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On Entire Solutions of a Class of Second-Order Algebraic Differential Equations

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Abstract—For second-order algebraic differential equations which have an explicit linear part, we describe all their possible solutions that are entire functions of finite order.

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1. HISTORICAL SURVEY. STATEMENT OF THE RESULTS

One of the directions in the study of algebraic differential equations is the description of their solutions that are meromorphic or entire functions. At the same time, most of the results obtained so far relate to linear equations (see, for example [1, Chap. 5]). For nonlinear equations, however, despite the fact that the first results were obtained in the 19th century (Hermite's theorem; see, for example [2, Chap. 2, Sec. 8]), the results available to date are mainly related to the study of specific equations. There are few statements describing meromorphic or entire solutions of any classes of nonlinear algebraic differential equations. Thus, meromorphic solutions of autonomous equations of Briot–Bouquet type $P(y, y^{(n)}) = 0$ are described (for a large number of cases, although not for all; see [3]). In addition, there exist a number of theorems that give conditions for the existence of solutions of algebraic differential equations that are either polynomials or entire functions with finitely many zeros [4, Chaps. 4, 5].

More detailed information about the available results in this area can be found in the survey [3] of Eremenko (relevant up to now) and in the monograph [4] of Gorbuzov. In recent years, in connection with the study of the arithmetical properties of the values of entire functions, the author of this paper has developed a certain technique [5] [6], which turned out to be applicable to problems of analyzing entire solutions of algebraic differential equations.

In the present paper, using this technique, possible entire solutions (solutions that are entire functions of finite order) for a class of second-order algebraic differential equations with an explicit linear part are described.

In what follows, by $E[\omega_0, \dots, \omega_n]$ we will denote the annulus of polynomials over the field E of variables $[\omega_0, \dots, \omega_n]$ and by $\mathbb{C}(z)$, the field of rational functions over the field of complex numbers $\mathbb{C}$.

If $\varphi(z)\colon \mathbb{C} \in \mathbb{C}$ is an entire function, then we put $M_\varphi(R) = \max_{|z|\le R} |\varphi(z)|$; by $N_\varphi(R)$ we denote the number of zeros (with their multiplicity taken into account) of the function $\varphi(z)$ on the disk $|z| \le R$; the order ρ of an entire function is determined by the equality

$$\rho = \varlimsup_{R\to+\infty} \frac{\ln\ln M_\varphi(R)}{\ln R}\,.$$

If $\rho < +\infty$, then the function $\varphi(z)$ is called an *entire function of finite order*.

The main result of this paper is the following statement.

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Theorem 1. *Let, for* $i = 0, 1, 2$, $b_i \in \mathbb{C}[z]$ *and* $b_2 \not\equiv 0$. *Let* $A \in \mathbb{C}[z, \omega_0, \omega_1, \omega_2]$ *and, at the same time,* $A(z, 0, \omega_1, -(b_1/b_2)\omega_1)$ *does not lie in* $\mathbb{C}(z)$. *Let an entire function of finite order* $y = f(z)$ *satisfy the differential equation*

$$b_0 y + b_1 y' + b_2 y'' + yA(z, y, y', y'') = 0.$$

Then there exist $A_1, A_2 \in \mathbb{C}[z]$ *and* $B_1, B_2 \in \mathbb{C}(z)$ *such that*

$$f(z) = B_1 e^{A_1} + B_2 e^{A_2}.$$

2. AUXILIARY STATEMENTS

Lemma 1 [5, Sec. 2, Corollary of Lemma 1]. *For any number* $H > 0$ *and any complex numbers* $\alpha_1, \dots, \alpha_n$, *it is possible to find, in the complex plane, a set of at most* n *disks whose sum of radii is at most* $2h$ *such that, for each point* z *lying outside these disks, the following estimate holds*:

$$\sum_{with J=1}^{h} \frac{1}{|z - \alpha_j|} < \frac{n(\ln N + 1)}{h}\,.$$

Lemma 2 [5, Sec. 2, Lemma 2]. *Let* $\delta \in (0; 1)$; *let* $R > 10^{1/\delta}$; *and let* B_R *be a finite set of disks whose sum of radii is at least* $2R^{1-\delta}$ *lying in the annulus*

$$C_R = \{2R \le |z| \le 3R\}.$$

Then there exists a number $R_1 \in (2R; 3R)$ *such that the circle* $\beta_{R_1} = \{z : /z/ = R_1\}$ *does not intersect the set* B_R.

The following statement is a slight enhancement of Lemma 3 from [5, Sec. 2].

