ISSN 0001-4346, Mathematical Notes, 2021, Vol. 109, No. 6, pp. 859–866. © Pleiades Publishing, Ltd., 2021.

Locally Weak Version of the Contraction Mapping Principle*

Priyam Chakraborty[1] and Binayak S. Choudhury[1]*****

[1] *Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah, 711103 India*

Received July 13, 2020; in final form, July 13, 2020; accepted October 22, 2020

Abstract—The Banach Contraction Mapping Principle has many generalizations and extensions in different directions. Here we define (ε, ψ)-Uniformly Local Weak Contractions and show that these mappings admit unique fixed points. We obtain a generalization of two existing results. We construct an example which illustrates of our main result in which the mapping is neither a ψ-weak contraction nor a local contraction.

DOI: 10.1134/S0001434621050199

Keywords: *metric space, weak contraction, fixed point, local contraction.*

1. INTRODUCTION

Metric fixed point theory is a vast and expanding branch of functional analysis. It is widely held that the theory has its origin in Banach's Contraction Principle [1],[2], which was established in the year of 1922. This result has served to establish several important results is different branches of mathematics [2]–[4]. It has several generalizations and extensions which were advanced over the years and still are being actively pursued [3],[5]–[10]. In this paper, we consider two such generalizations, one is the fixed point result of local contraction due to Edelstein [11] and the other is the fixed point result for weak contractions first proved by Alber et al. [12] in Hilbert spaces and subsequently established in the metric spaces by Rhoades [13]. The result in [13] has itself become a source of an area of fixed point theory where weak inequalities are considered [14]–[19]. Our result in this paper combines the above two ideas to lead to a unique fixed point result for uniform local weak contraction mappings that we define here.

2. PRELIMINARIES

In this section we recall some definitions and introduce the notion of local weak contraction.

In what follows, we denote by $B(x,r)$ the open ball of radius r centered at the point x, by $\mathbb{N}$ the set of natural numbers, and by $\mathbb{R}$ the set of real numbers.

Definition 1 (local contraction). [11] A map $T : X \to X$ is said to be *locally* (*Banach*) *contractive* if, for every $x \in X$, there exist ε, λ ($\in \mathbb{R}$, with $\varepsilon > 0,\ 0 \le \lambda < 1$), which may depend on x, such that $\forall\, p,\, q \in B(x,\varepsilon)$,

$$d(T(p),T(q)) < \lambda d(p,q).$$

Definition 2 (Uniform Local contraction). [11] A map $T : X \to X$ is said to be *uniformly locally contractive* if it is locally contractive and both ε and λ do not depend on x.

It is usually abbreviated as "$(\varepsilon, \lambda)-$uniformly locally contractive" when the role of ε and λ are to be stressed down.

*The article was submitted by the authors for the English version of the journal.

**E-mail: priyam.math123@gmail.com

***E-mail: binayak@math.iiests.ac.in

Definition 3 (ε-chainable metric space [11]). A metric space X is said to be ε-*chainable* ($\varepsilon \in \mathbb{R},\ > 0$) if, for any $x,\ y \in X$, there is an ε-chain from x to y, that is, there are finite number of points $a_0, a_1, ..., a_n$ in X, with $x = a_0,\ y = a_n$ such that $d(a_i, a_{i+1}) < \varepsilon,\ \forall i = 0, 1, ..., n-1$.

Note that ε-chainable metric spaces have appeared in the works like [11],[20]–[25].

Definition 4 (ψ-weak contraction). Let $\psi : [0,\infty) \to [0,\infty)$ be a given continuous function which is nondecreasing and satisfies $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$. A function $T : X \to X$ is said to be a ψ-*weak contraction* if $\forall x, y \in X$, $d(Tx, Ty) \le d(x, y) - \psi(d(x, y))$.

Definition 5 (weak contraction [13]). A map $T : X \to X$ is said to be a *weak contraction* if there is a continuous function $\psi : [0,\infty) \to [0,\infty)$ which is nondecreasing and satisfies $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$ and $\lim\limits_{t\to\infty} \psi(t) = \infty$, such that

$$x, y \in X \implies d(Tx, Ty) \le d(x, y) - \psi(d(x, y)).$$

Remark 1. There is a subtle difference between weak contraction and ψ-weak contraction. While the latter implies the former, it is possible that a weak contraction is not a ψ-weak contraction for a prior given ψ. If we consider $X = [0, 1]$, with the usual distance, $T(x) = x - \frac{1}{2}x^2$ and $\psi(t) = (1-\lambda)t$, where $0 < \lambda < 1$, then, although it is a weak contraction [15], but is not a ψ-weak contraction, for this given ψ.

