2)\,|f(x)|^2\,dx < \infty. \tag{1.17}$$

Then, by (1.7),

$$\begin{aligned} |(f,g)| &\le \left(\int_{-1}^{1} r(x)|f(x)|^2\,dx\right)^{1/2} \left(\int_{-1}^{1} \frac{|g(x)|^2}{r(x)}\,dx\right)^{1/2} \\ &\le \sqrt{\pi}\,\|f\|_{L_{2,r}}\|g\|_{H_{00}^{1/2}}, \qquad g \in H_{00}^{1/2}. \end{aligned} \tag{1.18}$$

It follows that $L_{2,r} \subset (H_{00}^{1/2})^*$, and $\|f\|_{(H_{00}^{1/2})^*} \le \sqrt{\pi}\,\|f\|_{L_{2,r}}$. Together with (1.15) this yields the following estimate of the solution:

$$\|u\|_{H_{00}^{1/2}} \le (2\pi+2)\sqrt{\pi}\,\|f\|_{L_{2,r}}. \tag{1.19}$$

Note that, in general, functions of class $H_{00}^{1/2}$ may be discontinuous. However, under the condition $f \in L_{2,r}$ we can expect that the solution is more regular. It turns out that it is inconvenient to study this question via the Fourier transform approach. We will study this problem using another integral transform.

2. THE INTEGRAL TRANSFORMATION $\mathscr{P}$

We need the integral transform $\mathscr{P}$ on the interval which was studied in detail in [7], [8]. Now we provide the basic information that will be needed below. In terms of $\mathscr{P}$, we introduce the scale of the Hilbert spaces $\widetilde{H}^s(-1,1)$; this material is new.

2.1. Definition of the Transform $\mathscr{P}$

Consider the space $\widetilde{L}_2(-1,1)$ consisting of all measurable functions on the interval $(-1,1)$ such that

$$\|u\|^2_{\widetilde{L}_2(-1,1)} := \int_{-1}^{1} \frac{|u(x)|^2}{1-x^2}\,dx < \infty.$$

For functions $u \in \widetilde{L}_2(-1,1)$ the transform $\mathscr{P}$ is defined as follows[4]:

$$\begin{cases} U(\xi) := \mathscr{P}[u](\xi) = \displaystyle\int_{-1}^{1} u(y)\left(\frac{1-y}{1+y}\right)^{i\xi} \frac{dy}{1-y^2}\,, & \xi \in \mathbb{R}, \\ u(x) = \mathscr{P}^{-1}[U](x) = \displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty} U(\xi)\left(\frac{1-x}{1+x}\right)^{-i\xi} d\xi, & x \in (-1,1). \end{cases} \tag{2.1}$$

Here $U \in L_2(\mathbb{R})$. The exact explanation of the meaning of relations (2.1) can be found in [7], [8]. In what follows, by default, the originals are denoted by lowercase letters, and the $\mathscr{P}$-images, by the corresponding uppercase letters.

2.2. The Parseval Identity

We have the following Parseval identity for the transform $\mathscr{P}$:

$$\int_{-1}^{1} u(y)\overline{g(y)}\,\frac{dy}{1-y^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} U(\xi)\overline{G(\xi)}\,d\xi. \tag{2.2}$$

By the Parseval identity, the relation $u \in \widetilde{L}_2(-1,1)$ is equivalent to the relation $U \in L_2(\mathbb{R})$. Thus, the operator $\pi^{-1/2}\mathscr{P}$ is a unitary mapping of the space $\widetilde{L}_2(-1,1)$ onto $L_2(\mathbb{R})$.

2.3. Relationship between $\mathscr{P}$ and the Fourier Transform

The transform $\mathscr{P}$ is related to the Fourier transform by the following change of variables:

$$x = \tanh\omega, \qquad \omega = \frac{1}{2}\ln\frac{1+x}{1-x}\,, \quad \omega \in \mathbb{R}, \qquad u(x) = u_1(\omega). \tag{2.3}$$

Then the $\mathscr{P}$-image of the function $u(x)$ and the Fourier-image of $u_1(\omega)$ satisfy

$$U(\xi) = \widehat{u}_1(2\xi). \tag{2.4}$$

It is easily seen that the linear mapping $A\colon \widetilde{L}_2(-1,1) \to L_2(\mathbb{R})$, defined by the rule $(Au)(\omega) = u_1(\omega)$, is an isometric isomorphism:

$$\|u\|_{\widetilde{L}_2(-1,1)} = \|Au\|_{L_2(\mathbb{R})}.$$

[4] In [7], [8], $\mathscr{P}$ was defined as the transform from the interval to the imaginary axis.

2.4. The $\mathscr{P}$-Transformation of Derivatives. The Spaces $\widetilde{H}^n$, $n \in \mathbb{Z}_+$

Now we define the space $\widetilde{H}^n(-1,1) =: \widetilde{H}^n$ of measurable functions $u(x)$ having generalized derivatives up to order n on the interval $(-1,1)$ and satisfying

$$|||u|||^2_{\widetilde{H}^n} := \int_{-1}^{1} \sum_{m=0}^{n} |(1-x^2)^m u^{(m)}(x)|^2 \, \frac{dx}{1-x^2} < \infty.$$

Then $\widetilde{H}^0(-1,1) = \widetilde{L}_2(-1,1)$. Obviously, $\widetilde{H}^n(-1,1) \subset H^n_{\mathrm{loc}}(-1,1)$.