Proposition 1. *Let* $h(z)$ *be an entire function of finite order* ρ. *Then, for any* $\varepsilon > 0$, *there exist numbers* $R_0 > 0$ *and* $\sigma > 0$ *such that the following assertion is valid*: *for any* $R > R_0$ *and* $H > 0$, *in the annulus* $C_R = \{2R \le |z| \le 3R\}$, *one can choose a finite set* E_R *of disks whose sum of radii is at most* $2H$, *so that, for any* $z \in C_R \setminus E_R$, *the following estimate holds*:

$$\left|\frac{h'(z)}{h(z)}\right| \le \sigma\left(1 + R^{\rho+\varepsilon-1} + \frac{R^{\rho+\varepsilon}}{H}\right).$$

Proof. Without loss of generality, we assume that $h(0) = 1$.

Indeed, if the assertion of the lemma is established for such functions, then, for $h_1(z) = Az^m h(z)$ ($A \in \mathbb{C}$, $m \ge 1$, $h(0) \ne 0$), for the values of z specified in the assertion of the proposition, we have

$$\left|\frac{h_1'(z)}{h_1(z)}\right| \le \left|\frac{h'(z)}{h(z)}\right| + \frac{m}{|z|} \le \left|\frac{h'(z)}{h(z)}\right| + \frac{m}{R}\,,$$

and the required inequality will also hold for the function $h_1(z)$.

We fix an arbitrary $\varepsilon > 0$. Then there exists an $R_0 > 0$ such that, for all $R > R_0$, the following estimate holds:

$$\ln M_h(R) < R^{\rho+\varepsilon/2}. \qquad (1)$$

Let us put

$$E_0(\omega) = 1 - \omega, \qquad E_m(\omega) = (1 - \omega)\exp\left(\sum_{k=1}^{m} \frac{\omega^k}{k}\right), \quad m \in \mathbb{N}.$$

Then

$$\frac{E_m(\omega)}{E_m(\omega)} = \frac{\omega^m}{\omega - 1}\,.$$

Denote $p = [\rho]$. Let $\{a_n\}$ be all the zeros of the function $h(z)$ (with their multiplicity taken into account). By Hadamard's theorem [7, Chap. 1], we have

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < +\infty \qquad \text{and} \qquad h(z) = e^{Q(z)} \prod_{n=1}^{\infty} E_p\biggl(\frac{z}{a_n}\biggr),$$

where $Q(z) \in \mathbb{C}[z]$ and $\deg Q \le R$. Then we obtain

$$\frac{h'(z)}{h(z)} = Q'(z) - \sum_{n=1}^{\infty} \frac{(z/a_n)^p}{a_n - z}\,. \tag{2}$$

Let $N(t)$ denote the number of zeros of $h(z)$ on the disk $|z| \le t$. Since $h(0) \ne 0$, it follows that, for some $\delta > 0$, $N(t) = 0$ for any $t \in [0, \delta]$.

Let us arbitrary fix $R > R_0$ and $H > 0$. By Lemma 1, inside the disk $|z| \le 4R$, there exists a finite set B_R of disks whose sum of radii is at most $2H$, so that, for each z satisfying the conditions

$$\begin{cases} |z| \le 4R, \\ z \notin B_R, \end{cases}$$

the following estimate holds:

$$\sum_{|a_n| \le 4R} \frac{1}{|z - a_n|} \le \frac{N(4R)(\ln N(4R) + 1)}{H}\,.$$

Let γ_j, $j = 0, 1, \dots$, denote constants depending only on ε and $h(z)$, and independent of R and H. Then, since $N(x) \le \gamma_0 \ln M_h(x)$ [7, Chap. 1], taking into account estimate (1), we find that, for each z such that

$$\begin{cases} |z| \le 4R, \\ z \notin B_R, \end{cases}$$

the following inequality holds:

$$\sum_{|a_n| \le 4R} \frac{1}{|z - a_n|} \le \gamma_0 \frac{R^{\rho + 2\varepsilon/3}}{H}\,. \tag{3}$$

We set

$$\Sigma_1 = \sum_{|a_n| \le R} \frac{|z/a_n|^p}{|a_n - z|}\,, \quad \Sigma_2 = \sum_{R < |a_n| \le 4R} \frac{|z/a_n|^p}{|a_n - z|}\,, \quad \Sigma_3 = \sum_{|a_n| > 4R} \frac{|z/a_n|^p}{|a_n - z|}\,.$$

Let us estimate $\Sigma_1, \Sigma_2, \Sigma_3$.

