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Uniqueness of the Solution of a Nonlocal Problem for an Elliptic-Hyperbolic Equation with Singular Coefficients

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Abstract—A boundary-value problem with nonlocal integral condition of Samarskii–Ionkin type is studied for a mixed-type equation with singular coefficients in a rectangular domain. A uniqueness criterion for the problem is established by the method of spectral analysis.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The theory of boundary-value problems for singular and degenerate equations is one of the most important sections of the modern theory of partial differential equations; this is due not only to its numerous applications in various fields of science and technology and the need to solve applied problems, but also to the intensive development of the theory of mixed-type equations. The first boundary-value problem for such elliptic equations with variable coefficients was first studied in [1]. A special place in this theory is occupied by the study of equations containing the Bessel differential operator. The study of this class of equations was begun in the works of Euler, Poisson, and Darboux and continued in the theory of the generalized axisymmetric potential [2]. The importance of this class of equations is due to their use in applications to various problems of gas dynamics and acoustics, jet theory in hydrodynamics, linearized Maxwell–Einstein equations, and elasticity-plasticity theory. An extensive study of boundary-value problems for equations of three main classes with Bessel operator was presented in [3]–[6].

In the domain

$$D = \{(x, y) \mid 0 < x < l,\ -\alpha < y < \beta\}$$

of the coordinate plane Oxy, where l, α, and β are given positive real numbers, we consider the elliptic-hyperbolic equation

$$Lu(x, y) \equiv u_{xx} + (\operatorname{sgn} y)u_{yy} + \frac{p}{x}u_x + \frac{q}{|y|}u_y = 0, \tag{1.1}$$

where p and q are given real numbers such that $|p| < 1$, $p \neq 0$, and $|q| < 1$, $q \neq 0$. Let us introduce the notation $D_+ = D \cap \{y > 0\}$ and $D_- = D \cap \{y < 0\}$.

Statement of the problem. It is required to find a function $u(x, y)$ that satisfies the following conditions:

$$u(x, y) \in C(\overline{D}) \cap C^2(D_+ \cup D_-), \tag{1.2}$$

$$Lu(x, y) \equiv 0, \qquad (x, y) \in D_+ \cup D_-, \tag{1.3}$$

$$\lim_{y \to 0+} y^q u_y(x, y) = \lim_{y \to 0-} (-y)^q u_y(x, y), \qquad 0 < x < l, \tag{1.4}$$

$$u(x, \beta) = \varphi(x), \qquad u(x, -\alpha) = \psi(x), \qquad 0 \le x \le l, \tag{1.5}$$

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$$\lim_{x\to 0+} x^p u_x(x,y) = 0, \qquad -\alpha \le y \le \beta, \tag{1.6}$$

$$\int_0^l x^p u(x,y)\,dx = A = \mathrm{const}, \qquad -\alpha \le y \le \beta, \tag{1.7}$$

where A is a given real number and $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions that satisfy the conditions $\lim_{x\to 0+} x^p\varphi'(x) = \lim_{x\to 0+} x^p\psi'(x) = 0$ and

$$\int_0^l x^p\varphi(x)\,dx = \int_0^l x^p\psi(x)\,dx = A. \tag{1.8}$$

The integral condition (1.7) previously appeared in the papers [7] and [8] dealing with the heat equation and also in [8], for example, in the study of the stability of a rarefied plasma, where the nonlocal condition means that the internal energy of the system is constant. At present, problems with integral conditions for equations of various classes have been studied in great detail (see, e.g., [9]–[12] and the bibliography given there). A nonlocal boundary-value problem for Eq. (1.1) with $q = 0$ was investigated in [13].