If $u \in \widetilde{H}^n$, then, integrating by parts, we obtain

$$\mathscr{P}\left[\left((1-y^2)\,\frac{d}{dy}\right)^n u(y)\right](\xi) = (2i\xi)^n U(\xi). \tag{2.5}$$

Let us explain this in details for $n = 1$. The condition $u \in \widetilde{H}^1$ means that

$$|||u|||^2_{\widetilde{H}^1} = \int_{-1}^{1} \left(\frac{|u(x)|^2}{1-x^2} + (1-x^2)|u'(x)|^2 \right) dx < \infty. \tag{2.6}$$

Since $\widetilde{H}^1 \subset H^1_{\mathrm{loc}}(-1,1)$, it follows that, by the Sobolev embedding theorem, any function $u \in \widetilde{H}^1$ is absolutely continuous inside the interval $(-1,1)$. We will show that it is continuous on the closed interval $[-1,1]$. Indeed, for any $x, y \in (-1,1)$, we have

$$\big||u(y)|^2 - |u(x)|^2\big| = \left| 2\,\mathrm{Re} \int_x^y u(t)\overline{u'(t)}\,dt \right| \le \int_x^y \left(\frac{|u(t)|^2}{1-t^2} + (1-t^2)|u'(t)|^2 \right) dt,$$

and from condition (2.6) it follows that $\big||u(y)|^2 - |u(x)|^2\big| \to 0$ as $x \to 1$ and $y \to 1$. Then, applying the Cauchy criterion, we conclude that there exists a finite limit $\lim_{y \to 1-0} |u(y)|^2$. This limit is equal to zero, since otherwise the integral $\int_{-1}^{1} (|u(t)|^2/(1-t^2))\,dt$ would be divergent. Hence $u(y)$ converges to zero as $y \to 1-0$ and we can put

$$u(1) := \lim_{y \to 1-0} u(y) = 0.$$

In a similar fashion, we check that the following limit exists:

$$u(-1) := \lim_{y \to -1+0} u(y) = 0.$$

Using the fact that u is continuous inside the interval $(-1,1)$, we obtain $u \in C[-1,1]$.

Integrating by parts and using the boundary conditions $u(-1) = u(1) = 0$, we obtain relation (2.5) with $n = 1$:

$$\mathscr{P}[(1-y^2)u'(y)](\xi) = \int_{-1}^{1} u'(y) \left(\frac{1-y}{1+y} \right)^{i\xi} dy = 2i\xi U(\xi), \qquad u \in \widetilde{H}^1. \tag{2.7}$$

Next, from (2.2), (2.6) and (2.7) it follows that

$$|||u|||^2_{\widetilde{H}^1} = \frac{1}{\pi} \int_{-\infty}^{\infty} (1+4\xi^2)|U(\xi)|^2 \, d\xi, \qquad u \in \widetilde{H}^1.$$

Relation (2.5) for $u \in \widetilde{H}^n$ with arbitrary $n \in \mathbb{N}$ is proved similarly. It turns out that $|||u|||^2_{\widetilde{H}^n}$ admits two-sided estimates by means of

$$\|u\|^2_{\widetilde{H}^n} := \frac{1}{\pi} \int_{-\infty}^{\infty} (1+4\xi^2)^n |U(\xi)|^2 \, d\xi. \tag{2.8}$$

Remark 1. The above arguments allow us to give another definition of the spaces $\widetilde{H}^n$ (for any $n \in \mathbb{Z}_+$) in terms of the transform $\mathscr{P}$: $\widetilde{H}^n(-1,1)$ is the class of functions $u \in \widetilde{L}_2(-1,1)$, for which the norm $\|u\|_{\widetilde{H}^n}$ given by (2.8) is finite. By (2.4), it is obvious that the mapping A (see Sec. 2.3) restricted to $\widetilde{H}^n(-1,1)$ is an isometric isomorphism of the space $\widetilde{H}^n(-1,1)$ onto the Sobolev space $H^n(\mathbb{R})$:

$$\|u\|_{\widetilde{H}^n} = \|Au\|_{H^n(\mathbb{R})}, \qquad u \in \widetilde{H}^n(-1,1).$$

2.5. The Spaces $\widetilde{H}^s$

We now introduce the space $\widetilde{H}^s(-1,1) =: \widetilde{H}^s$ with an arbitrary index $s \geq 0$ as the subspace of $\widetilde{L}_2(-1,1) = \widetilde{H}^0$ consisting of functions u such that

$$\|u\|^2_{\widetilde{H}^s} := \frac{1}{\pi}\int_{-\infty}^{\infty}(1+4\xi^2)^s|U(\xi)|^2\,d\xi < \infty.$$

Automatically, for $n \in \mathbb{Z}_+$, this agrees with the previous definition (see Remark 1). For any $s \geq 0$, the mapping A restricted to $\widetilde{H}^s(-1,1)$ is an isometric isomorphism of the space $\widetilde{H}^s(-1,1)$ onto the Sobolev space $H^s(\mathbb{R})$:

$$\|u\|_{\widetilde{H}^s} = \|Au\|_{H^s(\mathbb{R})}, \qquad u \in \widetilde{H}^s(-1,1).$$

Since $H^s(\mathbb{R})$, $s \geq 0$, forms an interpolational scale of Hilbert spaces, the same is true for the spaces $\widetilde{H}^s(-1,1)$, $s \geq 0$.