We now introduce a local version of the *weak contraction*.

Definition 6 ((ε, ψ)-uniform local weak contraction). Let $\psi : [0,\infty) \to [0,\infty)$ be a given continuous function which is nondecreasing and satisfies $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$. Let (X, d) be a metric space. A function $T : X \to X$ is said to be an (ε, ψ)-*uniform local weak contraction* if

$$d(x, y) < \varepsilon \implies d(Tx, Ty) \le d(x, y) - \psi(d(x, y)). \tag{1}$$

3. MAIN RESULTS

Theorem 1. *Let $\varepsilon > 0$ and X be a complete, ε-chainable metric space. Suppose $\psi : [0,\infty) \to [0,\infty)$ be a continuous, nondecreasing function such that $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$. Suppose also that $T : X \to X$ is a self-map which is an (ε, ψ)-uniform local weak contraction, that is, for each $x, y \in X$ if $d(x, y) < \varepsilon$, then*

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)).$$

Then T has a unique fixed point $\bar{x}$ in X.

Proof. Let $x \in X$ be an arbitrary element. We construct a sequence $\{x_n\}$ such that

$$x_0 := x,\ x_i := T^i x,\ \forall i \in \mathbb{N}. \tag{2}$$

As X is ε-chainable, let $x = a_0, a_1, ..., a_n = Tx$, be an an ε-chain from x to Tx, where

$$d(a_i, a_{i+1}) < \varepsilon,\ \forall i = 0, 1, .., n-1. \tag{3}$$

Since $d(a_i, a_{i+1}) < \varepsilon, \forall i = 0, 1, .., n-1$, equation-(1) is also satisfied for every pair of consecutive elements of the chain. Thus, we have

$$d(Ta_i, Ta_{i+1}) \le d(a_i, a_{i+1}) - \psi(d(a_i, a_{i+1})) \le d(a_i, a_{i+1}) < \varepsilon.$$

Inductively, we obtain $d(T^m a_i, T^m a_{i+1}) < \varepsilon$, for any $m \in \mathbb{N}$.

Let $R^i_m := d(T^m a_i, T^m a_{i+1})$. Then,

$$\begin{aligned} R^i_{m+1} = d(T^{m+1} a_i, T^{m+1} a_{i+1}) &\le d(T^m a_i, T^m a_{i+1}) - \psi(d(T^m a_i, T^m a_{i+1})) \\ &\le d(T^m a_i, T^m a_{i+1}) = R^i_m. \end{aligned}$$

Thus, $\{R_m^i\}$ is a nonincreasing sequence, and being bounded below (0 is a lower bound), it must be convergent. Suppose $\lim_{m\to\infty} R_m^i = R^i \geq 0$, for each $i = 0, 1, ..., n-1$.

Again, we have

$$\begin{aligned} R_{m+1}^i &\leq R_m^i - \psi(R_m^i) \\ &\Rightarrow \lim_{m\to\infty} R_{m+1}^i \leq \lim_{m\to\infty} R_m^i - \lim_{m\to\infty} \psi(R_m^i) \\ &\Rightarrow R^i \leq R^i - \psi(R^i) \qquad \text{[since } \psi \text{ is continuous]} \\ &\Rightarrow \psi(R^i) = 0 \\ &\Rightarrow R^i = 0. \end{aligned} \tag{4}$$

Now using triangle inequality, we have

$$d(x_m, x_{m+1}) = d(T^m x, T^m(Tx)) \leq \sum_{i=0}^{n-1} d(T^m a_i, T^m a_{i+1}) = \sum_{i=0}^{n-1} R_m^i. \tag{5}$$

Taking limit $m \longrightarrow \infty$, we obtain,

$$\lim_{m\to\infty} d(x_m, x_{m+1}) \leq \lim_{m\to\infty} \sum_{i=0}^{n-1} R_m^i = \sum_{i=0}^{n-1} \lim_{m\to\infty} R_m^i = 0. \tag{6}$$

We will show that $\{x_n\}$ is a Cauchy sequence.