a) The first of these sums can be estimated as follows:

$$\begin{aligned}
\Sigma_1 &\le \sum_{|a_n| \le R} \frac{(3R)^p}{|a_n|^p R} \le 3^p R^{p-1} \int_0^R \frac{dN(t)}{t^p} \\
&\le 3^p R^{p-1} \biggl(\int_0^{R_0} \frac{dN(t)}{t^p} + \int_{R_0}^R \frac{dN(t)}{t^p} \biggr) \\
&\le 3^p R^{p-1} \biggl(\int_0^{R_0} \frac{dN(t)}{t^p} + \frac{N(t)}{t^p}\bigg|_{R_0}^R + p \int_{R_0}^R \frac{N(t)}{t^{p+1}} dt \biggr) \\
&\le 3^p R^{p-1} \biggl(\int_0^{R_0} \frac{dN(t)}{t^p} + \gamma_1 R^{\rho + \varepsilon/2 - p} + p \int_{R_0}^R \gamma_1 t^{\rho + \varepsilon/2 - p - 1}\, dt \biggr) \\
&\le 3^p R^{p-1} \biggl(\int_0^{R_0} \frac{dN(t)}{t^p} + \gamma_1 R^{\rho + \varepsilon/2 - p} + \gamma_2 R^{\rho + \varepsilon/2 - p} \biggr) \le \gamma_3 R^{\rho + \varepsilon/2 - 1}.
\end{aligned}$$

b) Let us estimate the second sum. Using (3), we obtain

$$Sigma_2 \le \sum_{R<|a_n|\le 4R} \frac{|z/a_n|^p}{|a_n - z|} \le \gamma_4 \sum_{|a_n|\le 4R} \frac{1}{|a_n - z|} \le \gamma_5 \frac{R^{\rho+2\varepsilon/3}}{H}.$$

c) The third sum can be estimated as follows:

$$\Sigma_3 \le \sum_{|a_n|>4R} \frac{|z/a_n|^p}{|a_n| - |z|} \le \sum_{|a_n|>4R} \frac{(3R)^p}{(|a_n| - (3/4)|a_n|)|a_n|^p} \le \gamma_6 R^p \sum_{|a_n|>4R} \frac{1}{|a_n|^{p+1}}.$$

Without loss of generality, we assume that ε is quite small, so that $\rho + \varepsilon/2 - p - 1 < 0$. Then

$$\begin{aligned}\Sigma_3 &\le \gamma_6 R^p \sum_{|a_n|>4R} \frac{1}{|a_n|^{p+1}} = \gamma_6 R^p \int_{4R}^{\infty} \frac{dN(t)}{t^{p+1}} \\ &= \gamma_6 R^p \biggl(\frac{N(t)}{t^{p+1}}\bigg|_{4R}^{+\infty} + (p+1)\int_{4R}^{\infty} \frac{N(t)}{t^{p+2}}\,dt\biggr) \le \gamma_7 R^p \int_{4R}^{\infty} t^{\rho+\varepsilon/2-p-2}\,dt \le \gamma_8 R^{\rho+\varepsilon/2-1}.\end{aligned}$$

Using the estimates for $\Sigma_1, \Sigma_2, \Sigma_3$, from equality (2), we further find that, for any $z \notin B_R$,

$$\left|\frac{h'(z)}{h(z)}\right| \le \gamma_9 \biggl(R^{\rho+\varepsilon-1} + \frac{R^{\rho+\varepsilon}}{H}\biggr).$$

Putting $E_R = B_R \cap C_R$, we finally obtain the proof of Proposition 1. □

Corollary 1. *Under the assumptions of Proposition 1, for any natural n, for any $\varepsilon > 0$, there exist numbers $R_0 > 0$ and $d > 0$ such that, for any $R > R_0$ and $H > 0$, in the annulus $C_R = \{2R \le |z| \le 3R\}$, we can choose a finite set of disks B_R whose sum of radii is less than $2nH$, so that, for any $z \in C_R \setminus B_R$ and $k = 1, \dots, n$, the following inequality holds*:

$$\left|\frac{h^{(k)}(z)}{h(z)}\right| \le d\biggl(1 + R^{\rho+\varepsilon-1} + \frac{R^{\rho+\varepsilon}}{H}\biggr)^k.$$

Proof. Note that, for any $k = 0, 1, \dots, n$, the order of the entire function $h^{(k)}(z)$ is ρ [7, Chap. 1]. We fix arbitrary $\varepsilon > 0$ and $H > 0$. Let us apply Proposition 1 to each $h^{(k)}(z)$ (for $k = 0, 1, \dots, n-1$). Then, for any $k = 0, 1, \dots, n-1$, there exist positive $R_{0,k}$, σ_k such that, for any $R > R_{0,k}$, there exists a set E_{R_k} of disks whose sum of radii is less than $2H$ and, for any $z \in C_R \setminus E_{R_k}$, the following estimate holds:

$$\left|\frac{h^{(k+1)}(z)}{h^{(k)}(z)}\right| \le \sigma_k\biggl(1 + R^{\rho+\varepsilon-1} + \frac{R^{\rho+\varepsilon}}{H}\biggr).$$