Let us represent Eq. (1.1) in the form

$$x^{-p}\frac{\partial}{\partial x}\left(x^p\frac{\partial u}{\partial x}\right) - u_{yy} - \frac{q}{y}u_y = 0, \qquad y \in (-\alpha, 0),$$
$$x^{-p}\frac{\partial}{\partial x}\left(x^p\frac{\partial u}{\partial x}\right) + u_{yy} + \frac{q}{y}u_y = 0, \qquad y \in (0, \beta).$$

Let us multiply both equalities by x^p and integrate for fixed values $y \in (-\alpha, 0)$ and $y \in (0,\beta)$ with respect to the variable x over the interval from ε to $l-\varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. As a result, we obtain

$$\int_\varepsilon^{l-\varepsilon}\frac{\partial}{\partial x}\left(x^p\frac{\partial u}{\partial x}\right)dx - \int_\varepsilon^{l-\varepsilon} x^p u_{yy}\,dx - \int_\varepsilon^{l-\varepsilon} x^p\frac{q}{y}u_y\,dx = 0, \qquad y \in (-\alpha, 0),$$
$$\int_\varepsilon^{l-\varepsilon}\frac{\partial}{\partial x}\left(x^p\frac{\partial u}{\partial x}\right)dx + \int_\varepsilon^{l-\varepsilon} x^p u_{yy}\,dx + \int_\varepsilon^{l-\varepsilon} x^p\frac{q}{y}u_y\,dx = 0, \qquad y \in (0, \beta),$$

or

$$\left(x^p\frac{\partial u}{\partial x}\right)\Bigg|_\varepsilon^{l-\varepsilon} - \frac{d^2}{dy^2}\int_\varepsilon^{l-\varepsilon} x^p u(x,y)\,dx - \frac{q}{y}\frac{d}{dy}\int_\varepsilon^{l-\varepsilon} x^p u(x,y)\,dx = 0, \qquad y \in (-\alpha, 0),$$
$$\left(x^p\frac{\partial u}{\partial x}\right)\Bigg|_\varepsilon^{l-\varepsilon} + \frac{d^2}{dy^2}\int_\varepsilon^{l-\varepsilon} x^p u(x,y)\,dx + \frac{q}{y}\frac{d}{dy}\int_\varepsilon^{l-\varepsilon} x^p u(x,y)\,dx = 0, \qquad y \in (0, \beta).$$

In the last equalities, we pass to the limit as $\varepsilon \to 0$. In view of conditions (1.6) and (1.7), we obtain the following local boundary condition:

$$u_x(l,y) = 0, \qquad -\alpha \le y \le \beta. \tag{1.9}$$

In what follows, we will consider problem (1.2)–(1.6), (1.9).

2. UNIQUENESS OF THE SOLUTION OF THE PROBLEM

Substituting the function $u(x,y) = X(x)Y(y)$ into Eq. (1.1) and conditions (1.6), (1.9), after separating the variables, we obtain the spectral problem

$$X''(x) + \frac{p}{x}X'(x) + \lambda^2 X(x) = 0, \qquad 0 < x < l, \tag{2.1}$$

$$\lim_{x\to 0+} x^p X'(x) = 0, \qquad X'(l) = 0, \tag{2.2}$$

where λ^2 is the split constant.

The general solution of Eq. (2.1) for $|p| < 1$, $p \neq 0$, is given by the formula

$$X(x) = C_1 x^{(1-p)/2} J_{(1-p)/2}(\lambda x) + C_2 x^{(1-p)/2} J_{(p-1)/2}(\lambda x). \tag{2.3}$$

Since $(1-p)/2$ is not an integer, it follows that $J_{(1-p)/2}(\lambda x)$ and $J_{(p-1)/2}(\lambda x)$ are linearly independent solutions of Eq. (2.1). Here C_1 and C_2 are arbitrary constants.

From this formula we calculate

$$X'(x) = C_1 \lambda x^{(1-p)/2} J_{-(p+1)/2}(\lambda x) - C_2 \lambda x^{(1-p)/2} J_{(p+1)/2}(\lambda x).$$

Since, as $x \to 0+$, the equality $x^p X'(x) = O(C_1 + C_2 x^{p+1})$ holds, it follows that, for the function (2.3) to satisfy the first condition in (2.2), we must put $C_1 = 0$. Let $C_2 = 1$.

