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Inequalities for Eigenvalues of the Sub-Laplacian on Strictly Pseudoconvex CR Manifolds*

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Abstract—The sub-Laplacian plays a key role in CR geometry. In this paper, we investigate eigenvalues of the sub-Laplacian on bounded domains of strictly pseudoconvex CR manifolds, strictly pseudoconvex CR manifolds submersed in Riemannian manifolds. We establish some Levitin–Parnovski-type inequalities and Cheng–Huang–Wei-type inequalities for their eigenvalues. As their applications, we derive some results for the standard CR sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$, the Heisenberg group $\mathbb{H}^n$, a strictly pseudoconvex CR manifold submersed in a minimal submanifold in $\mathbb{R}^m$, domains of the standard sphere $\mathbb{S}^{2n}$ and the projective space $\mathbb{F}P^m$.

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1. INTRODUCTION

The study of eigenvalues of differential operators on manifolds is an important field in geometry and analysis. In the past decades, some progress has been made. Let Ω be a bounded domain in a Riemannian manifold M. The Dirichlet Laplacian problem is described by

$$\begin{cases} -\Delta u = \mu u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where Δ is the Laplacian. It has a real and discrete spectrum: $0 < \mu_1 < \mu_2 \le \cdots \le \mu_k \le \cdots$. When M is $\mathbb{R}^2$, Payne, Pólya and Weinberger [1] proved that the eigenvalues of problem (1) satisfy $\mu_2 \le 3\mu_1$ and $\mu_2 + \mu_3 \le 6\mu_1$. This led to the famous Payne–Pólya–Weinberger's conjecture on the lower order eigenvalues of problem (1) on a bounded domain $\Omega \subset \mathbb{R}^n$. Yau included this conjecture in his famous problem lists (cf. [2]). In 1992, Ashbaugh and Benguria [3] gave the proof of the first part of this conjecture. In 1993, they [4] proved that the second part of Payne–Pólya–Weinberger's conjecture holds under the assumption that Ω is invariant with respect to 90° rotations. In [4], they established a universal inequality

$$\sum_{i=1}^{n} \mu_{i+1} \le (n+4)\mu_1 \tag{2}$$

for $\Omega \subset \mathbb{R}^n$. In 2008, Sun, Cheng and Yang [5] obtained some universal inequalities for eigenvalues of problem (1) on bounded domains in the unit sphere, in complex projective space, and in compact complex submanifolds of complex projective spaces. Chen and Cheng [6] proved that (2) still holds when Ω is a bounded domain in an n-dimensional complete Riemannian manifold isometrically minimally immersed in a Euclidean space with mean curvature vector field $\mathbf{H}$. In fact, they obtained

$$\sum_{i=1}^{n} \widetilde{\mu}_{i+1} \le (n+4)\widetilde{\mu}_1, \qquad \text{where} \quad \widetilde{\mu}_i = \mu_i + \frac{n^2}{4}H_0^2, \quad H_0 = \max_{x\in\Omega}|\mathbf{H}(x)|. \tag{3}$$

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Since $\mathbb{R}^n$ can be seen as a totally geodesic minimal hypersurface in $\mathbb{R}^{n+1}$, we know that the result (2) of Ashbaugh and Benguria is included in (3). On the other hand, Levitin and Parnovski [7] generalized (2) to

$$\sum_{i=1}^{n} \mu_{i+j} \leq (n+4)\mu_j, \tag{4}$$

where j is any positive integer. A remarkable point of (4) is that it gives some estimates for the upper bounds of $\mu_{j+1} + \cdots + \mu_{j+n}$ in terms of μ_j. Moreover, it covers (2) when $j = 1$. This inequality has since then been referred to the Levitin–Parnovski inequality. On the other hand, for the clamped plate problem:

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

on a bounded domain Ω in an n-dimensional complete Riemannian manifold M, Cheng, Huang and Wei [8] derived

$$\sum_{i=1}^{n} (\Gamma_{i+1} - \Gamma_1)^{1/2} \leq \left[(2n+4)\Gamma_1^{1/2} + n^2 H_0^2\right]^{1/2} \left(4\Gamma_1^{1/2} + n^2 H_0^2\right)^{1/2}, \tag{5}$$

where H_0 is a nonnegative constant which only depends on M and Ω. Observe that (5) also gives a estimate for the lower eigenvalues in terms of the first eigenvalue.

In recent years, there is increasing interest in the research of the sub-Laplacian Δ_b on a strictly pseudoconvex CR manifold. A CR manifold is a differentiable manifold together with a subbundle of the complexified tangent bundle which is formally integrable and almost Lagrangian. The canonical examples of CR manifolds include the real $(2n+1)$-dimensional sphere as a submanifold of $\mathbb{C}^{n+1}$, and the Heisenberg group $\mathbb{H}^n$. Let (M, θ) be a strictly pseudoconvex CR manifold, where 1-form θ is called pseudo-Hermitian structure on M. The sub-Laplacian Δ_b is a second order differential operator on (M, θ), which is defined by

$$\Delta_b u = \mathrm{trace}_{G_\theta} \nabla du,$$

where ∇ the Tanaka–Webster connection on the tangent bundle TM and G_θ is the Levi form of θ. Similar to that played by the Laplacian in Riemannian geometry, the sub-Laplacian Δ_b plays a fundamental role in CR geometry. For example, in the famous CR Yamabe problem.

Some recent papers extended the results for the Laplacian to the sub-Laplacian on CR manifolds. For example, [9]–[12]. Noticing that the determination of the eigenvalues of the sub-Laplacian on the standard sphere is still an open problem, the research in this direction is significant. Let Ω be a bounded domain in a strictly pseudoconvex CR manifold (M, θ) of real dimension $2n+1$, V be a nonnegative continuous function, and $f : (M, \theta) \to \mathbb{R}^m$ be a semi-isometric C^2 map. Consider the following Dirichlet eigenvalue problem of the sub-Laplacian

$$\begin{cases} -\Delta_b u + Vu = \lambda u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{6}$$

It also has a real and discrete spectrum: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$. In 2013, Aribi and El Soufi [9] investigated eigenvalues of problem (6). For every $k \geq 1$ and $p \in \mathbb{R}$, they obtained

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^p \leq \frac{\max\{2, p\}}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{p-1} \left(\lambda_i + \frac{1}{4} D_\infty\right),$$

where $D_\infty = \sup\limits_{\Omega} \left(|H_b(f)|^2_{\mathbb{R}^m} - 4V\right)$. In this paper, we establish some Levitin–Parnovski-type inequalities and Cheng–Huang–Wei-type inequalities for lower order eigenvalues of problem (6) for the sub-Laplacian Δ_b on a strictly pseudoconvex CR manifold.