Remark 2. Recall that in the space $H^s(\mathbb{R})$ with $s \neq [s] =: k$, the following norm given in the internal terms is equivalent to the standard norm:

$$|u_1|^2_{\widetilde{H}^s(\mathbb{R})} = \sum_{j=0}^{k}\int_{-\infty}^{\infty}|u_1^{(j)}(\omega)|^2\,d\omega + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|u_1^{(k)}(\omega)-u_1^{(k)}(\tau)|^2}{|\omega-\tau|^{1+2\{s\}}}\,d\omega\,d\tau, \qquad \{s\} = s-k.$$

Using the isomorphism A, we see that the following norm given in the internal terms is equivalent to the standard norm in the space $\widetilde{H}^s(-1,1)$ with $s \neq [s] = k$:

$$|u|^2_{\widetilde{H}^s} = \sum_{j=0}^{k}\int_{-1}^{1}|(1-x^2)^j u^{(j)}(x)|^2\,\frac{dx}{1-x^2}$$
$$+ \int_{-1}^{1}\int_{-1}^{1}\frac{|(1-x^2)^k u^{(k)}(x)-(1-y^2)^k u^{(k)}(y)|^2}{|\ln((1-x)/(1+x))-\ln((1-y)/(1+y))|^{1+2\{s\}}}\,\frac{dx}{1-x^2}\,\frac{dy}{1-y^2}.$$

However, below we will not use this norm.

Using the isomorphism A, from the density of $C_0^\infty(\mathbb{R})$ in $H^s(\mathbb{R})$, we deduce the following statement.

Proposition 1. *For any $s \geq 0$, the set $C_0^\infty(-1,1)$ is dense in the space $\widetilde{H}^s(-1,1)$.*

We need the following statement, which is an analogue of the Sobolev embedding theorem.

Proposition 2. *Let $s > 1/2$. Then*

$$\widetilde{H}^s(-1,1) \subset C[-1,1].$$

Any function $u \in \widetilde{H}^s(-1,1)$ satisfies the boundary conditions

$$u(-1) = u(1) = 0 \tag{2.9}$$

and the estimate

$$\|u\|_{C[-1,1]} \leq C(s)\|u\|_{\widetilde{H}^s}, \qquad C(s) = \left(\frac{\Gamma(s-1/2)}{2\sqrt{\pi}\,\Gamma(s)}\right)^{1/2}. \tag{2.10}$$

Proof. We rely on the relation between $\mathscr{P}$ and the Fourier transform (see Sec. 2.3).

By the Sobolev embedding theorem, for $s > 1/2$, the space $H^s(\mathbb{R})$ is continuously embedded in the space of uniformly continuous functions. Next, a function $u_1(\omega)$ of class $H^s(\mathbb{R})$ satisfies $\widehat{u}_1 \in L_1(\mathbb{R})$, because

$$\int_{-\infty}^{\infty} |\widehat{u}_1(\xi)|\,d\xi \le \left(\int_{-\infty}^{\infty} (1+\xi^2)^{-s}\,d\xi\right)^{1/2} \left(\int_{-\infty}^{\infty} (1+\xi^2)^{s} |\widehat{u}_1(\xi)|^2\,d\xi\right)^{1/2} = C_s \|u_1\|_{H^s(\mathbb{R})} < \infty. \tag{2.11}$$

Here $C_s = (2\pi^{3/2}\Gamma(s-1/2)/\Gamma(s))^{1/2}$. Then, from the inversion formula

$$u_1(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi\omega}\widehat{u}_1(\xi)\,d\xi,$$

it follows by the Riemann–Lebesgue lemma that

$$\lim_{\omega\to\pm\infty} u_1(\omega) = 0. \tag{2.12}$$

Thus, $u_1(\omega)$ belongs to the class $C_0(\mathbb{R})$ of uniformly continuous functions on $\mathbb{R}$ satisfying conditions (2.12). By (2.11) and the inversion formula, we have the estimate

$$\max_{\omega\in\mathbb{R}} |u_1(\omega)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{u}_1(\xi)|\,d\xi \le \frac{1}{2\pi}\, C_s \|u_1\|_{H^s(\mathbb{R})}. \tag{2.13}$$

Now let $u \in \widetilde{H}^s(-1,1)$, $s > 1/2$. Substituting (2.3), we see that the function $(Au)(\omega) = u_1(\omega)$ belongs to $H^s(\mathbb{R})$, and $\|u_1\|_{H^s(\mathbb{R})} = \|u\|_{\widetilde{H}^s}$. Then, from the properties of the function $u_1(\omega)$ proved above, it follows that $u(x)$ is uniformly continuous on the closed interval $[-1,1]$ and satisfies the boundary conditions (2.9). The identities

$$\|u\|_{C[-1,1]} = \max_{x\in[-1,1]} |u(x)| = \max_{\omega\in\mathbb{R}} |u_1(\omega)|$$

and inequality (2.13) imply the estimate (2.10). □

2.6. Transformation of Distributions

The transform $\mathscr{P}$ for distributions is defined by duality, similarly to the definition of the Fourier transform for distributions. Consider the space $\mathscr{X}$ consisting of functions $\psi \in C^\infty[-1,1]$ such that the seminorms

$$\sup_{x\in[-1,1]} (1-x^2)^n |\psi^{(n)}(x)| \left(1 + \ln^2\left(\frac{1-x}{1+x}\right)\right)^k$$

are finite for all $n, k \in \mathbb{Z}_+$. Convergence in $\mathscr{X}$ is understood as convergence with respect to this set of seminorms. Let $\mathscr{X}'$ be the class of distributions dual to $\mathscr{X}$ with respect to the pairing in $\widetilde{L}_2(-1,1)$. The transformation $\mathscr{P}$ takes the class $\mathscr{X}$ onto the Schwartz class $\mathscr{S}(\mathbb{R})$. Let $u \in \mathscr{X}'$. Then $\mathscr{P}u \in \mathscr{S}'(\mathbb{R})$ is defined by

$$(\mathscr{P}u, \varphi)_{L_2(\mathbb{R})} = (u, \mathscr{P}^*\varphi)_{\widetilde{L}_2(-1,1)} = \pi(u, \mathscr{P}^{-1}\varphi)_{\widetilde{L}_2(-1,1)}, \qquad \varphi \in \mathscr{S}(\mathbb{R}).$$

Below we need the result of calculation of the $\mathscr{P}$-image of the distribution $v(y) = 1/y$ (understood in the mean value sense); see [7, (1.9)]:

$$\mathscr{P}\left[\frac{1}{y}\right](\xi) = \int_{-1}^{1} \frac{1}{y}\left(\frac{1-y}{1+y}\right)^{i\xi} \frac{dy}{1-y^2} = -i\pi \coth \pi\xi, \tag{2.14}$$

where the integral is understood in the mean value sense both at the point $y = 0$, and at the ends of the interval.