Then the solution of Eq. (2.1) satisfying the first condition of (2.2) is defined by the equality

$$X(x) = x^{(1-p)/2} J_{(p-1)/2}(\lambda x).$$

We now require that this function satisfy the second boundary condition from (2.2). Calculating

$$\frac{dX(x)}{dx}\bigg|_{x=l} = (x^{(1-p)/2} J_{(p-1)/2}(\lambda x))'\big|_{x=l} = -\lambda l^{(1-p)/2} J_{(p+1)/2}(\lambda l) = 0,$$

we obtain

$$\begin{gathered} \lambda_0 = 0, \\ J_{(p+1)/2}(\mu_n) = 0, \qquad \mu_n = \lambda_n l. \end{gathered} \tag{2.4}$$

Thus, the system of eigenfunctions of problem (2.1), (2.2) has the form

$$\widetilde{X}_0(x) = 1, \qquad \lambda_0 = 0, \tag{2.5}$$

$$\widetilde{X}_n(x) = x^{(1-p)/2} J_{(p-1)/2}\left(\frac{\mu_n x}{l}\right) = x^{(1-p)/2} J_{(p-1)/2}(\lambda_n x), \qquad n \in \mathbb{N}, \tag{2.6}$$

where the eigenvalues λ_n are defined as the zeros of Eq. (2.4).

Note that the system of eigenfunctions (2.5) and (2.6) of problem (2.1), (2.2) is orthogonal in the space $L_2[0, l]$ with weight x^p; moreover, it constitutes a complete system in this space [14, c. 343].

In further calculations, we will use the orthonormal system of functions

$$X_n(x) = \frac{1}{\|\widetilde{X}_n(x)\|} \widetilde{X}_n(x), \qquad n = 0, 1, 2, \dots, \tag{2.7}$$

where

$$\|\widetilde{X}_n(x)\|^2 = \int_0^l x^p \widetilde{X}_n^2(x)\, dx. \tag{2.8}$$

Let $u(x, y)$ be a solution of problem (1.2)–(1.6), (1.9). Following [13], we consider the functions

$$u_n(y) = \int_0^l u(x, y) x^p X_n(x)\, dx, \qquad n = 0, 1, 2, \dots, \tag{2.9}$$

where $X_n(x)$ is defined by (2.7).

On the basis of (2.9), we introduce auxiliary functions of the form

$$u_{n,\varepsilon}(y) = \int_\varepsilon^{l-\varepsilon} u(x, y) x^p X_n(x)\, dx, \qquad n = 0, 1, 2, \dots, \tag{2.10}$$

where $\varepsilon > 0$ is a sufficiently small number. Let us differentiate equality (2.10) twice with respect to the variable y for $y \in (-\alpha, 0) \cup (0, \beta)$. Taking into account Eq. (1.1), we obtain the equalities

$$\begin{aligned}
u''_{n,\varepsilon}(y) &= \int_\varepsilon^{l-\varepsilon} u_{yy}(x,y)x^p X_n(x)\,dx \\
&= \int_\varepsilon^{l-\varepsilon}\Big(-\Big(u_{xx}+\frac{p}{x}\,u_x\Big)-\frac{q}{y}\,u_y\Big)x^pX_n(x)\,dx \\
&= -\int_\varepsilon^{l-\varepsilon}\frac{\partial}{\partial x}(x^pu_x)X_n(x)\,dx-\frac{q}{y}\,\frac{d}{dy}\int_\varepsilon^{l-\varepsilon}u(x,y)x^px_n(x)\,dx \\
&= -\Big(x^pu_xX_n(x)\big|_\varepsilon^{l-\varepsilon}-\int_\varepsilon^{l-\varepsilon}x^pu_xX'_n(x)\,dx\Big) \\
&\qquad -\frac{q}{y}\frac{d}{dy}\int_\varepsilon^{l-\varepsilon}u(x,y)x^px_n(x)\,dx, \qquad y>0, \qquad (2.11)
\end{aligned}$$

$$\begin{aligned}
u''_{n,\varepsilon}(y) &= \int_\varepsilon^{l-\varepsilon} u_{yy}(x,y)x^p X_n(x)\,dx \\
&= x^pu_xX_n(x)\big|_\varepsilon^{l-\varepsilon}-\int_\varepsilon^{l-\varepsilon}x^pu_xX'_n(x)\,dx \\
&\qquad -\frac{q}{y}\frac{d}{dy}\int_\varepsilon^{l-\varepsilon}u(x,y)x^px_n(x)\,dx, \qquad y<0. \qquad (2.12)
\end{aligned}$$