Let $\varepsilon_1 > 0$ be chosen arbitrarily and take $\varepsilon_0 = \min\{\varepsilon, \varepsilon_1\} > 0$. As $\lim_{m\to\infty} d(x_m, x_{m+1}) = 0$, $\exists\, k \in \mathbb{N}$ such that

$$d(x_k, x_{k+1}) < \min\{\frac{\varepsilon_0}{2}, \psi(\frac{\varepsilon_0}{2})\}. \tag{7}$$

We note that if $d(y, x_k) < \varepsilon_0 \leq \varepsilon$, then (1) holds for $x = x_k$. Thus, if $y \in B(x_k, \frac{\varepsilon_0}{2})$, we have,

$$\begin{aligned} d(Ty, x_k) &\leq d(Ty, Tx_k) + d(Tx_k, x_k) \\ &\leq d(y, x_k) - \psi(d(y, x_k)) + d(x_k, x_{k+1}) \\ &< \frac{\varepsilon_0}{2} - \psi(d(y, x_k)) + \frac{\varepsilon_0}{2} \qquad [\because y \in B(x_k, \frac{\varepsilon_0}{2})] \\ &\leq \varepsilon_0 - \psi(d(y, x_k)) \\ &\leq \varepsilon_0. \end{aligned} \tag{8}$$

$\therefore Ty \in B(x_k, \varepsilon_0), \forall y \in B(x_k, \frac{\varepsilon_0}{2})$.

Also, if $\frac{\varepsilon_0}{2} \leq d(y, x_k) \leq \varepsilon_0$, by monotonicity of ψ, we have $\psi(\frac{\varepsilon_0}{2}) \leq \psi(d(y, x_k))$.

$$\begin{aligned} d(Ty, x_k) &\leq d(Ty, Tx_k) + d(Tx_k, x_k) \\ &\leq d(y, x_k) - \psi(d(y, x_k)) + d(x_k, x_{k+1}) \\ &\leq d(y, x_k) - \psi(\frac{\varepsilon_0}{2}) + \psi(\frac{\varepsilon_0}{2}) \\ &\leq d(y, x_k) \\ &\leq \varepsilon_0. \end{aligned} \tag{9}$$

By the above two cases we have $Ty \in B(x_k, \varepsilon_0), \forall y \in B(x_k, \varepsilon_0)$, which implies $x_m \in B(x_k, \varepsilon_0)$ for all $m \geq k$. Hence $d(x_m, x_k) < \varepsilon_0 \leq \varepsilon_1$ for all $m \geq k$, and thus $\{x_n\}$ is a Cauchy sequence. X being complete, $\{x_n\}$ must converge to some $\bar{x} \in X$. Again for any $\varepsilon_2 > 0$ let $\delta = \min\{\varepsilon, \varepsilon_2\}$. Now, if $d(x, y) < \delta$, then $d(x, y) < \varepsilon$ and thus $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \leq d(x, y) < \delta \leq \varepsilon_2$, which means that T is continuous. Therefore, $\bar{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T\bar{x}$. Therefore, $\bar{x}$ is a fixed point of T.

We now prove that $\bar{x}$ is unique. If not, let $\exists\ \bar{y}\ (\neq \bar{x}) \in X$ such that $\bar{y} = T\bar{y}$. Then $d(\bar{x}, \bar{y}) > 0$. Now consider the ε-chain from $\bar{x}$ to $\bar{y}$. Let $\bar{x} = b_0, b_1, ..., b_p = \bar{y}$ be an an ε-chain from $\bar{x}$ to $\bar{y}$. Thus, $d(b_i, b_{i+1}) < \varepsilon, \forall i = 0, 1, .., p-1$. By the same argument as we reached (5) we obtain that

$$d(\bar{x}, \bar{y}) = d(T^m\bar{x}, T^m\bar{y}) \leq \sum_{i=0}^{p-1} d(T^m b_i, T^m b_{i+1}). \tag{10}$$

Taking limit $m \longrightarrow \infty$ we obtain, $d(\bar{x}, \bar{y}) = 0$, which is a contradiction. Hence the fixed point would be unique. □

Remark 2. The above result generalizes the result of Edelstein[11] and the result of Rhoades [13] in the context of ε-chainable metric spaces.

The next result shows that instead of assuming the whole space X to be ε-chainable, if $T(X)$ only is assumed to be so, the conclusion of the previous result still remains valid.