2.7. The Convolution Formulas

The following convolution formulas are valid for the transform $\mathscr{P}$:

$$\mathscr{P}\left[\int_{-1}^{1} u(x)\, v\left(\frac{y-x}{1-xy}\right) \frac{dx}{1-x^2}\right](\xi) = U(\xi)V(\xi), \tag{2.15}$$

$$\mathscr{P}\left[\int_{-1}^{1} u'(x)v\left(\frac{y-x}{1-xy}\right) dx\right](\xi) = 2i\xi U(\xi)V(\xi). \tag{2.16}$$

In (2.15) it is assumed that $u \in \widetilde{L}_2(-1,1)$ and $v \in \widetilde{L}_1(-1,1) \cap \widetilde{L}_2(-1,1)$. The class $\widetilde{L}_1(-1,1)$ is distinguished by the condition $\int_{-1}^{1} |v(t)|(1-t^2)^{-1}\, dt < \infty$. Identity (2.16) is valid under the same conditions on v and for $u \in \widetilde{H}^1$. However, as in the case of the Fourier transform, the conditions of applicability of relations (2.15), (2.16) can be expanded significantly. In particular, one can relax the requirements on v, assuming v to be a distribution and, if necessary, imposing more restrictive conditions on u. We will not go into details here.

3. THE WEAK SOLUTION OF THE PRANDTL EQUATION. APPROACH VIA THE TRANSFORM $\mathscr{P}$

3.1. "Another" Definition of a Weak Solution

Now, we apply the transform $\mathscr{P}$ to the study of problem (1.1). As above, we start from formal considerations. Using the boundary conditions $u(-1) = u(1) = 0$ and the identity $1 - x^2 = 1 - xt + x(t - x)$, we transform the expression in (1.1):

$$-(1-x^2)\int_{-1}^{1} \frac{u'(t)}{t-x}\, dt = \int_{-1}^{1} u'(t)\frac{1-xt}{x-t}\, dt. \tag{3.1}$$

The form of the right-hand side allows us to apply the convolution formula (2.16) with $v(t) = t^{-1}$ in order to calculate the $\mathscr{P}$-image of the function (3.1). Taking (2.14) into account, we obtain

$$\mathscr{P}\left[-(1-x^2)\frac{1}{2\pi}\int_{-1}^{1} \frac{u'(t)}{t-x}\, dt\right](\xi) = \xi \coth \pi\xi\, U(\xi). \tag{3.2}$$

Multiplying (1.1) by some function $\overline{g(x)}$, integrating over the interval, and using (2.2) and (3.2), we arrive at

$$\int_{-1}^{1} V(x)u(x)\overline{g(x)}\, dx + \frac{1}{\pi}\int_{-\infty}^{\infty} \xi \coth \pi\xi\, U(\xi)\overline{G(\xi)}\, d\xi = \int_{-1}^{1} f(x)\overline{g(x)}\, dx. \tag{3.3}$$

The sesquilinear form in the left-hand side of (3.3) is denoted by

$$[u,g]_1 := \int_{-1}^{1} V(x)u(x)\overline{g(x)}\, dx + \frac{1}{\pi}\int_{-\infty}^{\infty} \xi \coth \pi\xi\, U(\xi)\overline{G(\xi)}\, d\xi. \tag{3.4}$$

The natural class to look for the weak solution is the space $\widetilde{H}^{1/2}(-1,1)$; see Sec. 2.5. The form $[u,u]_1^{1/2}$ defines the norm in $\widetilde{H}^{1/2}$ equivalent to the standard norm. Indeed, by the estimates

$$\frac{1}{\pi^2} + \frac{2}{3}\xi^2 \le \xi^2 \coth^2 \pi\xi \le \frac{1}{\pi^2} + \xi^2, \qquad \xi \in \mathbb{R}, \tag{3.5}$$

$$\int_{-1}^{1} V(x)|u(x)|^2\, dx \le M\int_{-1}^{1} \frac{|u(x)|^2}{1-x^2}\, dx = \frac{M}{\pi}\int_{-\infty}^{\infty} |U(\xi)|^2\, d\xi \le M\|u\|^2_{\widetilde{H}^{1/2}}, \qquad u \in \widetilde{H}^{1/2},$$

we see that

$$\frac{1}{\pi}\|u\|^2_{\widetilde{H}^{1/2}} \le [u,u]_1 \le \left(M + \frac{1}{2}\right)\|u\|^2_{\widetilde{H}^{1/2}}, \qquad u \in \widetilde{H}^{1/2}.$$

Then the sesquilinear form $[u, g]_1$ given by (3.4) can be taken as the inner product in $\widetilde{H}^{1/2}$. By Proposition 1, the set $C_0^\infty(-1,1)$ is dense in $\widetilde{H}^{1/2}$.