By virtue of Eq. (2.1), from (2.10) we obtain

$$\begin{aligned}
u_{n,\varepsilon}(y) &= -\frac{1}{\lambda_n^2}\int_\varepsilon^{l-\varepsilon}u(x,y)x^p\Big[X''_n(x)+\frac{p}{x}\,X'_n(x)\Big]\,dx \\
&= -\frac{1}{\lambda_n^2}\int_\varepsilon^{l-\varepsilon}u(x,y)\frac{d}{dx}(x^pX'_n(x))\,dx \\
&= -\frac{1}{\lambda_n^2}\Big[u(x,y)x^pX'_n(x)\big|_\varepsilon^{l-\varepsilon}-\int_\varepsilon^{l-\varepsilon}x^pu_xX'_n(x)\,dx\Big],
\end{aligned}$$

whence we have

$$\int_\varepsilon^{l-\varepsilon}x^pu_xX'_n(x)\,dx=\lambda_n^2u_{n,\varepsilon}(y)+u(x,y)x^pX'_n(x)\big|_\varepsilon^{l-\varepsilon}.$$

Substituting this expression into (2.11) and (2.12), we obtain

$$\begin{aligned}
u''_{n,\varepsilon}(y) &= -\Big(x^pu_xX_n(x)\big|_\varepsilon^{l-\varepsilon}-\lambda_n^2u_{n,\varepsilon}(y)-u(x,y)x^pX'_n(x)\big|_\varepsilon^{l-\varepsilon}\Big) \\
&\qquad -\frac{q}{y}\frac{d}{dy}\int_\varepsilon^{l-\varepsilon}u(x,y)x^px_n(x)\,dx, \qquad y>0, \\
u''_{n,\varepsilon}(y) &= x^pu_xX_n(x)\big|_\varepsilon^{l-\varepsilon}-\lambda_n^2u_{n,\varepsilon}(y)-u(x,y)x^pX'_n(x)\big|_\varepsilon^{l-\varepsilon} \\
&\qquad -\frac{q}{y}\frac{d}{dy}\int_\varepsilon^{l-\varepsilon}u(x,y)x^px_n(x)\,dx, \qquad y<0.
\end{aligned}$$

Since we have $-1<k<1$ and $k\neq 0$, it follows by virtue of (1.2) that, in the last two equalities, we can pass to the limit as $\varepsilon\to 0$, which yields, due to conditions (1.6), (1.9), and (2.2), the differential equation

$$u''_n(y)+\frac{q}{y}\,u'_n(y)-(\operatorname{sgn} y)\lambda_n^2u_n(y)=0, \qquad y\in(-\alpha,0)\cup(0,\beta), \qquad (2.13)$$

from which we find the functions (2.9). For $|q|<1$ and $q\neq 0$, the general solution of Eq. (2.13) is of the form

$$u_n(y)=\begin{cases}a_n y^{(1-q)/2}I_{(1-q)/2}(\lambda_n y)+b_n y^{(1-q)/2}I_{(q-1)/2}(\lambda_n y), & y>0,\\ c_n(-y)^{(1-q)/2}J_{(1-q)/2}(-\lambda_n y)+d_n(-y)^{(1-q)/2}J_{(q-1)/2}(-\lambda_n y), & y<0,\end{cases}\tag{2.14}$$

where $J_\nu(\xi)$ and $J_{-\nu}(\xi)$ are the Bessel functions of the first kind, $\nu=(1-k)/2$, $I_\nu(\xi)$ and $I_{-\nu}(\xi)$ are the modified Bessel functions, and a_n, b_n, c_n, and d_n are arbitrary constants.