The paper is organized as follows: In Section 3, we first consider problem (6) for the sub-Laplacian on a bounded domain Ω in a strictly pseudoconvex CR manifold (M, θ) of real dimension $2n+1$.

Let $f : (M, \theta) \to \mathbb{R}^m$ be a semi-isometric C^2 map. In Theorem 1, we derive the following Levitin–Parnovski-type inequality

$$\sum_{i=1}^{2n} \lambda_{j+i} \leq (2n+4)\lambda_j + \sup_{\Omega}\big(|H_b(f)|^2_{\mathbb{R}^m} - 4V\big),$$

where $H_b(f)$ is the Levi tension of f and $j \in \mathbb{N}$. The Heisenberg group and real hypersurfaces of complex manifolds are two important models for CR manifolds. In Corollaries 1 and 2, by using Theorem 1, we obtain some Levitin–Parnovski-type inequalities for the standard CR sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$ and the Heisenberg group. In Section 4, we establish some Cheng–Huang–Wei-type inequalities (cf. [8] and [13]) for lower order eigenvalues of problem (6). In Theorem 2, for a bounded domain Ω in a strictly pseudoconvex CR manifold (M, θ) of real dimension $2n+1$, we prove that the following inequality

$$\sum_{i=1}^{2n} (\lambda_{i+1} - \lambda_1)^{1/2} \leq \sqrt{2n}\left[4\lambda_1 + \sup_{\Omega}\big(|H_b(f)|^2_{\mathbb{R}^m} - 4V\big)\right]^{1/2}$$

holds. We also obtain some results for the standard CR sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$ and the Heisenberg group in Corollaries 3 and 4. In Section 5, as applications of Theorems 1 and 2, we concern Riemannian submersions over submanifolds of the Euclidean space. Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n+1$ and let $f : (M, \theta) \to N$ be a Riemannian submersion over a Riemannian manifold of dimension $2n$ such that $df(\xi) = 0$. In Theorem 3, we derive a Levitin–Parnovski-type inequality and a Cheng–Huang–Wei-type inequality for the eigenvalues of problem (6) for the sub-Laplacian Δ_b on a bounded domain $\Omega \subset M$ by using Theorems 1 and 2. In Corollary 5, by using Theorem 3, we give some results for a minimal submanifold in $\mathbb{R}^m$, the standard sphere $\mathbb{S}^{2n}$ and the projective space $\mathbb{F}P^m$.

2. PRELIMINARIES

In this section, we give some definitions and basic facts about strictly pseudoconvex CR manifolds and sub-Laplacians. For more details, we refer to [9], [14]–[16].

Let M be an orientable CR manifold of CR dimension n. That is to say, M is an orientable manifold of real dimension $2n+1$ equipped with a pair $(H(M), J)$, where $H(M)$ is a subbundle of the tangent bundle TM of real rank $2n$ and J is an integrable complex structure on $H(M)$. $H(M)$ is called a *Levi distribution*.

Since M is orientable, there exists a nonzero 1-form $\theta \in \Gamma(T^*M)$ such that $Ker\theta = H(M)$. Such 1-form θ is called a *pseudo-Hermitian structure* on M. To each pseudo-Hermitian structure θ we associate its Levi form G_θ defined on $H(M)$ by $G_\theta(X, Y) = \theta([JX, Y])$ for any $X, Y \in \Gamma(H(M))$. The CR manifold is said to be strictly pseudoconvex if the *Levi form* G_θ of a compatible pseudo-Hermitian structure θ is positive definite for the pseudo-Hermitian structure θ. The *Reeb vector field* of θ is the unique tangent vector field determined by the pseudo-Hermitian structure θ, which satisfies $\theta(\xi) = 1$ and $\xi \lrcorner d\theta = 0$. It is also called characteristic direction of θ.

The *Tanaka–Webster connection* of a strictly pseudoconvex CR manifold (M, θ) is the unique affine connection ∇ on TM satisfying:

(1) $\nabla\theta = 0$, $\nabla d\theta = 0$, and $\nabla J = 0$.

(2) The torsion T_∇ of ∇ is such that, for all $X, Y \in H(M)$,

$$T_\nabla(X, Y) = -\theta([X, Y])\xi \qquad \text{and} \qquad T_\nabla(\xi, JX) = -JT_\nabla(\xi, X) \in H(M).$$

Take a local G_θ-orthonormal frame $\{X_1, \cdots, X_{2n}\}$ of Levi distribution $H(M)$. Then one has

$$\Delta_b u = \sum_{i=1}^{2n}\{X_iX_iu - (\nabla_{X_i}X_i)u\} = \sum_{i=1}^{2n}\langle \nabla_{X_i}\nabla^H u, X_i\rangle_{G_\theta},$$

where $\nabla^H u \in H(M)$ is the horizontal gradient of u defined by

$$Xu = G_\theta(X, \nabla^H u) \qquad \text{for any} \quad X \in \Gamma(H(M)).$$

The contact form θ induces the volume form $\vartheta_\theta = 1/(2^n n!)\theta \wedge (d\theta)^n$ on M. For every compactly supported smooth function u, integration by parts yields

$$\int_M u\Delta_b u\vartheta_\theta = -\int_M |\nabla^H u|^2_{G_\theta}\vartheta_\theta. \tag{7}$$

Let $f : (M, \theta) \to (N, \zeta)$ be a smooth map, where (N, ζ) is a Riemannian manifold. Let η_f be a vector valued 2-form on $H(M)$ given by

$$\eta_f(X, Y) = \nabla^f_X df(Y) - df(\nabla_X Y),$$

where ∇^f is the connection induced on the bundle $f^{-1}TN$ by the Levi-Cività connection of (N, ζ). $H_b(f) = \mathrm{trace}_{G_\theta}\eta_f$ is said to be the *Levi tension* of f. In fact, $H_b(f)$ is a vector field defined similarly to the tension vector field in the Riemannian case. Then one has

$$H_b(f) = \sum_{i=1}^{2n} \nabla^f_{X_i} df(X_i) - df(\nabla_{X_i} X_i).$$

The map $f : (M, \theta) \to (N, \zeta)$ is said to be *semi-isometric* if it preserve lengths in the horizontal directions as well as the orthogonality between $H(M)$ and Reeb vector field ξ of θ. That is to say, $\forall X \in H(M)$, we have

$$|df(X)|_\zeta = |X|_{G_\theta} \text{ and } \langle df(X), df(\xi)\rangle_\zeta = 0.$$