Relation (3.3) can be written as

$$[u, g]_1 = (f,\, g).$$

A natural class for f is the space $(\widetilde{H}^{1/2})^*$ dual to $\widetilde{H}^{1/2}$ with respect to the pairing in $L_2(-1,1)$. (It is defined similarly to the space $(H_{00}^{1/2})^*$.) The norm in $(\widetilde{H}^{1/2})^*$ is given by

$$\|f\|_{(\widetilde{H}^{1/2})^*} = \sup_{0\neq u\in\widetilde{H}^{1/2}} \frac{|(f,u)|}{\|u\|_{\widetilde{H}^{1/2}}}\,.$$

Remark 3. For $u, g \in C_0^\infty(-1,1)$, we have $[u, g]_1 = [u, g]$, where $[u, g]$ is given by (1.6). Indeed, both expressions $[u, g]$ and $[u, g]_1$ are equal to the integral of the left-hand side of (1.1) multiplied by $\overline{g(x)}$. Using the fact that $C_0^\infty(-1,1)$ is dense in $H_{00}^{1/2}$, as well as in $\widetilde{H}^{1/2}$, we see that $[u,u]^{1/2}$ defines a norm in $H_{00}^{1/2}$ equivalent to the standard one and $[u,u]_1^{1/2}$ defines a norm in $\widetilde{H}^{1/2}$ equivalent to the standard one, so we conclude that $\widetilde{H}^{1/2} = H_{00}^{1/2}$ and $[u,g]_1 = [u,g]$ for any $u, g \in H_{00}^{1/2}$. Hence the space $(\widetilde{H}^{1/2})^*$ coincides with $(H_{00}^{1/2})^*$.

Now, on the basis of identity (3.3), we can give a definition of the weak solution of problem (1.1), which is independent of Definition 1.

Definition 2. Let $f \in (\widetilde{H}^{1/2})^*$. An element $u \in \widetilde{H}^{1/2}$ satisfying the integral identity (3.3) for any $g \in \widetilde{H}^{1/2}$ is called the *weak solution of problem* (1.1).

The following theorem is equivalent to Theorem 1 and is proved in a similar way (with the help of the Riesz theorem).

Theorem 2. *Suppose that $V(x)$ satisfies* (1.2). *Then, for any $f \in (\widetilde{H}^{1/2})^*$, there exists a unique weak solution $u \in \widetilde{H}^{1/2}$ of problem* (1.1). *The solution satisfies the estimate*

$$\|u\|_{\widetilde{H}^{1/2}} \le \pi \|f\|_{(\widetilde{H}^{1/2})^*}. \tag{3.6}$$

Example. Suppose that $f \in L_{2,r}(-1,1)$, see (1.17). Then

$$|(f,g)| \le \|f\|_{L_{2,r}} \|g\|_{\widetilde{H}^{1/2}}, \qquad g \in \widetilde{H}^{1/2},$$

cf. (1.18). Hence $L_{2,r} \subset (\widetilde{H}^{1/2})^*$, and $\|f\|_{(\widetilde{H}^{1/2})^*} \le \|f\|_{L_{2,r}}$. Together with (3.6), this implies that

$$\|u\|_{\widetilde{H}^{1/2}} \le \pi \|f\|_{L_{2,r}},$$

cf. (1.19). In what follows, we will also need the estimate

$$\int_{-1}^{1} V(x)|u(x)|^2\,dx \le \frac{\pi}{4}\|f\|_{L_{2,r}}^2, \tag{3.7}$$

which follows from the identity $[u,u]_1 = (f,u)$. Indeed, we have

$$\begin{aligned}
&\int_{-1}^{1} V(x)|u(x)|^2\,dx + \frac{1}{\pi}\int_{-\infty}^{\infty} \xi\coth\pi\xi\,|U(\xi)|^2\,d\xi \\
&\quad = \int_{-1}^{1} f(x)\overline{u(x)}\,dx \le \|f\|_{L_{2,r}}\|u\|_{\widetilde{L}_2} = \|f\|_{L_{2,r}}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}|U(\xi)|^2\,d\xi\right)^{1/2} \\
&\quad \le \|f\|_{L_{2,r}}\left(\int_{-\infty}^{\infty}\xi\coth\pi\xi\,|U(\xi)|^2\,d\xi\right)^{1/2} \le \frac{\pi}{4}\|f\|_{L_{2,r}}^2 + \frac{1}{\pi}\int_{-\infty}^{\infty}\xi\coth\pi\xi\,|U(\xi)|^2\,d\xi.
\end{aligned}$$

Here we have used the Parseval identity (2.2), the lower estimate (3.5), and the elementary inequality $ab \le \alpha a^2 + (1/4\alpha)b^2$ (for positive numbers a, b with arbitrary $\alpha > 0$).

In the next subsection, we will show that, for $f \in L_{2,r}$, the solution is more regular.

3.2. Improvement of Regularity of the Solution

In this subsection, it is assumed that $f \in L_{2,r}(-1,1)$, i.e., (1.17) is satisfied. Obviously, the space $L_{2,r}(-1,1)$ is dual to $\widetilde{L}_2(-1,1)$ with respect to the pairing in $L_2(-1,1)$. In other words, $L_{2,r}(-1,1) = (\widetilde{H}^0)^*$ and

$$\|f\|_{(\widetilde{H}^0)^*} = \|f\|_{L_{2,r}}.$$

The integral identity (3.3) can be written as

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \xi \coth \pi\xi \, U(\xi)\overline{G(\xi)}\, d\xi = \int_{-1}^{1} (f(x) - V(x)u(x))\overline{g(x)}\, dx, \qquad g \in \widetilde{H}^{1/2}. \tag{3.8}$$