Theorem 2. *Let $\varepsilon > 0$, and let X be a complete metric space. Suppose $\psi : [0, \infty) \to [0, \infty)$ be a continuous, nondecreasing function such that $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$. Also let $T : X \to X$ be a self-map which is (ε, ψ)-uniform local weak contraction, that is, for each $x, y \in X$ if $d(x, y) < \varepsilon$, then*

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)).$$

If $T(X)$ is closed and ε-chainable, then T has a unique fixed point.

Proof. Let $z \in X$ be arbitrary, then $Tz \in T(X)$. We write $x_0 = x = Tz$ and set $x_n = T^n x$, for $n = 1, 2, \dots$. Then repeating the same argument as in Theorem 1, we assure the existence of a fixed point in $T(X)$ and hence in X. For the uniqueness part, we note that if z is any fixed point of T, then $z = T(z) \in T(X)$. Then following the same argument for the uniqueness part as in Theorem 1, we assure the uniqueness of the fixed point. □

4. ILLUSTRATIVE EXAMPLE

In this section we present an example to show that our theorem is applicable to a function which is neither a uniform local (Banach) contraction nor a ψ-weak contraction on the whole space, for a preassigned ψ.

Example 1. Let $X = A \cup B$, where $A = \big\{(x(t), y(t)) :\ x(t) = 1/2 - t, y(t) = 0,\ 0 \leq t \leq 1/2\big\}$ and $B = \big\{(x(s), y(s)) :\ x(s) = 0,\ y(s) = s + 1/4,\ 0 \leq s \leq 1/4\big\}$.

X is clearly a complete subspace of the metric space $\mathbb{R}^2$ with the usual distance d. Further we note that, X is η-chainable for any $\eta > 1/4$.

Let $\psi : [0, \infty) \to [0, \infty)$ be defined by $\psi(u) = u^2/2$. Clearly, ψ is a continuous, nondecreasing function such that $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$.

Let $T : X \to X$ be defined by

$$T(x(u), y(u)) = \begin{cases} \left(\frac{1}{2} - \left(u - \frac{1}{2}u^2\right),\ 0\right) & if\ (x, y) \in A \\ \left(\frac{1}{2} - \left\{\left(u + \frac{3}{4}\right) - \frac{1}{2}\left(u + \frac{3}{4}\right)^2\right\},\ 0\right) & if\ (x, y) \in B. \end{cases}$$

First, we show that T is not a uniform local (Banach) contraction. If possible, let T be a uniform local contraction. Then there exist ε, λ (with $\varepsilon > 0, 0 \leq \lambda < 1$) such that, for all $U, V \in X$,

$$\text{if} \quad d(U, V) < \varepsilon, \qquad \text{then} \quad d(T(U), T(V)) < \lambda\, d(U, V). \tag{11}$$

Let $t = \min\{\varepsilon,\ (1 - \lambda)/2\}$. Now if we consider the points $U(1/2, 0)$ and $V(1/2 - t, 0)$ then $d(U, V) = t$ and

$$d(TU, TV) = d\left(\left(\frac{1}{2}, 0\right), \left(\frac{1}{2} - \left\{t - \frac{1}{2}t^2\right\}, 0\right)\right) = t - \frac{1}{2}t^2.$$

Since $d(U,V)=t<\varepsilon$, relation-(11) is satisfied. Thus, we obtain

$$\begin{aligned} & t-\frac{1}{2}t^2 <\lambda t, \\ \text{or}\quad & t-\frac{1}{2}t^2 <(1-2t)t \quad [\text{because } t\le(1-\lambda)/2 \implies \lambda\le 1-2t], \\ \text{or}\quad & -\frac{1}{2}t^2 <-2t^2. \end{aligned} \tag{12}$$

Relation (12) is absurd. So we arrive at a contradiction, and thus T is not a uniform local contraction.

Now, we show that T is not a ψ-weak contraction. For this, consider a pair of points $U(1/2,0)$ and $V(0,1/2)$ of X corresponding to $t=0$ and $s=1/4$, respectively. Then

$$d(U,V)=\frac{\sqrt{2}}{2}, \qquad d(U,V)-\psi(d(U,V))=\frac{\sqrt{2}}{2}-\frac{1}{2}\times\frac{1}{2}=\frac{2\sqrt{2}-1}{4}.$$

Again,

$$\begin{aligned} T\left(\left(\frac{1}{2},0\right)\right) &= \left(\frac{1}{2}-\left(0-\frac{1}{2}\times 0^2\right),0\right)=\left(\frac{1}{2},0\right), \\ T\left(\left(0,\frac{1}{2}\right)\right) &= \left(0,\frac{1}{2}-\left(1-\frac{1}{2}\times 1^2\right)\right)=(0,0). \end{aligned}$$

Thus,

$$d(T(U),T(V))=\frac{1}{2}\nleq\frac{2\sqrt{2}-1}{4}=d(U,V)-\psi(d(U,V)).$$

Hence relation (1) is not satisfied for this pair of points, which shows that it is not a ψ-weak contraction.