Now let $R > \max_k R_{0,k}$. Set $B_R = \bigcup_{k=0}^{n-1} E_{R,k}$. Then, for any $z \in C_R \setminus B_R$ and any $k = 1, \dots, n$,

$$\left|\frac{h^{(k)}(z)}{h(z)}\right| = \prod_{l=0}^{k-1}\left|\frac{h^{(l+1)}(z)}{h^{(l)}(z)}\right| \le \biggl(\prod_{l=0}^{k-1}\sigma_l\biggr)\biggl(1 + R^{\rho+\varepsilon-1} + \frac{R^{\rho+\varepsilon}}{H}\biggr)^k.$$

By putting $d = \prod_{l=0}^{n-1}(\max\{1, \sigma_l\})$, we obtain the proof of Corollary1. □

Lemma 3. *Let $f(z)$ be an entire function of finite order; let $P \in \mathbb{C}[z, \omega]$ be a polynomial such that $\partial P/\partial\omega \not\equiv 0$; $P(z, f(z)) = 0$ for any $z \in \mathbb{C}$. Then $f(z)$ is a polynomial from the annulus $\mathbb{C}[z]$.*

Proof. Let

$$P = a_m(z)\omega^m + \dots + a_n(z)\omega^n,$$

where all the $a_i(z)$ belong to $\mathbb{C}(z)$, let $0 \le m \le n$, and let $a_m(z)$, $a_n(z)$ be nonzero polynomials. If $f(z)$ is nonzero, then $m < n$ and

$$a_m(z) + f(z)(a_{m+1}(z) + \dots + a_n(z)f^{n-m}(z)) \equiv 0 \qquad \text{in} \quad \mathbb{C}.$$

Therefore, all the zeros of $f(z)$ are included among the zeros of the polynomial $a_m(z)$; therefore, by Hadamard's theorem, there exist polynomials $b_1(z), b_2(z) \in \mathbb{C}[z]$ such that $f(z) = b_1(z)e^{b_2(z)}$.

Suppose that $b_2(z)$ is not a constant. We have

$$a_n(z)(b_1(z)e^{b_2(z)})^{n-m} + \cdots + a_{m+1}(z)b_1(z)e^{b_2(z)} + a_m(z) \equiv 0 \quad \text{in} \quad \mathbb{C}. \tag{4}$$

Let $R > 0$ be large enough for all the zeros of all the polynomials $\{a_i(z)\}$, $b_1(z)$, $b_2(z)$ to lie inside the disk $|z| < R$. Let $z_R \in \mathbb{C}$ be a point such that $|z_R| = R$ and

$$\max_{|z| \le R} |b_1(z)e^{b_2(z)}| = |b_1(z_R)e^{b_2(z_R)}|.$$

If $b_2(z) = \alpha_d z^d + \cdots + \alpha_0$, $\alpha_d \ne 0$, $d \ge 1$, then

$$|b_1(z_R)e^{b_2(z_R)}| \ge e^{\gamma_1 R^d} \tag{5}$$

for some constant $\gamma_1 > 0$ independent of R. But then it follows from (4) that

$$1 = \frac{-1}{a_n(z_R)}\left[a_{n-1}(z_R)\frac{1}{b_1(z_R)e^{b_2(z_R)}} + \cdots + \frac{a_m(z_R)}{(b_1(z_R)e^{b_2(z_R)})^{n-m}}\right],$$

whence, by virtue of (5), we have

$$1 \le \frac{1}{|a_n(z_R)|}(|a_{n-1}(z_R)|e^{-\gamma_1 R^d} + \cdots + |a_m(z_R)|e^{-\gamma_1(n-m)R^d}) \le e^{-\gamma_2 R^d}$$

for a constant $\gamma_2 > 0$ independent of R; but this is impossible, because $R \to +\infty$. Therefore, $b_2(z)$ does not depend on z and, therefore, $f(z)$ is a polynomial from $\mathbb{C}[z]$. Thus, Lemma 3 is proved. □

Remark 1. Note that the assertion of Lemma 3 is also valid for entire functions of infinite order. This can be deduced from the fact that a univalent algebraic function must be rational (see, for example [8, pp. 215, 224]).

Lemma 4. *Let an entire function $y = f(z)$, other than a constant, satisfy the differential equation $y' = R(z)y$, where $R(z) \in \mathbb{C}(z)$. Then there exist $z_0 \in \mathbb{C}$ and $Q_1, Q_2 \in \mathbb{C}[z]$ such that*

1) $e^{\int_{z_0}^{z} R(\omega)d\omega} = Q_1(z)e^{Q_2(z)}$;

2) $f(z) = \gamma Q_1(z)e^{Q_2(z)}$ *for some* $\gamma \in \mathbb{C}$.

Lemma 4 (in stronger form) was proved by Shidlovskii [8, Chap. 5, Lemma 11].