When (N, ζ) is the standard space $\mathbb{R}^m$, we have

$$H_b(f) = (\Delta_b f_1, \cdots, \Delta_b f_m). \tag{8}$$

3. LEVITIN–PARNOVSKI-TYPE INEQUALITIES FOR THE SUB-LAPLACIAN

In this section, we establish some Levitin–Parnovski-type inequalities for problem (6) for the sub-Laplacian Δ_b on some strictly pseudoconvex CR manifolds. We first state the following theorem:

Theorem 1. *Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n + 1$ and let $f : (M, \theta) \to \mathbb{R}^m$ be a semi-isometric C^2 map. Let V be a nonnegative continuous function on a bounded domain $\Omega \subset M$. Denote by λ_i the ith eigenvalue of problem* (6) *for the sub-Laplacian Δ_b on Ω. Then we have*

$$\sum_{i=1}^{2n} \lambda_{j+i} \le (2n + 4)\lambda_j + \sup_\Omega\big(|H_b(f)|^2_{\mathbb{R}^m} - 4V\big), \tag{9}$$

where $H_b(f)$ is the Levi tension of f and $j \in \mathbb{N}$.

In order to prove Theorem 1, we need the following abstract formula established by Levitin and Parnovski (see Theorem 2.2 of [7]).

Lemma 1. *Let $\mathcal{H}$ be a complex Hilbert space with a given inner product $\langle , \rangle$. Let $E : \mathcal{D} \subset \mathcal{H} \longrightarrow \mathcal{H}$ be a self-adjoint operator defined on a dense domain $\mathcal{D}$ which is semibounded below and has a discrete spectrum $\mu_1 \le \mu_2 \le \mu_3 \le \cdots$. Let $\{G_\alpha : E(\mathcal{D}) \longrightarrow \mathcal{H}\}_{\alpha=1}^N$ be a collection of symmetric operators which leave $\mathcal{D}$ invariant. Denote by u_i the normalized eigenvectors of E and u_i corresponding to the ith eigenvalue μ_i. Moreover, this family of eigenvectors is further assumed to be an orthonormal basis for $\mathcal{H}$. For any positive integer j, we have*

$$\sum_{k=1}^{\infty} \frac{\|\langle [E, G_\alpha]u_j, u_k\rangle\|^2}{\mu_k - \mu_j} = -\frac{1}{2}\langle [[E, G_\alpha], G_\alpha]u_j, u_j\rangle, \tag{10}$$

where $[E, G_\alpha] := EG_\alpha - G_\alpha E$ is the commutator of E and G_α. Here we have $\langle [E, G_\alpha]u_j, u_k\rangle = 0$ if $\mu_k = \mu_j$ for $k \ne j$.

The proof of Theorem 1 is based on the observation that estimates in the proof of Lemma 1 can be sharpened. Using Lemma 1, the properties of the sub-Laplacian and a strictly pseudoconvex CR manifold, we can give the proof of Theorem 1.

Proof of Thereom 1 Let $f:(M,\theta)\to\mathbb{R}^m$ be a semi-isometric map and let $f_1,\cdots,f_m$ be its Euclidean components. For each $\alpha=1,\cdots,m$, we still use f_α to denote the multiplication operator naturally associated with f_α. Let u_i be the orthonormal eigenfunction corresponding to the ith eigenvalue λ_i of problem (6). We know that u_i satisfies

$$\int_\Omega u_i u_j \vartheta_\theta = \delta_{ij}. \tag{11}$$

For each j fixed, we consider a $m\times m$ matrix $Q=(q_{\alpha\beta})_{m\times m}$, where $q_{\alpha\beta}=\langle[-\Delta_b+V,f_\alpha]u_j,u_{j+\beta}\rangle$. According to the QR-factorization theorem, we know that there exists an orthogonal $m\times m$ matrix $P=(p_{\alpha\beta})_{m\times m}$ such that $B=PQ=(b_{\alpha\beta})_{m\times m}$ is an upper triangular matrix. That is to say, for $1\le\beta<\alpha\le m$, we have

$$b_{\alpha\beta}=\sum_{\gamma=1}^m p_{\alpha\gamma}q_{\gamma\beta}=\langle[-\Delta_b+V,\sum_{\gamma=1}^m p_{\alpha\gamma}f_\gamma]u_j,u_{j+\beta}\rangle=0. \tag{12}$$

Define the functions φ_α by $\varphi_\alpha=\sum_{\gamma=1}^m p_{\alpha\gamma}f_\gamma$. Therefore, we can choose the functions $\varphi_1,\cdots,\varphi_m$ as the standard coordinates functions of $\mathbb{R}^m$ such that

$$\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_{j+\beta}\rangle=0,\ \text{ for } 1\le\beta<\alpha\le m. \tag{13}$$

Rewriting the summation index k, and using (13), we find that

$$\sum_{k=j+1}^{j+\alpha-1}\frac{\|\langle-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2}{\lambda_k-\lambda_j}=\sum_{\beta=1}^{\alpha-1}\frac{\|\langle-\Delta_b+V,\varphi_\alpha]u_j,u_{j+\beta}\rangle\|_{L^2}^2}{\lambda_{j+\beta}-\lambda_j}=0. \tag{14}$$

Taking $E=-\Delta_b+V$ and $G_\alpha=\varphi_\alpha$ in (10), we have

$$\sum_{k=1}^\infty\frac{\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2}{\lambda_k-\lambda_j}=-\frac{1}{2}\langle[[-\Delta_b+V,\varphi_\alpha],\varphi_\alpha]u_j,u_j\rangle. \tag{15}$$

Utilizing (14), we can deduce an inequality. In fact, rewriting the summation index, one can obtain

$$\begin{aligned}\sum_{k=1}^\infty\frac{\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2}{\lambda_k-\lambda_j}&=\sum_{k=1}^{j-1}\frac{\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2}{\lambda_k-\lambda_j}\\&\quad+\sum_{k=j+1}^{j+\alpha-1}\frac{\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2}{\lambda_k-\lambda_j}\\&\quad+\sum_{k=j+\alpha}^{\infty}\frac{\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2}{\lambda_k-\lambda_j}.\end{aligned} \tag{16}$$