Denote

$$Q(\xi) = \mathscr{P}[(1-x^2)(f(x) - V(x)u(x))](\xi) = \int_{-1}^{1} (f(x) - V(x)u(x))\left(\frac{1-x}{1+x}\right)^{i\xi} dx.$$

It is easily seen that the function $(1-x^2)(f(x) - V(x)u(x))$ belongs to $\widetilde{L}_2(-1,1)$, whence $Q \in L_2(\mathbb{R})$. Indeed, by (1.2), (2.2) and (3.7),

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty} |Q(\xi)|^2\, d\xi &= \int_{-1}^{1} (1-x^2)|f(x) - V(x)u(x)|^2\, dx \\ &\le 2\int_{-1}^{1} (1-x^2)|f(x)|^2\, dx + 2\int_{-1}^{1} (1-x^2)V(x)\cdot V(x)|u(x)|^2\, dx \\ &\le C_1 \int_{-1}^{1} (1-x^2)|f(x)|^2\, dx, \end{aligned} \tag{3.9}$$

where $C_1 = 2 + (\pi/2)M$.

Now, using (2.2), we represent identity (3.8) as

$$\int_{-\infty}^{\infty} \xi \coth \pi\xi \, U(\xi)\, \overline{G(\xi)}\, d\xi = \int_{-\infty}^{\infty} Q(\xi)\, \overline{G(\xi)}\, d\xi, \quad G \in \mathscr{P}[\widetilde{H}^{1/2}]. \tag{3.10}$$

For $N > 0$, we define the cut-off function

$$w(\xi, N) = \begin{cases} 1, & |\xi| < N, \\ 0, & |\xi| > N+1, \\ -\xi + N + 1, & N < \xi < N+1, \\ \xi + N + 1, & -N-1 < \xi < -N, \end{cases}$$

and take the test function of the form

$$G_N(\xi) = \xi \coth \pi\xi \, U(\xi) w(\xi, N). \tag{3.11}$$

Let us check that $G_N \in \mathscr{P}[\widetilde{H}^{1/2}]$. Let $g_N = \mathscr{P}^{-1}(G_N)$. Then it follows from (3.5) that

$$\begin{aligned}\|g_N\|^2_{\widetilde{H}^{1/2}} &= \frac{1}{\pi}\int_{-\infty}^{\infty} (1+4\xi^2)^{1/2}|G_N(\xi)|^2\, d\xi \\ &\le \frac{1}{\pi}\int_{-N-1}^{N+1} (1+4\xi^2)^{1/2}\xi^2 \coth^2 \pi\xi |U(\xi)|^2\, d\xi \le C(N)\|u\|^2_{\widetilde{H}^{1/2}}, \end{aligned} \tag{3.12}$$

where $C(N) = 1/\pi^2 + (N+1)^2$. Thus, $g_N \in \widetilde{H}^{1/2}$, and we can substitute $G_N \in \mathscr{P}[\widetilde{H}^{1/2}]$ as a test function in identity (3.10). We obtain

$$\int_{-\infty}^{\infty} \xi^2 \coth^2 \pi\xi |U(\xi)|^2 w(\xi, N)\, d\xi = \int_{-\infty}^{\infty} \xi \coth \pi\xi \, Q(\xi) w(\xi, N)\overline{U(\xi)}\, d\xi. \tag{3.13}$$

Next, by analogy with (3.12), it is easily seen that the function $\widetilde{G}_N(\xi) = Q(\xi)w(\xi, N)$ belongs to the class $\mathscr{P}[\widetilde{H}^{1/2}]$. From identity (3.10) with $G(\xi) = \widetilde{G}_N(\xi)$ it follows that

$$\int_{-\infty}^{\infty} \xi \coth \pi\xi \, Q(\xi)w(\xi, N)\overline{U(\xi)} \, d\xi = \int_{-\infty}^{\infty} |Q(\xi)|^2 w(\xi, N) \, d\xi. \tag{3.14}$$

Together with (3.13) this yields

$$\int_{-\infty}^{\infty} \xi^2 \coth^2 \pi\xi \, |U(\xi)|^2 w(\xi, N) \, d\xi = \int_{-\infty}^{\infty} |Q(\xi)|^2 w(\xi, N) \, d\xi. \tag{3.15}$$

Consequently,

$$\int_{-\infty}^{\infty} \xi^2 \coth^2 \pi\xi |U(\xi)|^2 w(\xi, N) \, d\xi \le \int_{-\infty}^{\infty} |Q(\xi)|^2 \, d\xi. \tag{3.16}$$

Letting N tend to infinity and applying the Fatou theorem, we conclude that the integral

$$\int_{-\infty}^{\infty} \xi^2 \coth^2 \pi\xi |U(\xi)|^2 \, d\xi$$

converges and satisfies the estimate

$$\int_{-\infty}^{\infty} \xi^2 \coth^2 \pi\xi |U(\xi)|^2 \, d\xi \le \int_{-\infty}^{\infty} |Q(\xi)|^2 \, d\xi. \tag{3.17}$$

From (3.9) and (3.17) it follows that

$$\int_{-\infty}^{\infty} \xi^2 \coth^2 \pi\xi |U(\xi)|^2 \, d\xi \le \pi C_1 \int_{-1}^{1} (1 - x^2)|f(x)|^2 \, dx.$$

Combining this with (2.8) and (3.5), we obtain that $u \in \widetilde{H}^1$ and

$$\|u\|^2_{\widetilde{H}^1} \le C_2 \int_{-1}^{1} (1 - x^2)|f(x)|^2 \, dx, \qquad C_2 = \pi^2 C_1 = \pi^2\biggl(2 + \frac{\pi}{2} M\biggr).$$

By Proposition 2, the solution is continuous on $[-1, 1]$ and satisfies the boundary conditions $u(-1) = u(1) = 0$. Note that the solution $u \in \widetilde{H}^1$ satisfies identity (3.3) for any test function $g \in \widetilde{H}^0$. Moreover, for this solution, the initial statement of problem (1.1) can be used (the equation is satisfied almost everywhere, the boundary conditions are fulfilled and, therefore, there is no need to replace the problem by the integral identity). Such a solution is called a *strong solution*.