Lemma 5. *Let $P \in (\mathbb{C}[z, \omega_0, \omega_1] \setminus \mathbb{C}[z])$ be a nonzero homogeneous (with respect to the variables ω_0, ω_1) polynomial. Let $f(z)$ be an entire function of finite order such that $P(z, f(z), f'(z)) = 0$ in $\mathbb{C}$. Then $f(z) = Q_1(z)e^{Q_2(z)}$ for some $Q_1(z), Q_2(z) \in \mathbb{C}[z]$.*

Proof. If $\partial P/\partial\omega_0 \equiv 0$, then $P \in \mathbb{C}[z, \omega_1]$ and $P(z, f'(z)) \equiv 0$ in $\mathbb{C}$. Then, by Lemma 3, $f'(z)$ is a polynomial from $\mathbb{C}[z]$, and hence $f(z)$ is also a polynomial.

Now let $\partial P/\partial\omega_0 \not\equiv 0$. Taking out the maximum possible power $\omega_0^{m_0}\omega_1^{m_1}$ (m_0, m_1 are nonnegative integers), we find that $P = \omega_0^{m_0}\omega_1^{m_1}Q$, where $Q \in \mathbb{C}(z)[\omega_0, \omega_1]$ is homogeneous in ω_0, ω_1 and either $Q \in \mathbb{C}[\omega_0]$, or $Q \in \mathbb{C}[\omega_1]$, or

$$Q = a_N\omega_0^N + \cdots + a_0\omega_1^N,$$

and $a_0, a_N \in \mathbb{C}[z]$ is nonzero. Then either $f(z) \equiv 0$, or $f'(z) \equiv 0$, or

$$a_N(f(z))^N + \cdots + a_0(f'(z))^N \equiv 0 \quad \text{in} \quad \mathbb{C}.$$

In the first two cases, $f(z) \in \mathbb{C}[z]$. Let us consider the third case. We assume that $f(z)$ is not a constant. Suppose that there exists a $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$; $a_0(z_0) \ne 0$. Then $f(z) = (z - z_0)^d\varphi(z)$ for a natural number $d \ge 1$ and $\varphi(z_0) \ne 0$.

Then the function

$$h(z) = \frac{f(z)}{f'(z)} = \frac{(z - z_0)\varphi(z)}{d\varphi(z) + (z - z_0)\varphi'(z)}$$

is holomorphic in some $O_\delta(z_0)$, $h(z_0) = 0$, and

$$a_N(z)(h(z))^N + \cdots + a_1(z)h(z) + a_0(z) = 0$$

in this neighborhood $O_\delta(z_0)$. But then, for $z = z_0$, we have $a_0(z_0) = 0$.

The resulting contradiction with the choice of z_0 means that any zero of the function $f(z)$ is a zero of the polynomial $a_0(z)$. Therefore (because $f(z)$ is an entire function of finite order), there exist polynomials $Q_1, Q_2 \in \mathbb{C}[z]$ such that $f(z) = Q_1(z)e^{Q_2(z)}$. Thus, Lemma 5 is proved. □

Remark 2. Lemma 5 can be proved without the assumption that the order of the entire function $f(z)$ is finite (using arguments similar to those of Shidlovskii in his proof of Lemma 1 [8, Chap. 6]).

Lemma 6. *Let $T \in \mathbb{C}(z)[\omega_0, \omega_1]$, $T \notin \mathbb{C}(z)$, and let T be irreducible in $\mathbb{C}(z)[\omega_0, \omega_1]$. Let $y = f(z)$ be an entire function of finite order satisfying the system of equations*

$$\begin{cases} T(z, y, y') = 0, \\ y'' = \alpha_1 y' + \alpha_0 y, \end{cases} \tag{6}$$

where $\alpha_0, \alpha_1 \in \mathbb{C}(z)$. Then there exist $A_1, A_2 \in \mathbb{C}[z]$ and $B_1, B_2 \in \mathbb{C}(z)$ such that

$$f(z) = B_1 e^{A_1} + B_2 e^{A_2}.$$

Proof. Let Δ be the differential operator (in $\mathbb{C}(z)[\omega_0, \omega_1]$) defined by the equality

$$\Delta = \frac{\partial}{\partial z} + \omega_1 \frac{\partial}{\partial \omega_0} + (\alpha_0 \omega_0 + \alpha_1 \omega_1) \frac{\partial}{\partial \omega_1}\,.$$

Then, for any solution y of the equation $y'' - \alpha_1 y' - \alpha_0 y = 0$ and any polynomial $B \in \mathbb{C}(z)[\omega_0, \omega_1]$, the following equality holds:

$$\Delta B(z, y, y') = \frac{d}{dz}(B(z, y, y')).$$

If the polynomial T from the assertion of Lemma 6 does not depend on ω_0 or on ω_1, then, applying Lemma 3, we obtain $f(z) \in \mathbb{C}[z]$.