Moreover, noticing that the spectrum of problem (6) is non-decreasing, one can find that

$$\sum_{k=1}^{j-1}\frac{|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle|^2}{\lambda_k-\lambda_j}\le 0. \tag{17}$$

Combining (14), (16) and (17), we derive

$$\begin{aligned}\sum_{k=1}^\infty\frac{\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2}{\lambda_k-\lambda_j}&\le\sum_{k=j+\alpha}^{\infty}\frac{\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2}{\lambda_k-\lambda_j}\\&\le\frac{1}{\lambda_{j+\alpha}-\lambda_j}\sum_{k=1}^\infty\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|_{L^2}^2.\end{aligned} \tag{18}$$

Furthermore, Parseval's identity implies

$$\sum_{k=1}^{\infty}\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|^2_{L^2}=\|[-\Delta_b+V,\varphi_\alpha]u_j\|^2_{L^2}. \tag{19}$$

Combining (18) and (19), we obtain

$$\sum_{k=1}^{\infty}\frac{\|\langle[-\Delta_b+V,\varphi_\alpha]u_j,u_k\rangle\|^2_{L^2}}{\lambda_k-\lambda_j}\le\frac{1}{\lambda_{j+\alpha}-\lambda_j}\|[-\Delta_b+V,\varphi_\alpha]u_j\|^2_{L^2}. \tag{20}$$

Substituting (20) into (15), and taking sum on α from 1 to m, we derive

$$-\frac{1}{2}\sum_{\alpha=1}^{m}(\lambda_{j+\alpha}-\lambda_j)\langle[[-\Delta_b+V,\varphi_\alpha],\varphi_\alpha]u_j,u_j\rangle\le\sum_{\alpha=1}^{m}\|[-\Delta_b+V,\varphi_\alpha]u_j\|^2_{L^2}. \tag{21}$$

Because

$$\Delta_b(\varphi_\alpha u_j)=u_j\Delta_b\varphi_\alpha+\varphi_\alpha\Delta_b u_j+2\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle_{G_\theta},$$

we have

$$[-\Delta_b+V,\varphi_\alpha]u_j=-\Delta_b\varphi_\alpha u_j-2\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle_{G_\theta}. \tag{22}$$

From (22), we derive

$$\langle[[-\Delta_b+V,\varphi_\alpha],\varphi_\alpha]u_j,u_j\rangle=-2\int_\Omega u_j^2|\nabla^H\varphi_\alpha|^2_{G_\theta}\vartheta_\theta. \tag{23}$$

On the other hand, we have

$$\big[-\Delta_b+V,\varphi_\alpha\big]u_j=-\,u_j\Delta_b\varphi_\alpha-2\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle_{G_\theta}.$$

From this, we derive

$$\|[-\Delta_b+V,\varphi_\alpha]u_j\|^2_{L^2}=\int_\Omega\left[u_j^2(\Delta_b\varphi_\alpha)^2+4\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle^2_{G_\theta}+4u_j\Delta_b\varphi_\alpha\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle_{G_\theta}\right]\vartheta_\theta. \tag{24}$$

Therefore, substituting (23) and (24) into (21), we derive

$$\begin{aligned}&\sum_{\alpha=1}^{m}(\lambda_{j+\alpha}-\lambda_j)\int_\Omega u_j^2|\nabla^H\varphi_\alpha|^2_{G_\theta}\vartheta_\theta\\&\qquad\le\sum_{\alpha=1}^{m}\int_\Omega\left[u_j^2(\Delta_b\varphi_\alpha)^2+4\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle^2_{G_\theta}+4u_j\Delta_b\varphi_\alpha\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle_{G_\theta}\right]\vartheta_\theta.\end{aligned} \tag{25}$$

Now we calculate the terms in (25) by using the geometric properties of a strictly pseudoconvex CR manifold. According to the isometry property of f with respect to horizontal directions and the orthogonal property of the matrix Q, one can obtain

$$\begin{aligned}\sum_{\alpha=1}^{m}\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle^2_{G_\theta}&=\sum_{\alpha=1}^{m}\langle\nabla^H f_\alpha,\nabla^H u_j\rangle^2_{G_\theta}=\sum_{\alpha=1}^{m}\langle\nabla f_\alpha,\nabla^H u_j\rangle^2_{G_\theta}\\&=|df(\nabla^H u_j)|^2_{\mathbb{R}^m}=|\nabla^H u_j|^2_{G_\theta}.\end{aligned} \tag{26}$$

This yields

$$\begin{aligned}\sum_{\alpha=1}^{m}\int_\Omega\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle^2_{G_\theta}\vartheta_\theta&=\int_\Omega|\nabla^H u_j|^2_{G_\theta}\vartheta_\theta=\int_\Omega u_j(-\Delta_b+V)u_j\vartheta_\theta-\int_\Omega Vu_j^2\vartheta_\theta\\&=\lambda_j-\int_\Omega Vu_j^2\vartheta_\theta.\end{aligned} \tag{27}$$

Noticing that Levi tension of f satisfies (8), we have

$$\sum_{\alpha=1}^{m}\int_{\Omega}u_j^2(\Delta_b\varphi_\alpha)^2\vartheta_\theta=\sum_{\alpha=1}^{m}\int_{\Omega}u_j^2(\Delta_b f_\alpha)^2\vartheta_\theta=\int_{\Omega}|H_b(f)|^2_{\mathbb{R}^m}u_j^2\vartheta_\theta. \tag{28}$$

Denote by $\{E_\alpha\}$ the standard basis of $\mathbb{R}^m$. Using Lemma 2.1 in [9], we obtain

$$\begin{aligned}\sum_{\alpha=1}^{m}\Delta_b\varphi_\alpha\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle_{G_\theta}&=\sum_{\alpha=1}^{m}\Delta_b f_\alpha\langle\nabla^H f_\alpha,\nabla^H u_j\rangle_{G_\theta}\\&=\Big\langle\sum_{\alpha=1}^{m}\Delta_b f_\alpha E_\alpha,\sum_{\alpha=1}^{m}\langle\nabla f_\alpha,\nabla^H u_j\rangle_{G_\theta}E_\alpha\Big\rangle_{\mathbb{R}^m}\\&=\langle H_b(f),df(\nabla^H u_j)\rangle_{\mathbb{R}^m}=0.\end{aligned} \tag{29}$$