We now show that T is a (η,ψ)-uniformly local weak contraction map, for some $\eta>0$.

Let us now consider the following cases.

Case I: Let us consider two points $P(x(t),y(t))\in A$ and $Q((x(s),y(s))\in B$, where $0\le t\le 1/2$ and $0\le s\le 1/4$. Then

$$d(P,Q)=\sqrt{\left(\frac{1}{2}-t\right)^2+\left(s+\frac{1}{4}\right)^2}.$$

Thus,

$$d(P,Q)-\psi\left(d(P,Q)\right)=\sqrt{\left(\frac{1}{2}-t\right)^2+\left(s+\frac{1}{4}\right)^2}-\frac{1}{2}\left(\left(\frac{1}{2}-t\right)^2+\left(s+\frac{1}{4}\right)^2\right).$$

Now,

$$\begin{aligned} d\left(T(P),T(Q)\right) &= d\left(\left(\frac{1}{2}-(t-\frac{1}{2}t^2),0\right),\left(\frac{1}{2}-\left\{\left(s+\frac{3}{4}\right)-\frac{1}{2}\left(s+\frac{3}{4}\right)^2\right\},0\right)\right) \\ &= \left\{\left(s+\frac{3}{4}\right)-\frac{1}{2}\left(s+\frac{3}{4}\right)^2\right\}-\left(t-\frac{1}{2}t^2\right). \end{aligned}$$

Consider the function $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, defined by

$$\begin{aligned} F(t,s) &= \left\{\sqrt{\left(\frac{1}{2}-t\right)^2+\left(s+\frac{1}{4}\right)^2}-\frac{1}{2}\left(\left(\frac{1}{2}-t\right)^2+\left(s+\frac{1}{4}\right)^2\right)\right\} \\ &\quad -\left\{\left\{\left(s+\frac{3}{4}\right)-\frac{1}{2}\left(s+\frac{3}{4}\right)^2\right\}-\left(t-\frac{1}{2}t^2\right)\right\}. \end{aligned}$$

Now $F(1/2, 0) > 0$ and F is continuous at $(1/2, 0)$. Thus, by the Neighborhood property of Continuous functions, there is a $\delta - neighborhood$ of the point $(1/2, 0)$ where the function assumes only positive values.

Thus, for $t = 1/2$ and $s = 0$, the corresponding pair of points $(0, 0) \in A$ and $(0, 1/4) \in B$ satisfies the relation (1) and there is some δ with $1/4 > \delta > 0$, such that, for all pair of points $R(x(t), y(t)) \in A$ and $S((x(s), y(s)) \in B)$, where $t \in (1/2 - \delta, 1/2]$ and $s \in [0,\ \delta)$, relation (1) remains satisfied.

Now let us consider $\alpha \geq 0$ be such that $\alpha \leq \delta$. The points corresponding to parametric values $t = 1/2 - \alpha$ and $s = \alpha$ are $P\,(\alpha, 0)$ and $Q(0, 1/4 + \alpha)$, respectively, and the distance between them is

$$d(P, Q) = d\left((\alpha, 0)\,, \left(0, \frac{1}{4} + \alpha\right)\right) = \sqrt{\alpha^2 + \left(\frac{1}{4} + \alpha\right)^2}.$$

$$\therefore d(P, Q) - \psi\,(d(P, Q)) = \sqrt{\alpha^2 + \left(\frac{1}{4} + \alpha\right)^2} - \frac{1}{2}\left\{\alpha^2 + \left(\frac{1}{4} + \alpha\right)^2\right\} = g(\alpha)(say).$$