Now let $\partial T/\partial \omega_0 \not\equiv 0$, $\partial T/\partial \omega_1 \not\equiv 0$. If T and ΔT are coprime, then, consideannulus the resultant $B = \operatorname{Res}_{\omega_1}(T, \Delta T)$, we obtain $B \in \mathbb{C}(z)[\omega_0]$, $B \not\equiv 0$ and $B(f(z)) \equiv 0$ in $\mathbb{C}$, whence, by Lemma 3, we see that $f(z) \in \mathbb{C}[z]$.

Suppose that T and ΔT are not coprime. Then, by virtue of the irreducibility of T, there exists a $\lambda(z) \in \mathbb{C}(z)$ such that $\Delta T = \lambda(z)T$ (if $\Delta T = 0$, then $\lambda(z) \equiv 0$).

Let $T = \sum_{i=0}^{N} P_i$, where each of the P_i is a homogeneous (in ω_0, ω_1) polynomial of degree i.

Note that the operator Δ takes any homogeneous (with respect to the variables ω_0, ω_1) polynomial from $\mathbb{C}(z)[\omega_0, \omega_1]$ to a homogeneous polynomial of the same degree or to zero. Since $T \notin \mathbb{C}(z)$, it follows that there exists a number $i_0 \geq 1$ such that P_{i_0} is a nonzero polynomial. Then $\Delta P_{i_0} = \lambda P_{i_0}$.

Suppose that $f(z)$ has infinitely many zeros. By assumption, $y = f(z)$ satisfies the equation $y'' - \alpha_1 y' - \alpha_0 y = 0$, where α_0, α_1 are rational functions of z. Let us choose the point z_0 so that $f(z_0) = 0$ and z_0 will not be a singular point of $\alpha_0(z)$, $\alpha_1(z)$, $\lambda(z)$. By the properties of solutions of a linear homogeneous equation, there exists a neighborhood $O_\delta(z_0)$ of the point z_0 and a holomorphic function $g(z)$ in it such that the following conditions hold in $O_\delta(z_0)$:

a) $\alpha_0(z)$, $\alpha_1(z)$, $\lambda(z)$ are holomorphic;

b) $g(z)$ and $f(z)$ are linearly independent over $\mathbb{C}$;

c) $y = C_1 f(z) + C_2 g(z)$ is a solution in $O_\delta(z_0)$ of the equation $y'' - \alpha_1 y' - \alpha_0 y = 0$ for any $C_1, C_2 \in \mathbb{C}$.

For each $\overline{C} = (C_1, C_2) \in \mathbb{C}^2$, we set

$$\Phi_{\overline{C}}(z) = P_{i_0}(z, C_1 f + C_2 g, C_1 f' + C_2 g'). \tag{7}$$

Then (from the condition $\Delta P_{i_0} = \lambda P_{i_0}$) we find that $\Phi'_{\overline{C}}(z) = \lambda(z)\Phi_{\overline{C}}(z)$ for any $z \in O_\delta(z_0)$. Hence

$$\Phi_{\overline{C}}(z) = A(C_1, C_2)h(z), \qquad \text{where} \quad h(z) = e^{\int_{z_0}^{z} \lambda(\omega)\,d\omega};$$

$A(C_1, C_2)$ depends only on C_1, C_2 and does not depend on z.

Note that (due to the holomorphy in $O_\delta(z_0)$ of the function $\lambda(z)$) $h(z_0) \neq 0$ (if $\lambda(z) \equiv 0$, then we consider $h(z) \equiv 1$).

Thus,

$$P_{i_0}(z, C_1 f(z) + C_2 g(z), C_1 f'(z) + C_2 g'(z)) = A(C_1, C_2)h(z). \tag{8}$$

For $g(z)$ we can take the function $g(z) = f(z)h(z)$, where

$$h'(z) = \frac{1}{f^2(z)}\, R_1 e^{R_2}$$

with some rational functions R_1, R_2 of z; to do this, it is enough to consider the following differential equation for the Wronskian

$$W = \begin{vmatrix} f(z) & g(z) \\ f'(z) & g'(z) \end{vmatrix}.$$

It follows from equality (8) that $A(C_1, C_2)$ is a polynomial.

If $A(C_1, C_2) = \gamma$ is a constant, then (putting $C_2 = 0$ in (8)) we obtain

$$C_1^N P_{i0}(z, f(z), f'(z)) = \gamma h(z)$$

for any complex C_1, whence $\gamma = 0$. In that case, $P_{i0}(z, f(z), f'(z)) \equiv 0$, whence (by Lemma 5) $f(z) = Q_1 e^{Q_2}$ for some $Q_1, Q_2 \in \mathbb{C}[z]$.