Therefore, using (27)–(29), we can write

$$\begin{aligned}&\sum_{\alpha=1}^{m}\int_{\Omega}\Big[u_j^2(\Delta_b\varphi_\alpha)^2+4\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle^2_{G_\theta}+4u_j\Delta_b\varphi_\alpha\langle\nabla^H\varphi_\alpha,\nabla^H u_j\rangle_{G_\theta}\Big]\vartheta_\theta\\&\qquad=4\lambda_j+\int_{\Omega}\big(|H_b(f)|^2_{\mathbb{R}^m}-4V\big)u_j^2\vartheta_\theta.\end{aligned} \tag{30}$$

Furthermore, since Q is an orthogonal matrix and f preserves the Levi form with respect to a G_θ-orthonormal frame $\{e_i\}$ of $H_p(M)$, one has

$$\begin{aligned}\sum_{\alpha=1}^{m}|\nabla^H\varphi_\alpha|^2_{G_\theta}&=\sum_{\alpha=1}^{m}|\nabla^H f_\alpha|^2_{G_\theta}=\sum_{\alpha=1}^{m}\sum_{i=1}^{2n}\langle\nabla^H f_\alpha,e_i\rangle^2_{G_\theta}\\&=\sum_{i=1}^{2n}|df(e_i)|^2_{\mathbb{R}^m}=\sum_{i=1}^{2n}|e_i|^2_{G_\theta}=2n.\end{aligned} \tag{31}$$

The inequality $|\nabla^H f_\alpha|^2_{G_\theta}\le 1$ implies that

$$|\nabla^H\varphi_\alpha|^2_{G_\theta}\le 1,\quad \text{for } \alpha=1,\cdots,m. \tag{32}$$

According to (31) and (32), we deduce

$$\begin{aligned}\sum_{\alpha=1}^{m}(\lambda_{j+\alpha}-\lambda_j)|\nabla^H\varphi_\alpha|^2_{G_\theta}&\ge\sum_{i=1}^{2n}(\lambda_{j+i}-\lambda_j)|\nabla^H\varphi_i|^2_{G_\theta}+(\lambda_{j+2n}-\lambda_j)\sum_{\beta=2n+1}^{m}|\nabla^H\varphi_\beta|^2_{G_\theta}\\&=\sum_{i=1}^{2n}(\lambda_{j+i}-\lambda_j)|\nabla^H\varphi_i|^2_{G_\theta}+(\lambda_{j+2n}-\lambda_j)\sum_{i=1}^{2n}(1-|\nabla^H\varphi_i|^2_{G_\theta})\\&\ge\sum_{i=1}^{2n}(\lambda_{j+i}-\lambda_j)|\nabla^H\varphi_i|^2_{G_\theta}+\sum_{i=1}^{2n}(\lambda_{j+i}-\lambda_j)(1-|\nabla^H\varphi_i|^2_{G_\theta})\\&=\sum_{i=1}^{2n}(\lambda_{j+i}-\lambda_j).\end{aligned} \tag{33}$$

Substituting (30) and (33) into (25), we derive

$$\sum_{i=1}^{2n}\lambda_{j+i}\le(2n+4)\lambda_j+\int_{\Omega}\big(|H_b(f)|^2_{\mathbb{R}^m}-4V\big)u_j^2\vartheta_\theta. \tag{34}$$

Taking the supremum of $|H_b(f)|^2_{\mathbb{R}^m}-4V$ on Ω in (34), we obtain (9). This concludes the proof of Theorem 1.

Using Theorem 1, we can obtain some results for two important models of CR manifolds: real hypersurfaces of complex manifolds and the Heisenberg group. Denote by $\mathbb{S}^{2n+1}$ the standard CR sphere in $\mathbb{C}^{n+1}$. As well known, the standard embedding $j:\mathbb{S}^{2n+1}\to\mathbb{C}^{n+1}$ satisfies $|H_b(j)|^2_{\mathbb{C}^{n+1}}=4n^2$. Hence we obtain the following corollary by using Theorem 1.

Corollary 1. *Let Ω ba a domain in the standard CR sphere $\mathbb{S}^{2n+1}\subset\mathbb{C}^{n+1}$. Let V be a nonnegative continuous function on a bounded domain $\Omega\subset\mathbb{S}^{2n+1}$. Denote by λ_i the ith eigenvalue of problem* (6) *for the sub-Laplacian Δ_b on Ω. Then we have*

$$\sum_{i=1}^{2n}\lambda_{j+i}\le(2n+4)\lambda_j+4n^2-4V_0,$$

where $j\in\mathbb{N}$ and $V_0=\inf\limits_{\Omega}V$.

Denote by $\mathbb{H}^n\cong\mathbb{C}^n\times\mathbb{R}$ the Heisenberg group endowed with its standard CR structure. The corresponding sub-Laplacian on $\mathbb{H}^n$ is

$$\Delta_{\mathbb{H}^n}=\frac{1}{4}\sum_{j\le n}(X_j^2+Y_j^2).$$

Noticing that the standard projection $\mathbb{H}^n$ is semi-isometric with zero Levi tension, we obtain the following corollary.

Corollary 2. *Let Ω ba a domain in the Heisenberg group $\mathbb{H}^n$. Let V be a nonnegative continuous function on a bounded domain $\Omega\subset\mathbb{H}^n$. Denote by λ_i the ith eigenvalue of problem* (6) *for the sub-Laplacian Δ_b on Ω. Then we have*

$$\sum_{i=1}^{2n}\lambda_{j+i}\le(2n+4)\lambda_j-4V_0,$$

where $j\in\mathbb{N}$ and $V_0=\inf\limits_{\Omega}V$.

4. CHENG–HUANG–WEI-TYPE INEQUALITIES FOR THE SUB-LAPLACIAN

In this section, we establish some Cheng–Huang–Wei-type inequalities for lower order eigenvalues of problem (6) for the sub-Laplacian Δ_b.

Theorem 2. *Under the same assumptions as Theorem* 1*,*

$$\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}\le\sqrt{2n}\Big[4\lambda_1+\sup_{\Omega}\big(|H_b(f)|^2_{\mathbb{R}^m}-4V\big)\Big]^{1/2},\tag{35}$$

where $H_b(f)$ is the Levi tension of f.

In order to prove Theorem 2, we need the following abstract formula established by Sun and Zeng [17].