As a result, we arrive at the following theorem about regularity of the solution.

Theorem 3. *Suppose that $V(x)$ satisfies conditions* (1.2). *Suppose that $f(x)$ is subject to condition* (1.17). *Then the weak solution $u(x)$ of problem* (1.1) *belongs to the class $\widetilde{H}^1(-1, 1)$ and satisfies the estimate*

$$\int_{-1}^{1} \biggl(\frac{|u(x)|^2}{1 - x^2} + (1 - x^2)|u'(x)|^2\biggr) \, dx \le C_2 \int_{-1}^{1} (1 - x^2)|f(x)|^2 \, dx.$$

The constant $C_2 = \pi^2(2 + (\pi/2)M)$ depends only on the constant M from condition (1.2).

3.3. Interpolation

Let R be the operator taking f into the solution of problem (1.1). Theorems 2 and 3 show that the operator R is continuous from $(\widetilde{H}^{1/2})^*$ to $\widetilde{H}^{1/2}$ and from $(\widetilde{H}^0)^*$ to $\widetilde{H}^1$. We have

$$\|R\|_{(\widetilde{H}^{1/2})^* \to \widetilde{H}^{1/2}} \le \pi, \tag{3.18}$$

$$\|R\|_{(\widetilde{H}^0)^* \to \widetilde{H}^1} \le \sqrt{C_2}\,. \tag{3.19}$$

Denote by $(\widetilde{H}^s)^*$ the space dual to $\widetilde{H}^s$ with respect to the pairing in $L_2(-1,1)$. Interpolating between (3.18) and (3.19), we obtain

$$\|R\|_{(\widetilde{H}^{1/2-\theta})^*\to\widetilde{H}^{1/2+\theta}} \le C_\theta = \pi^{1-2\theta}C_2^\theta, \qquad 0\le\theta\le\frac{1}{2}\,.$$

We arrive at the following result.

Theorem 4. *Suppose that $V(x)$ satisfies conditions* (1.2). *Let $0\le\theta\le 1/2$ and $f\in(\widetilde{H}^{1/2-\theta})^*$. Then the weak solution $u(x)$ of problem* (1.1) *belongs to the class $\widetilde{H}^{1/2+\theta}(-1,1)$ and satisfies the estimate*

$$\|u\|_{\widetilde{H}^{1/2+\theta}} \le C_\theta\|f\|_{(\widetilde{H}^{1/2-\theta})^*}.$$

The constant C_θ depends only on the constant M from condition (1.2) *and on θ.*

Theorem 4 and Proposition 2 imply the following corollary.

Corollary 1. *Under the condition $f\in(\widetilde{H}^{1/2-\theta})^*$, where $0<\theta\le 1/2$, the solution $u(x)$ of problem* (1.1) *is continuous on the closed interval $[-1,1]$ and satisfies conditions $u(-1)=u(1)=0$. We have*

$$\|u\|_{C[-1,1]} \le C\left(\frac{1}{2}+\theta\right)C_\theta\|f\|_{(\widetilde{H}^{1/2-\theta})^*}.$$

Here $C(1/2+\theta)$ is the constant $C(s)$ from (2.10) *with $s=1/2+\theta$.*

3.4. Duality

For completeness, we consider the so-called "very weak" solution from the class $\widetilde{L}_2(-1,1)$, assuming that $f\in(\widetilde{H}^1)^*$.

Definition 3. Let $f\in(\widetilde{H}^1)^*$. A function $u\in\widetilde{L}_2(-1,1)$ satisfying identity (3.3) for any test function $g\in\widetilde{H}^1(-1,1)$ is called a *very weak solution of problem* (1.1).

Theorem 5. *Suppose that $V(x)$ satisfies conditions* (1.2). *For any $f\in(\widetilde{H}^1)^*$, there exists a unique very weak solution $u\in\widetilde{L}_2(-1,1)$ of problem* (1.1). *We have*

$$\|u\|_{\widetilde{L}_2} \le \sqrt{C_2}\,\|f\|_{(\widetilde{H}^1)^*}. \tag{3.20}$$

Proof. Let $R\colon(\widetilde{H}^0)^*\to\widetilde{H}^1$ be the resolving operator from Theorem 3. Then the continuous adjoint operator $R^*\colon(\widetilde{H}^1)^*\to\widetilde{H}^0$ is correctly defined by the relation

$$(Rg,f)=(g,R^*f), \qquad g\in(\widetilde{H}^0)^*, \quad f\in(\widetilde{H}^1)^*. \tag{3.21}$$

Fix $f\in(\widetilde{H}^1)^*$. Let us check that $u=R^*f\in\widetilde{H}^0$ is the very weak solution of problem (1.1).