But if the polynomial $A(C_1, C_2)$ is not a constant, then there exist complex numbers C_1^0, C_2^0, not all equal to zero, such that $A(C_1^0, C_2^0) = 0$. Then, from (8), we obtain

$$P_{i0}(z, C_1^0 f(z) + C_2^0 g(z), C_1^0 f'(z) + C_2 g'(z)) = 0.$$

If $C_2^0 = 0$, then, by Lemma 5, we obtain the required form for $f(z)$. If $C_2^0 \neq 0$, then (dividing the last equality, by $C_1^0 f + C_2^0 g$), we see that

$$T(z, \varphi(z)) = 0$$

for some polynomial $T \in \mathbb{C}[z, w]$, other than a constant, where

$$\varphi(z) = \frac{C_1^0 f'(z) + C_2^0 g'(z)}{C_1^0 f(z) + C_2^0 g(z)} = (\ln(C_1^0 f(z) + C_2^0 g(z)))'.$$

Also note that

$$\varphi(z) = \frac{C_1^0 f'(z) + C_2^0 (f(z)h(z))'}{C_1^0 f(z) + C_2^0 g(z)} = \frac{C_1^0 f(z)f'(z) + C_2^0 f'(z)g(z) + C_2^0 R_1 e^{R_2}}{(C_1^0 f(z) + C_2^0 g(z))f(z)}.$$

Thus, the function $\varphi(z)$ is an algebraic function.

In view of the equalities

$$g(z) = f(z)h(z) \qquad \text{and} \qquad h'(z) = \frac{1}{f^2(z)}\, R_1 e^{R_2},$$

if $g(z)$ has critical points, then these points can only be singular points for the rational functions R_1 and R_2. But, in that case (integrating the corresponding Laurent series for $h'(z)$), we find that all the branch points from $\mathbb{C}$ of the function $\varphi(z)$ will necessarily be logarithmic, i.e., $\varphi(z)$ in $\mathbb{C}$ does not have any branch points of finite order. Therefore, $\varphi(z)$ is a rational function, i.e.,

$$(\ln(C_1^0 f(z) + C_2^0 g(z)))' \in \mathbb{C}(z).$$

Then, decomposing the rational function into the sum of polynomials and simple partial fractions, we obtain

$$C_1^0 f(z) + C_2^0 g(z) = Q_1 e^{Q_2}, \qquad \text{where} \quad Q_1, Q_2 \in \mathbb{C}(z).$$

Hence $Q_1 e^{Q_2}$ is also a solution of the linear equation from (6).

Since $C_2^0 \neq 0$, it follows that, taking the function $Q_1 e^{Q_2}$ for $g(z)$ and repeating the above arguments, we find that

$$f(z) = D_1 Q_1 e^{Q_2} + D_2 Q_3 e^{Q_4} \qquad \text{under some} \quad D_1, D_2 \in \mathbb{C}, \quad Q_3, Q_4 \in \mathbb{C}(z).$$

Consider the following system of equations (for e^{Q_2}, e^{Q_4}):

$$\begin{cases} (D_1 Q_1 e^{Q_2})' + (D_2 Q_3 e^{Q_4})' = f'(z), \\ D_1 Q_1 e^{Q_2} + D_2 Q_3 e^{Q_4} = f(z). \end{cases}$$

If the determinant Δ of this system is identically zero, then we have the equation $f'(z) = \lambda f(z)$ (for some $\lambda \in \mathbb{C}(z)$), whence (by Lemma 4) $f(z) = R_1 e^{R_2}$ with $R_1, R_2 \in \mathbb{C}[z]$. If $\Delta \neq 0$, then

$$e^{Q_2} = \alpha_1 f'(z) + \alpha_2 f(z), \qquad e^{Q_4} = \beta_1 f'(z) + \beta_2 f(z)$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}(z)$. But then e^{Q_2}, e^{Q_4} cannot have finite essentially singular points, whence $Q_2, Q_4 \in \mathbb{C}[z]$.

Thus, Lemma 6 is proved. □

Remark 3. Note that the proof of Lemma 6 uses ideas expressed earlier by K. Siegel and A. B. Shidlovskii (see, for example [2, pp. 192, 213, 214, 222]).

3. PROOF OF THEOREM 1

Let ρ be the order of the function $f(z)$. Let us put $\varepsilon = 1/2$. Then, by the Corollary of Proposition 1, there exists an $R_0 > 0$ such that, for any $R > R_0$, from the annulus $C_R = \{2R < |z| < 3R\}$ we can throw out a set B_R of finitely many disks whose sum of radii is at most $4R^{1-\varepsilon}$ (we assume $H = R^{1-\varepsilon}$), so that, for any $z \in C_R \setminus B_R$ and $k = 1, 2$, the following estimate holds:

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \delta\left(1 + R^{\rho+\varepsilon-1} + \frac{R^{\rho+\varepsilon}}{H}\right)^2 \leq \gamma_1 R^{2\rho},$$

where γ_1 does not depend on R.