Lemma 2. *Let $\mathcal{H}$ be a complex Hilbert space with a given inner product $\langle,\rangle$ and corresponding norm $\|\cdot\|$. We let $A:\mathcal{D}\subset\mathcal{H}\longrightarrow\mathcal{H}$ be a self-adjoint operator defined on a dense domain $\mathcal{D}$ which is semibounded below and has a discrete spectrum $\mu_1\le\mu_2\le\cdots$. Let $\{T_\alpha:\mathcal{D}\longrightarrow\mathcal{H}\}_{\alpha=1}^m$ be a collection of skew-symmetric operators and $\{B_\alpha:A(\mathcal{D})\longrightarrow\mathcal{H}\}_{\alpha=1}^m$ be a collection of symmetric operators which leave $\mathcal{D}$ invariant. Denote by u_i the normalized eigenvectors corresponding to the ith eigenvalues μ_i of A. This family of eigenvectors are further assumed to be an orthonormal basis for $\mathcal{H}$. If the operators $\{B_\alpha\}_{\alpha=1}^m$ satisfy*

$$\langle B_\alpha u_1,u_{\beta+1}\rangle=0,\quad \textit{for}\quad 1\le\beta<\alpha\le m,\tag{36}$$

then

$$\sum_{\alpha=1}^m(\mu_{\alpha+1}-\mu_1)^{1/2}\langle[T_\alpha,B_\alpha]u_1,u_1\rangle\le2\bigg\{\sum_{\alpha=1}^m\langle[A,B_\alpha]u_1,B_\alpha u_1\rangle\sum_{\alpha=1}^m\|T_\alpha u_1\|^2\bigg\}^{1/2}.\tag{37}$$

Proof of Thereom 2 Let $f:(M,\theta)\to\mathbb{R}^m$ be a semi-isometric map and let $f_1,\cdots,f_m$ be its Euclidean components. In order to make use of Lemma 2, we construct some functions satisfying (36) from f_α. Similar to the proof of Theorem 1, we can prove that there exists a series of functions h_α which satisfy

$$\langle h_\alpha u_1, u_{\beta+1}\rangle = 0, \text{ for } 1\le\beta<\alpha\le m.$$

In fact, we consider an $m\times m$ matrix $S=\Big(\int_\Omega f_\alpha u_1 u_{\beta+1}\vartheta_\theta\Big)_{m\times m}$.

According to the QR-factorization theorem, we know that there exists an orthogonal $m\times m$ matrix $T=\big(t_{\alpha\beta}\big)_{m\times m}$ such that $U=TS$ is an upper triangular matrix. Namely we have

$$\sum_{\gamma=1}^m t_{\alpha\gamma}\int_\Omega f_\gamma u_1 u_{\beta+1}\vartheta_\theta=0, \quad \text{for} \quad 1\le\beta<\alpha\le m.$$

Defining the functions ψ_α by $\psi_\alpha=\sum_{\gamma=1}^m t_{\alpha\gamma}f_\gamma$. Thus we infer

$$\langle\psi_\alpha u_1, u_{\beta+1}\rangle=\int_\Omega \psi_\alpha u_1 u_{\beta+1}\vartheta_\theta=0, \quad \text{for} \quad 1\le\beta<\alpha\le m. \tag{38}$$

In other words, the functions ψ_α satisfy (36). Hence, taking

$$A=-\Delta_b+V, \qquad B_\alpha=\psi_\alpha, \qquad T_\alpha=[\Delta_b-V,\psi_\alpha]$$

in (37), we have

$$\begin{aligned}&\sum_{\alpha=1}^m(\lambda_{\alpha+1}-\lambda_1)^{1/2}\langle\big[[\Delta_b-V,\psi_\alpha],\psi_\alpha\big]u_1,u_1\rangle\\ &\qquad\le 2\Big\{\sum_{\alpha=1}^m\langle\big[-\Delta_b+V,\psi_\alpha\big]u_1,\psi_\alpha u_1\rangle\sum_{\alpha=1}^m\|[\Delta_b-V,\psi_\alpha]u_1\|_{L^2}^2\Big\}^{1/2}.\end{aligned} \tag{39}$$

Now we calculate the terms of (39). Similar to (27)–(29), according to the isometry property of f with respect to horizontal directions and the orthogonal property of the matrix T, we obtain

$$\sum_{\alpha=1}^m\int_\Omega\langle\nabla^H\psi_\alpha,\nabla^H u_1\rangle_{G_\theta}^2\vartheta_\theta=\lambda_1-\int_\Omega Vu_1^2\vartheta_\theta, \tag{40}$$

$$\sum_{\alpha=1}^m\int_\Omega u_1^2(\Delta_b\psi_\alpha)^2\vartheta_\theta=\int_\Omega|H_b(f)|_{\mathbb{R}^m}^2u_1^2\vartheta_\theta, \tag{41}$$

$$\sum_{\alpha=1}^m\Delta_b\psi_\alpha\langle\nabla^H\psi_\alpha,\nabla^H u_j\rangle_{G_\theta}=\langle H_b(f),df(\nabla^H u_j)\rangle_{\mathbb{R}^m}=0. \tag{42}$$

Then, using (40)–(42), we have

$$\begin{aligned}&\sum_{\alpha=1}^m\|[\Delta_b-V,\psi_\alpha]u_1\|_{L^2}^2\\ &\quad=\sum_{\alpha=1}^m\int_\Omega\Big[u_1^2(\Delta_b\psi_\alpha)^2+4\langle\nabla^H\psi_\alpha,\nabla^H u_1\rangle_{G_\theta}^2+4u_1\Delta_b\psi_\alpha\langle\nabla^H\psi_\alpha,\nabla^H u_1\rangle_{G_\theta}\Big]\vartheta_\theta\\ &\quad=4\lambda_1+\int_\Omega\big(|H_b(f)|_{\mathbb{R}^m}^2-4V\big)u_1^2\vartheta_\theta.\end{aligned} \tag{43}$$

Moreover, just as (23), we can write

$$\langle[[\Delta_b-V,\psi_\alpha],\psi_\alpha]u_1,u_1\rangle=-2\int_\Omega u_1^2|\nabla^H\psi_\alpha|_{G_\theta}^2\vartheta_\theta. \tag{44}$$