$d(P, Q) - \frac{1}{2}(d(P, Q))^2 = g(\alpha)$ is a strictly increasing function of $d(P, Q)$ when $d(P, Q) < 1$. Thus, $g(\alpha)$ is increasing with respect to α, for $0 \leq \alpha < 1/4$. Hence the minimum value of $g(\alpha)$ in $[0, 1/4)$ is $7/32$, corresponding to $\alpha = 0$. Now,

$$d\,(T(P), T(Q)) = \left\{\left(\frac{3}{4} + \alpha\right) - \frac{1}{2}\left(\frac{3}{4} + \alpha\right)^2\right\} - \left\{\left(\frac{1}{2} - \alpha\right) - \frac{1}{2}\left(\frac{1}{2} - \alpha\right)^2\right\} = \frac{3}{32} + \frac{3}{4}\alpha = f(\alpha)$$

(say), which is clearly a continuous and increasing function of α. In particular, for $\alpha = 0.1$, we have $d\,(T(P), T(Q)) = 27/160 < 7/32$, the minimum value of the R. H. S of (1) in this case. Thus, $f(\alpha) < f(0.1) = 27/160 < 7/32 = g(0) < g(\alpha)$. Thus, for $\alpha \leq 0.1$ the inequality (1) is satisfied. Hence we can choose $\delta = 0.1$

Thus, in this case we see that the pair of points P, Q (where $P \in A, Q \in B$), whose distance is less than

$$\eta = \sqrt{0.1^2 + \left(\frac{1}{4} + 0.1\right)^2} \approx 0.3640,$$

satisfies relation (1).

Case II: $P(x(s_1), y(s_1)), Q((x(s_2), y(s_2)) \in B), 0 \leq s_1 \leq s_2 \leq 1/4$.

$$\begin{aligned} d(T(P), T(Q)) &= \left\{\left(s_2 + \frac{3}{4}\right) - \frac{1}{2}\left(s_2 + \frac{3}{4}\right)^2\right\} - \left\{\left(s_1 + \frac{3}{4}\right) - \frac{1}{2}\left(s_1 + \frac{3}{4}\right)^2\right\} \\ &= (s_2 - s_1) - \frac{1}{2}\left\{\left(s_2 + \frac{3}{4}\right)^2 - \left(s_1 + \frac{3}{4}\right)^2\right\} \\ &= (s_2 - s_1) - \frac{1}{2}\left\{\left(s_2{}^2 - s_1{}^2\right) + 2 \times \frac{3}{4}\,(s_2 - s_1)\right\} = \frac{1}{4}\,(s_2 - s_1) - \frac{1}{2}\left(s_2{}^2 - s_1{}^2\right). \end{aligned}$$

Now, $d(P, Q) = s_2 - s_1$. Therefore,

$$d(P, Q) - \psi(d(P, Q)) = d(P, Q) - \frac{1}{2}(d(P, Q))^2 = (s_2 - s_1) - \frac{1}{2}(s_2 - s_1)^2.$$

Since $0 \leq s_1 \leq s_2$, we note that $s_2{}^2 - s_1{}^2 = (s_2 + s_1)(s_2 - s_1) \geq (s_2 - s_1)(s_2 - s_1) = (s_2 - s_1)^2$. Thus, we have

$$\frac{1}{4}(s_2 - s_1) - \frac{1}{2}\left\{s_2{}^2 - s_1{}^2\right\} \leq (s_2 - s_1) - \frac{1}{2}(s_2 - s_1)^2.$$

Therefore, (1) is satisfied. In particular, if $d(P, Q) < \eta$, then (1) is also satisfied.

Case III: $P(x(t_1), y(t_1)), Q((x(t_2), y(t_2)) \in A), 0 \le t_1 \le t_2 \le \frac{1}{2}$. Then

$$d(T(P), T(Q)) = \left\{t_2 - \frac{1}{2}{t_2}^2\right\} - \left\{t_1 - \frac{1}{2}{t_1}^2\right\} = (t_2 - t_1) - \frac{1}{2}\left\{{t_2}^2 - {t_1}^2\right\}.$$

Also, $d(P, Q) = t_2 - t_1$. Therefore,

$$d(P, Q) - \psi(d(P, Q)) = d(P, Q) - \frac{1}{2}(d(P, Q))^2 = (t_2 - t_1) - \frac{1}{2}(t_2 - t_1)^2.$$

Since $0 \le t_1 \le t_2$, we note that

$${t_2}^2 - {t_1}^2 = (t_2 + t_1)(t_2 - t_1) \ge (t_2 - t_1)(t_2 - t_1) = (t_2 - t_1)^2.$$

Thus, we have

$$(t_2 - t_1) - \frac{1}{2}\left\{{t_2}^2 - {t_1}^2\right\} \le (t_2 - t_1) - \frac{1}{2}(t_2 - t_1)^2.$$

Thus, (1) is satisfied. In particular, if $d(P, Q) < \eta$, then (1) also holds.