Now, bearing in mind (1.2), we choose the constants a_n, b_n, c_n, and d_n in (2.14) so that the following matching conditions hold:

$$u_n(0+)=u_n(0-),\qquad u_n'(0+)=u_n'(0-).\tag{2.15}$$

Calculate

$$u_n'(y)=\begin{cases}a_n\lambda_n y^{(1-q)/2}I_{-(q+1)/2}(\lambda_n y)+b_n\lambda_n y^{(1-q)/2}I_{(q+1)/2}(\lambda_n y), & y>0,\\ c_n\lambda_n(-y)^{(1-q)/2}J_{-(q+1)/2}(-\lambda_n y)-d_n\lambda_n(-y)^{(1-q)/2}J_{(q+1)/2}(-\lambda_n y), & y<0.\end{cases}\tag{2.16}$$

Taking into account the formula $I_\nu(z)=e^{-\nu\pi i/2}J_\nu(iz)$[15] and using (2.14) and (2.16), we obtain

$$u_n(y)=\begin{cases}a_n e^{((q-1)\pi/4)i}y^{(1-q)/2}J_{(1-q)/2}(i\lambda_n y)\\ \qquad +b_n e^{((1-q)\pi/4)i}y^{(1-q)/2}J_{(q-1)/2}(i\lambda_n y), & y>0,\\ c_n(-y)^{(1-q)/2}J_{(1-q)/2}(-\lambda_n y)+d_n(-y)^{(1-q)/2}J_{(q-1)/2}(-\lambda_n y), & y<0,\end{cases}\tag{2.17}$$

$$u_n'(y)=\begin{cases}a_n e^{((q+1)\pi/4)i}\lambda_n y^{(1-q)/2}J_{-(q+1)/2}(i\lambda_n y)\\ \qquad +b_n e^{-((q+1)\pi/4)i}\lambda_n y^{(1-q)/2}J_{(q+1)/2}(i\lambda_n y), & y>0,\\ c_n\lambda_n(-y)^{(1-q)/2}J_{-(q+1)/2}(-\lambda_n y)-d_n\lambda_n(-y)^{(1-q)/2}J_{(q+1)/2}(-\lambda_n y), & y<0.\end{cases}$$

This implies that conditions (2.15) hold for

$$a_n=e^{-((q+1)\pi/4)i}c_n,\qquad b_n=e^{((q-1)\pi/4)i}d_n,\qquad n=0,1,2,\dots.$$

Substituting the expressions for a_n and b_n into (2.17), we obtain

$$u_n(y)=\begin{cases}c_n e^{-(\pi/2)i}y^{(1-q)/2}J_{(1-q)/2}(i\lambda_n y)+d_n y^{(1-q)/2}J_{(q-1)/2}(i\lambda_n y), & y>0,\\ c_n(y)^{(1-q)/2}J_{(1-q)/2}(-\lambda_n y)+d_n(-y)^{(1-q)/2}J_{(q-1)/2}(-\lambda_n y), & y<0,\end{cases}$$

or

$$u_n(y)=\begin{cases}-c_n i y^{(1-q)/2}J_{(1-q)/2}(i\lambda_n y)+d_n y^{(1-q)/2}J_{(q-1)/2}(i\lambda_n y), & y>0,\\ c_n(-y)^{(1-q)/2}J_{(1-q)/2}(-\lambda_n y)+d_n(-y)^{(1-q)/2}J_{(q-1)/2}(-\lambda_n y), & y<0.\end{cases}\tag{2.18}$$

Now we substitute (2.9) into the boundary conditions (1.5):

$$u_n(\beta)=\int_0^l\varphi(x)x^pX_n(x)\,dx=\varphi_n,\qquad u_n(-\alpha)=\int_0^l\psi(x)x^pX_n(x)\,dx=\psi_n.\tag{2.19}$$

From (2.18) and (2.19) we obtain the following system for finding the constants c_n and d_n:

$$\begin{cases}c_n iJ_{(1-q)/2}(i\lambda_n\beta)-d_nJ_{(q-1)/2}(i\lambda_n\beta)=-\varphi_n\beta^{(q-1)/2},\\ c_nJ_{(1-q)/2}(\lambda_n\alpha)+d_nJ_{(q-1)/2}(\lambda_n\alpha)=\psi_n\alpha^{(q-1)/2}.\end{cases}\tag{2.20}$$