Let $g\in(\widetilde{H}^0)^*$. Then $v:=Rg\in\widetilde{H}^1$ satisfies the identity

$$[v,u]_1=(g,u) \qquad \text{for all} \quad u\in\widetilde{H}^0.$$

Here the sesquilinear form $[v,u]_1$ (see (3.4)) is extended to the pairs $v\in\widetilde{H}^1$, $u\in\widetilde{H}^0$. In other words,

$$[Rg,u]_1=(g,u) \qquad \text{for all} \quad g\in(\widetilde{H}^0)^*, \quad u\in\widetilde{H}^0. \tag{3.22}$$

Substituting the function $u=R^*f\in\widetilde{H}^0$ in (3.22), we obtain

$$[Rg,u]_1=(g,R^*f)=(Rg,f) \qquad \text{for all} \quad g\in(\widetilde{H}^0)^*. \tag{3.23}$$

Here we have used (3.21). Note that, if g runs over $(\widetilde{H}^0)^*$, then $v = Rg$ runs over $\widetilde{H}^1$. Therefore, identity (3.23) can be written as

$$[v, u]_1 = (v, f) \qquad \text{for all} \quad v \in \widetilde{H}^1.$$

Since the form (3.4) is Hermitian, this is equivalent to the identity

$$[u, v]_1 = (f, v) \qquad \text{for all} \quad v \in \widetilde{H}^1.$$

This means that $u = R^* f \in \widetilde{H}^0$ is the very weak solution of problem (1.1).

To prove uniqueness, assume that $u \in \widetilde{L}_2(-1, 1)$ is the very weak solution of problem (1.1) with $f = 0$. By analogy with (3.8)–(3.10), we represent the identity for u in the form

$$\int_{-\infty}^{\infty} \xi \coth \pi\xi \, U(\xi)\overline{G(\xi)}\, d\xi = \int_{-\infty}^{\infty} Q(\xi)\overline{G(\xi)}\, d\xi, \qquad G \in \mathscr{P}[\widetilde{H}^1]. \tag{3.24}$$

We have $Q \in L_2(\mathbb{R})$ and $\|Q\|_{L_2(\mathbb{R})} \le \sqrt{\pi} M \|u\|_{\widetilde{L}_2}$. Substituting the test function $G_N(\xi)$ of the form (3.11) (which belongs to $\mathscr{P}[\widetilde{H}^1]$) in (3.24), by analogy with (3.13)–(3.17) we prove that $u \in \widetilde{H}^1$. Now Theorem 2 (the uniqueness part) implies that $u = 0$.

The estimate (3.20) follows from (3.19) and the relation $\|R^*\|_{(\widetilde{H}^1)^* \to \widetilde{H}^0} = \|R\|_{(\widetilde{H}^0)^* \to \widetilde{H}^1}$. □

From the uniqueness of the very weak solution it follows that the resolving operator $R^* \colon (\widetilde{H}^1)^* \to \widetilde{H}^0$ is an extension of the operator $R\colon (\widetilde{H}^{1/2})^* \to \widetilde{H}^{1/2}$. Keeping the same notation R for the extended operator, we rewrite (3.20) as

$$\|R\|_{(\widetilde{H}^1)^* \to \widetilde{H}^0} \le \sqrt{C_2}\,. \tag{3.25}$$

Interpolating between (3.25) and (3.19), we obtain

$$\|R\|_{(\widetilde{H}^{1-s})^* \to \widetilde{H}^s} \le \sqrt{C_2}\,, \qquad 0 \le s \le 1.$$

We arrive at the following final result, which combines the statements of all the previous theorems.

Theorem 6. *Suppose that $V(x)$ satisfies conditions* (1.2). *Let $0 \le s \le 1$. For any $f \in (\widetilde{H}^{1-s})^*$, there exists a unique solution $u \in \widetilde{H}^s(-1, 1)$ of problem* (1.1). *We have*

$$\|u\|_{\widetilde{H}^s} \le \sqrt{C_2}\, \|f\|_{(\widetilde{H}^{1-s})^*}.$$

In Theorem 6, for $0 \le s < 1/2$, the solution is understood as the very weak solution in the sense of Definition 3, for $1/2 \le s < 1$, the solution is understood as the weak solution in the sense of Definition 2 and, for $s = 1$, the solution is understood as the strong solution.

ACKNOWLEDGMENTS

The authors are grateful to A. I. Nazarov and F. V. Petrov for useful discussions.

FUNDING

This work was supported by the Russian Science Foundation under grant 17-11-01069.

REFERENCES

1. V. V. Golubev, *Lectures on Wing Theory* (Gostekhizdat, Moscow, 1949) [in Russian].
2. K. Stewartson, "A note on lifting line theory," Quart. J. Mech. Appl. Math. **13**, 49–56 (1960).
3. I. P. Krasnov, *Calculation Methods for Ship Magnetism and Electrical Engineering* (Sudostroenie, Leningrad, 1986) [in Russian].
4. A. I. Kalandiya, *Mathematical Methods of Two-Dimensional Elasticity* (Mir, Moscow, 1975) [English transl.].
5. J. I. Diaz, D. Gomez-Castro, and J. L. Vazquez, "The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach," Nonlinear Anal. **177**, 325–360 (2018), Part A.
6. M. M. Fall, "Regularity estimates for nonlocal Schrödinger equations," Discrete Contin. Dyn. Syst. **39** (3), 1405–1456 (2019).
7. V. É. Petrov, "Integral transform on a segment," J. Math. Sci. (New York) **132** (4), 451–481 (2006).
8. V. E. Petrov, "The generalized singular Tricomi equation as a convolution equation," Dokl. Math. **74** (3), 901–905 (2006).
9. I. V. Andronov and V. E. Petrov, "Diffraction by an impedance strip at almost grazing incidence," IEEE Trans. Antennas and Propagation **64** (8), 3562–3572 (2016).
10. V. E. Petrov, "On exact solutions of Hankel equations," St. Petersburg Math. J. **30** (1), 123–148 (2019).
11. J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary-Value Problems and Applications* (Springer-Verlag, Berlin–Heidelberg, 1972), Vol. 1.