Furthermore, since

$$\big[-\Delta_b+V,\psi_\alpha\big]u_1=-u_1\Delta_b\psi_\alpha-2\langle\nabla^H\psi_\alpha,\nabla^H u_1\rangle_{G_\theta},$$

we obtain

$$\langle\big[-\Delta_b+V,\psi_\alpha\big]u_1,\psi_\alpha u_1\rangle=-\int_\Omega\psi_\alpha u_1^2\Delta_b\psi_\alpha\vartheta_\theta-\frac{1}{2}\int_\Omega\langle\nabla^H\psi_\alpha^2,\nabla^H u_1^2\rangle_{G_\theta}\vartheta_\theta. \tag{45}$$

Then, using (45) and

$$\int_\Omega\langle\nabla^H\psi_\alpha^2,\nabla^H u_1^2\rangle_{G_\theta}\vartheta_\theta=-2\int_\Omega\psi_\alpha u_1^2\Delta_b\psi_\alpha\vartheta_\theta-2\int_\Omega u_1^2|\nabla^H\psi_\alpha|_{G_\theta}^2\vartheta_\theta,$$

we derive

$$\sum_{\alpha=1}^m\langle\big[-\Delta_b+V,\psi_\alpha\big]u_1,\psi_\alpha u_1\rangle=\sum_{\alpha=1}^m\int_\Omega u_1^2|\nabla^H\psi_\alpha|_{G_\theta}^2\vartheta_\theta=2n. \tag{46}$$

Substituting (43), (44) and (46) into (39), we obtain

$$\sum_{\alpha=1}^m(\lambda_{\alpha+1}-\lambda_1)^{1/2}\int_\Omega u_1^2|\nabla^H\psi_\alpha|_{G_\theta}^2\vartheta_\theta\le\sqrt{2n}\bigg[4\lambda_1+\int_\Omega\big(|H_b(f)|_{\mathbb{R}^m}^2-4V\big)u_1^2\vartheta_\theta\bigg]^{1/2}. \tag{47}$$

It follows from fact that f preserves the Levi form with respect to the G_θ-orthonormal frame $\{e_i\}$ of $H_p(M)$ that

$$\sum_{\alpha=1}^m|\nabla^H\psi_\alpha|_{G_\theta}^2=2n.$$

Then, similar to the proof of (33), we deduce

$$\begin{aligned}\sum_{\alpha=1}^m(\lambda_{\alpha+1}-\lambda_1)^{1/2}|\nabla^H\psi_\alpha|_{G_\theta}^2&\ge\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}|\nabla^H\psi_i|_{G_\theta}^2+(\lambda_{2n+1}-\lambda_1)^{1/2}\sum_{\beta=2n+1}^m|\nabla^H\psi_\beta|_{G_\theta}^2\\&\ge\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}|\nabla^H\psi_i|_{G_\theta}^2+\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}(1-|\nabla^H\psi_i|_{G_\theta}^2)\\&=\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}.\end{aligned} \tag{48}$$

Combining (47) with (48), we obtain (35). This finishes the proof of Theorem 2.

From Theorem 2, we can derive the following corollaries for problem (6) of the sub-Laplacian on a bounded domain Ω in the standard CR sphere $\mathbb{S}^{2n+1}\subset\mathbb{C}^{n+1}$, a bounded domain Ω in the Heisenberg group $\mathbb{H}^n$.

Corollary 3. *Under the same assumptions as in Corollary* 1*, the following result for problem* (6) *for the sub-Laplacian Δ_b on a bounded domain Ω in the standard CR sphere $\mathbb{S}^{2n+1}\subset\mathbb{C}^{n+1}$ holds*:

$$\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}\le 2\sqrt{2n}(\lambda_1+n^2-V_0)^{1/2}.$$

Corollary 4. *Under the same assumptions as in Corollary 2, the following result for problem* (6) *for the sub-Laplacian Δ_b on a bounded domain Ω in the Heisenberg group $\mathbb{H}^n$ hods*:

$$\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}\le 2\sqrt{2n}(\lambda_1-V_0)^{1/2}.$$

5. APPLICATIONS OF THEOREMS 1 AND 2: RIEMANNIAN SUBMERSIONS OVER SUBMANIFOLDS OF THE EUCLIDEAN SPACE

Let $f:(M,\theta)\to N$ be a Riemannian submersion over a Riemannian manifold N of dimension $2n$. The manifold N admits infinitely many isometric immersions into Euclidean spaces. As applications of Theorems 1 and 2, we can derive the following results for Riemannian submersions.

Theorem 3. *Let (M,θ) be a strictly pseudoconvex CR manifold of real dimension $2n+1$, and let $f:(M,\theta)\to N$ be a Riemannian submersion over a Riemannian manifold of dimension $2n$ such that $df(\xi)=0$. Denote by $\mathcal{I}(\mathbb{N},\mathbb{R}^m)$ the set of all C^2-isometric immersions from N to the m-dimensional Euclidean space $\mathbb{R}^m$, where $m\geq 2n$. Set*

$$H^{euc}(N)=\inf_{\phi\in\cup_{m\in\mathbb{N}}\mathcal{I}(\mathbb{N},\mathbb{R}^m)}\|H(\phi)\|_\infty^2,$$

where $H(\phi)$ stands for the mean curvature vector field of ϕ. Denote by λ_i the ith eigenvalue of problem (6) *for the sub-Laplacian Δ_b on a bounded domain $\Omega\subset M$. Then*

$$\sum_{i=1}^{2n}\lambda_{j+i}\leq(2n+4)\lambda_j+\sup_\Omega\bigl(H^{euc}(N)-4V\bigr),\tag{49}$$

$$\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}\leq\sqrt{2n}\Bigl[4\lambda_1+\sup_\Omega\bigl(H^{euc}(N)-4V\bigr)\Bigr]^{1/2}.\tag{50}$$

Proof. According Nash's famous embedding theorem [18], we know that each complete Riemannian manifold can be isometrically immersed into a Euclidean space. Let $\phi:N\to\mathbb{R}^m$ be any isometric immersion. We can know that the map $\phi\circ f:(M,\theta)\to\mathbb{R}^m$ is semi-isometric. Denote by B_ϕ the second fundamental form of ϕ. Then according to Corollary 2.1 of [9], we know that $\beta_f=0$ and $H_b(f)=0$. Thus, we have

$$\beta_{\phi\circ f}(X,Y)=d\phi(\beta_f(X,Y))+B_\phi(df(X),df(Y))=B_\phi(df(X),df(Y)).$$

For any $x\in M$, the differential of f induces an isometry between $H_x(M)$ and $T_{f(x)}N$. Hence if $X_1,\cdots,X_{2n}$ is a local orthonormal frame of $H(M)$, then $df(X_1),\cdots,df(X_{2n})$ is also an orthonormal frame of TN. This yields

$$H_b(\phi\circ f)=H(\phi).$$

Therefore, applying Theorem 1 and Theorem 2 to $\phi\circ f$, and taking the infimum with respect to ϕ, we obtain (49) and (50). This finishes the proof of Theorem 3. ☐

By using Theorem 3, we can now state the following results for a minimal submanifold in $\mathbb{R}^m$, the standard sphere $\mathbb{S}^{2n}$ and the projective space $\mathbb{F}P^m$.