Hence, from the above cases we can conclude that the function T is an (η, ψ)- uniformly locally weak contraction for $\eta = 0.36$. The space X is 0.36-chainable. Thus, all the conditions of Theorem 1 are satisfied. The point $(1/2, 0)$ is a fixed point of T.

ACKNOWLEDGMENTS

The first author acknowledges the Indian Institute of Engineering Science of Technology, Shibpur , for supporting him as JRF.

REFERENCES

1. S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," Fund. Math. **3**, 133–181 (1922).
2. Md. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory* (John Wiley & Sons, New York, 2001).
3. V. Pata, *Fixed Point Theorems and Applications* (Springer International Publishing, Switzerland, 2019).
4. K. C. Border, *Fixed point theorems with applications to economics and game theory* (Cambridge Univ. Press, Cambridge, 1989).
5. P. V. Subrahmanyam, *Elementary Fixed Point Theorems* (Springer International Publishing, Singapore, 2018).
6. D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," Proc. Amer. Math. Soc. **20** (2), 458–464 (1969).
7. J. D. Weston, "A characterization of metric completeness," Proc. Amer. Math. Soc. **64** (1), 186–188 (1977).
8. B. E. Rhoades, "A comparison of various definitions of contractive mappings," Trans. Amer. Math. Soc. **226**, 257–290 (1977).
9. T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," Proc. Amer. Math. Soc. **136**, 1861–1869 (2008).
10. D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric space," Fixed Point Theory and Appl. **94** (2012).
11. M. Edelstein, "An extension of Banach's contraction principle," Proc. Amer. Math. Soc. **12** (1), 7–10 (1961).
12. Y. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," New Results in Operator Theory and Its Applications, Operator Theory: Advances and Applications(ed. by I. Goldberg and Yu. Lyubich), Birkhäuser Verlag **98**, 7-22 (1997).
13. B. E. Rhoades, "Some theorems on weakly contractive maps," Nonlinear Analysis: Theory, Methods and Appl. **47** (4), 2683–2693 (200l).
14. P. Chakraborty, B. S. Choudhury, and M. De la Sen, "Relation-Theoretic Fixed Point Theorems for Generalized Weakly Contractive Mappings," Symmetry **12** (1), 29 (2020).
15. P. N. Dutta and B. S. Choudhury, "A generalization of contraction principle in metric spaces," Fixed Point Theory Appl. **406368** (2008).
16. B. S. Choudhury, P. Konar, B. E. Rhoades, and N. Metiya, "Fixed point theorems for generalized weakly contractive mappings," Nonlinear Analysis: Theory, Methods & Applications **74** (6), 2116–2126. (2011).

17. J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," Nonlinear Analysis: Theory, Methods & Applications **71** (7–8), 3403–3410. (2009).
18. L. A. Salazar and S. Reich, "A remark on weakly contractive mappings," J. Nonlinear Convex Anal. **16**, 767–773. (2015).
19. S. Cho, "Fixed point theorems for generalized weakly contractive mappings in metric spaces with applications," Fixed Point Theory and Appl. **3** (2018).
20. G. Beer, "Which connected metric spaces are compact?," Proc. Amer. Math. Soc. **83** (4) (1981).
21. K. Shrivastava and G. Agrawal, "Characterization of ε-chainable sets in metric spaces," Indian J. Pure Appl. Math **33**, 933–940. (2002).
22. S. Kundu , M. Aggarwal, and S. Hazra, "Finitely chainable and totally bounded metric spaces: Equivalent characterizations," Topology and its Applications **216**, 59–73 (2017).
23. T. Hu and W. A. Kirk, "Local contractions in metric spaces," Proc. Amer. Math. Soc. **68** (1) (1978).
24. T. Hu and H. Rosen, "Locally contractive and expansive mappings," Proc. Amer. Math. Soc. **86** (4) (1982).
25. A. Azam and Md. Arshad, "Fixed points of a sequence of locally contractive multivalued maps," Computers and Mathematics with Appl. **57**, 96—100 (2009).