If, for all $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the determinant of system (2.20) satisfies

$$\Delta_n(\alpha,\beta)=iJ_{(q-1)/2}(\lambda_n\alpha)J_{(1-q)/2}(i\lambda_n\beta)+J_{(1-q)/2}(\lambda_n\alpha)J_{(q-1)/2}(i\lambda_n\beta)\neq 0,\tag{2.21}$$

then the system has a unique solution, which is given by

$$c_n = \frac{-\varphi_n \beta^{(q-1)/2} J_{(q-1)/2}(\lambda_n \alpha) + \psi_n \alpha^{(q-1)/2} J_{(q-1)/2}(i\lambda_n \beta)}{\Delta(n)},$$
$$d_n = \frac{\varphi_n \beta^{(q-1)/2} J_{(1-q)/2}(\lambda_n \alpha) + \psi_n i \alpha^{(q-1)/2} J_{(1-q)/2}(i\lambda_n \beta)}{\Delta(n)}.$$

Substituting the obtained values of c_n and d_n into (2.18), we find the final form of the functions:

$$u_n(y) = \begin{cases} \dfrac{\varphi_n \sqrt{(\alpha y)^{1-q}}\, \Delta_n(\alpha, y) + \psi_n \sqrt{(\beta y)^{1-q}}\, A_n(y, \beta)}{\Delta_n(\alpha, \beta)\sqrt{(\alpha\beta)^{1-q}}}, & y > 0, \\ \dfrac{\varphi_n \sqrt{(-\alpha y)^{1-q}}\, B_n(\alpha, -y) + \psi_n \sqrt{(-\beta y)^{1-q}}\, \Delta_n(-y, \beta)}{\Delta_n(\alpha, \beta)\sqrt{(\alpha\beta)^{1-q}}}, & y < 0, \end{cases} \tag{2.22}$$

where

$$A_n(y, \beta) = (-J_{(q-1)/2}(i\lambda_n \beta) J_{(1-q)/2}(i\lambda_n y) + J_{(1-q)/2}(i\lambda_n \beta) J_{(q-1)/2}(i\lambda_n y))i,$$
$$B_n(\alpha, -y) = -J_{(q-1)/2}(\lambda_n \alpha) J_{(1-q)/2}(-\lambda_n y) + J_{(1-q)/2}(\lambda_n \alpha) J_{(q-1)/2}(-\lambda_n y).$$

Using the obtained partial solutions, we write the solution of problem (1.2)–(1.6), (1.9) formally as the Fourier–Bessel series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) X_n(x),$$

where the functions $u_n(t)$ are defined by (2.22) and $X_n(x)$, $n = 0, 1, 2, \dots$, are defined by (2.7), after which it is not difficult to show the equivalence of problems (1.2)–(1.6), (1.9), and (1.2)–(1.7), provided that conditions (1.8) are fulfilled.

If condition (2.21) holds, then problem (1.2)–(1.6), (1.9) has a unique solution. Indeed, let $\varphi(x) = \psi(x) \equiv 0$. Then it follows from (2.19) and (2.22) that $u_n(y) = 0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By virtue of (2.9), we have

$$\int_0^l u(x, y) x^p X_n(x)\, dx = 0.$$

Hence the completeness of the system $X_n(x)$ in the space $L_2[0, l]$ with weight x^p implies that $u(x, y) = 0$ almost everywhere on the interval $x \in [0, l]$ for any $y \in [-\alpha, \beta]$. Since, in view of (1.2), the function $u(x, y)$ belongs to $C(\overline{D})$, we have $u(x, y) \equiv 0$ in $\overline{D}$.

Thus, we have proved the following theorem.

Theorem (uniqueness criterion for solutions)**.** *If there exists a solution of problem* (1.2)–(1.6)*,* (1.9)*, then it is unique if and only if condition* (2.21) *holds for all* $n \in \mathbb{N}_0$.

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