Let us fix an arbitrary sufficiently large $R > 0$. Applying Lemma 2 (for $\delta = 2\varepsilon/3$), we find that, for some $R_1 \in (2R; 3R)$, the circle $\beta_{R_1} = \{z : /z/ = R_1\}$ does not intersect the set of discarded disks B_R, whence, for $k = 1, 2$,

$$\max_{z \in \beta_{R_1}} \left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \gamma_2 R^{2\rho}. \tag{9}$$

By assumption, the function $y = f(z)$ satisfies the equation

$$b_0 y + b_1 y' + b_2 y'' + y A(z, y, y', y'') = 0.$$

Then, for $z \in \beta_{R_1}$, we can write

$$b_0 + b_1 \frac{y'}{y} + b_2 \frac{y''}{y} = -A(z, y, y', y''),$$

whence, in view of (9), we have

$$\max_{|z|\le R} |A(z,y,y',y'')| \le \max_{z\in\beta_{R_1}} |A(z,y,y',y'')| \le R^a,$$

where $a>0$ is a constant independent of R. But, in that case, by Liouville's theorem (see, for example [2, Chap. 1, Sec. 1]), the entire function $A(z,f(z),f'(z),f''(z))$ is a polynomial, i.e., there exists a $q(z)\in\mathbb{C}[z]$ such that, for $y=f(z)$, the following equalities hold:

$$\begin{cases} b_0y+b_1y'+b_2y''=q(z)y, \\ A(z,y,y',y'')=-q(z). \end{cases}$$

We put

$$T=A\biggl(z,\omega_0,\omega_1,-\biggl(\frac{b_0-q}{b_2}\omega_0+\frac{b_1}{b_2}\omega_1\biggr)\biggr)+q(z).$$

Then $T\in\mathbb{C}(z)[\omega_0,\omega_1]$ and $T\notin\mathbb{C}(z)$ (because, otherwise, for $\omega_0=0$, we have

$$A(z,0,\omega_1,-(b_1/b_2)\omega_1)\in\mathbb{C}(z),$$

which contradicts the assumptions of the theorem).

Let us put

$$\alpha_0=-\frac{b_0-q}{b_2}, \qquad \alpha_1=-\frac{b_1}{b_2}.$$

Then, for $y=f(z)$, the following equalities hold:

$$\begin{cases} T(z,y,y')=0, \\ y''=\alpha_0y+\alpha_1y'. \end{cases}$$

At the same time (replacing, if necessary, the polynomial T by its divisor), we can assume that T is irreducible in $\mathbb{C}(z)[\omega_0,\omega_1]$. But then, applying Lemma 6, we see that the assertion of Theorem 1 is valid.

4. CONCLUSIONS

In conclusion, we give a few remarks.

Remark 4. It is not difficult to give examples of differential equations of the kind described in Theorem 1, such as the following ones:

a) $2zy''-(4z^2+2)y'+y((4z^2+2)y^2-(y'')^2)=0$; $y=e^{z^2}$ is an entire solution of this equation;

b) $(zy)''-3(zy)'+2zy+(z^2+1)y(3((zy)'-2zy)^2-(zy)''+zy)=0$; $y=(e^{2z}-e^z)/z$ is an entire solution of this equation.

Remark 5. When proving Theorem 1, we essentially use the fact that $f(z)$ is an entire function of finite order. In general, the solutions of algebraic differential equations can also be entire functions of infinite order (for example, $f=e^{e^z}$ is the solution of the equation $y''y-(y')^2-yy'=0$). Nevertheless, it seems to be a fair hypothesis that the possible solutions of equations with an explicit linear part (considered in Theorem 1) will necessarily be of finite order if they are entire functions.

Remark 6. Statements similar to Theorem 1 proved in this paper can be obtained for some other classes differential equations, for example, for equations of the form

$$(\beta_2y''+\beta_1y'+\beta_0y)^n+y^nA(z,y,y',y'')=0$$

with an arbitrary natural number n (the proof of such a statement practically repeats the proof of the theorem in the paper).

Remark 7. The paper dealt with equations with a selected linear part for $n = 2$:

$$\sum_{i=0}^{n} \beta_i y^{(i)} + yA(z, y, y', \dots, y^{(n)}) = 0.$$

The description of entire solutions of such equations for an arbitrary n is not known to the author. (At present, it is possible to do this only for some special cases, for example, for equations of the form $\sum_{i=0}^{n} \beta_i y^{(i)} + y^2 A(z, y, y', \dots, y^{(n)}) = 0$.)

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