Corollary 5. *Let (M,θ) be a strictly pseudoconvex CR manifold of real dimension $2n+1$ and let $f:(M,\theta)\to N$ be a Riemannian submersion over a Riemannian manifold of dimension $2n$ such that $df(\xi)=0$. Denote by λ_i the ith eigenvalue of problem* (6) *for the sub-Laplacian Δ_b on a bounded domain $\Omega\subset M$. Set $V_0=\inf_\Omega V$. Then*:

(1) *If N is an open set of $\mathbb{R}^{2n}$, or a minimal submanifold in $\mathbb{R}^m$, then*

$$\sum_{i=1}^{2n}\lambda_{j+i}\leq(2n+4)\lambda_j-4V_0,\tag{51}$$

$$\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}\leq 2\sqrt{2n}\Bigl(\lambda_1-V_0\Bigr)^{1/2}.\tag{52}$$

(2) *If N is a domain D of the standard sphere $\mathbb{S}^{2n}$, then*

$$\sum_{i=1}^{2n}\lambda_{j+i}\le(2n+4)\lambda_j+4\big(n^2-V_0\big),\tag{53}$$

$$\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}\le 2\sqrt{2n}\Big[\lambda_1+(n^2-V_0)\Big]^{1/2}.\tag{54}$$

(3) *Denote by $\mathbb{F}P^m$ the projective space of dimension m over $\mathbb{F}$ (or real dimension $2n$, i.e., $m=2n/d_{\mathbb{F}}$). That is to say, if $\mathbb{F}=\mathbb{R}$, then $\mathbb{F}P^m$ is the m-dimensional real projective space; if $\mathbb{F}=\mathbb{C}$, then $\mathbb{F}P^m$ is the complex real projective space of real dimension $2m$; if $\mathbb{F}=\mathbb{Q}$, then $\mathbb{F}P^m$ is the quanternionic projective space of real dimension $4m$. If N is a domain D of the projective space $\mathbb{F}P^m$ of real dimension $2n$, then*

$$\sum_{i=1}^{2n}\lambda_{j+i}\le(2n+4)\lambda_j+4n(2n+d_{\mathbb{F}})-4V_0,\tag{55}$$

$$\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)^{1/2}\le 2\sqrt{2n}\Big[\lambda_1+n(2n+d_{\mathbb{F}})-V_0\Big]^{1/2},\tag{56}$$

where

$$d_{\mathbb{F}}=\dim_{\mathbb{R}}\mathbb{F}=\begin{cases}1 & \text{if } \mathbb{F}=\mathbb{R},\\ 2 & \text{if } \mathbb{F}=\mathbb{C},\\ 4 & \text{if } \mathbb{F}=\mathbb{Q}.\end{cases}$$

Proof. (1) If N is an open set of $\mathbb{R}^{2n}$, or a minimal submanifold in R^m, then

$$H^{euc}(N)=0.\tag{57}$$

Substituting (57) into (49) and (50), we obtain (51) and (52).

(2) If N is a domain of the standard sphere $\mathbb{S}^{2n}$, then

$$H^{euc}(N)=4n^2,\tag{58}$$

which follows from the fact that $|H(\tau)|^2_{\mathbb{R}^{2n+1}}=4n^2$, where $\tau:\mathbb{S}^{2n}\to\mathbb{R}^{2n+1}$ is the natural embedding of the standard sphere into the Euclidean space. Substituting (58) into (49) and (50), we obtain (53) and (54).

(3) As we know, the projective space $\mathbb{F}P^m$ carries a natural metric such that the Hopf fibration $\pi:\mathbb{S}^{d_{\mathbb{F}}(m+1)-1}\subset\mathbb{F}^{m+1}\to\mathbb{F}P^m$ is a Riemannian fibration. Let

$$\mathcal{H}_{m+1}(\mathbb{F})=\{A\in\mathcal{M}_{m+1}(\mathbb{F})|A^*:=^t\overline{A}=A\}$$

be the vector space of $(m+1)\times(m+1)$ Hermitian matrices with coefficients in $\mathbb{F}$, endowed with the inner product $\langle A,B\rangle=\frac{1}{2}\mathrm{trace}(AB)$. The map $\Psi:\mathbb{S}^{d_{\mathbb{F}(m+1)}-1}\subset\mathbb{F}^{m+1}\to\mathcal{H}_{m+1}(\mathbb{F})$ given by

$$\Psi(z)=\begin{pmatrix}|z_0|^2 & z_0\overline{z_1} & \cdots & z_0\overline{z_m}\\ z_1\overline{z_0} & |z_1|^2 & \cdots & z_1\overline{z_m}\\ \cdots & \cdots & \cdots & \cdots\\ z_m\overline{z_0} & z_m\overline{z_1} & \cdots & |z_m|^2\end{pmatrix}$$

induces through the Hopf fibration an isometric embedding ϕ from $\mathbb{F}P^m$ into $\mathcal{H}_{m+1}(\mathbb{F})$. Moreover, $\phi(\mathbb{F}P^m)$ is a minimal submanifold of the hypersurfaces $\mathbb{S}(\frac{I}{m+1},\sqrt{\frac{m}{2m+1}})$ of $\mathcal{H}_{m+1}(\mathbb{F})$ of radius $\sqrt{\frac{m}{2m+1}}$ centered at $\frac{I}{m+1}$. One deduces that the mean curvature $H(\phi)$ satisfies

$$|H(\phi)|^2=2m(m+1)d_{\mathbb{F}}^2.$$

Hence we know that

$$H^{euc}(\mathbb{F}P^m) \leq 2m(m+1)d_{\mathbb{F}}^2. \tag{59}$$

Therefore, we can obtain (55) and (56) by using (49), (50) and (59). This completes the proof of Corollary 5. □

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