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Underlying Manifolds of High-Dimensional Morse–Smale Diffeomorphisms with Two Saddle Periodic Points

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Abstract—The paper describes the topological structure of closed manifolds of dimension ≥ 4 that admit Morse–Smale diffeomorphisms whose nonwandering sets contain arbitrarily many sink periodic points, arbitrarily many source periodic points, and two saddle periodic points. The underlying manifolds of Morse–Smale diffeomorphisms with fewer saddle periodic points are also described.

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INTRODUCTION

Morse–Smale systems are structurally stable dynamical systems with zero topological entropy. These systems are remarkable in that there is a deep relationship between their dynamical properties and the topological structure of the underlying manifolds (see the relatively recent survey [1]) and in that they exist on any (smooth) closed manifolds [2], [3]. In this article, we restrict ourselves to discrete-time Morse–Smale systems. A discrete-time dynamical system is generated by a diffeomorphism of the underlying manifold, and such a system is the set of iterations of the generating diffeomorphism. In the case of Morse–Smale systems, this generating diffeomorphism is called a Morse–Smale diffeomorphism. (See the next section for precise definitions.)

Clearly, the structure of the underlying manifold does not change if the original diffeomorphism is replaced by some of its iterations. Therefore, without loss of generality, we can (and will) consider Morse–Smale diffeomorphisms whose periodic points are fixed points. By $MS(M^n; a, b, c)$ we denote the set of Morse–Smale diffeomorphisms of a closed smooth n-dimensional manifold M^n such that the nonwandering set $NW(f)$ of any diffeomorphism $f \in MS(M^n; a, b, c)$ consists of a sink fixed points (sinks), b source fixed points (sources), and c saddle fixed points (saddles). Smale [2] proved that $a \geq 1$ and $b \geq 1$. In the dimension $n = 1$, one always has $c = 0$ and $a = b$, and the underlying manifold is the circle $M^1 = S^1$. For the dimensions $n = 2, 3$, there are quite a few papers on the relationship between the topological structure of the underlying manifolds and the dynamics of Morse–Smale diffeomorphisms (see the surveys [1] and [4] and the book [5], which contains an extensive bibliography). Therefore, we consider underlying manifolds of dimension $n \geq 4$ in what follows. (See the remark at the end of the article.)

It is well known that if $c = 0$, then $a = b = 1$ and the underlying manifold M^n is the n-dimensional sphere $\mathbb{S}^n$ [6]. There exists a description of the topological structure of the underlying manifold M^n and the triples (a, b, c) for the case of $c = 1$; see Proposition 4. The present research deals with the topological structure of closed underlying manifolds M^n, $n \geq 4$, and triples (a, b, c) in the case of $c = 2$.

First, we consider the case in which both saddles of the diffeomorphism $f \in MS(M^n; a, b, 2)$ are saddles of codimension 1. (That is, one of the invariant manifolds of the saddles is one-dimensional.) In the following theorem, $N \otimes S^1$ denotes the total space of a locally trivial bundle with fiber N over S^1. The manifold $N \otimes S^1$ is obtained from $N \times [0; 1]$ by identifying $N \times \{0\}$ with $N \times \{1\}$ via

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some diffeomorphism $\tau\colon N \to N$. Throughout the paper, $\mathbb{D}^k$ is the k-dimensional closed disk and $\mathbb{S}^k$ is the k-dimensional sphere.

Theorem 1. *Let both saddles of a diffeomorphism $f \in MS(M^n; a, b, 2)$, $n \geq 4$, be saddles of codimension 1. Then one of the following two conditions is satisfied:*

(1) *$f \in MS(M^n; 1, 1, 2)$ and M^n is homeomorphic to the union of two copies of $\mathbb{D}^{n-1} \otimes S^1$;*

(2) $f \in MS(\mathbb{S}^n, 2, 2, 2) \cup MS(\mathbb{S}^n, 1, 3, 2) \cup MS(\mathbb{S}^n, 3, 1, 2)$.

The following result is proved for the case of only one saddle of codimension 1.

Theorem 2. *Assume that one saddle of a diffeomorphism $f \in MS(M^n; a, b, 2)$, $n \geq 4$, is a saddle of codimension 1 and the other saddle is not a saddle of codimension 1. If the closures of one-dimensional separatrices of the saddle of codimension 1 form a segment, then*

(1) *the dimension n of the underlying manifold can only take the values $n \in \{4, 8, 16\}$;*

(2) $f \in MS(M^n; 1, 2, 2) \cup MS(M^n; 2, 1, 2)$;

(3) *the manifold M^n is the disjoint union of an open ball $\mathbb{B}^n$ and an $n/2$-dimensional sphere $S^{n/2}$ topologically embedded in M^n, and if $n \in \{8, 16\}$, then the embedding of $S^{n/2}$ in M^n is locally flat.*

We prove the following result for the case in which there are no saddles of codimension 1.

Theorem 3. *Let a diffeomorphism $f \in MS(M^n; a, b, 2)$, $n \geq 4$, have no saddles of codimension 1. Then $f \in MS(M^n; 1, 1, 2)$, and the manifold M^n is simply connected.*

Remark 1. All sets of Morse–Smale diffeomorphisms in the above statements are nonempty:

$$MS(\mathbb{D}^{n-1} \otimes S^1 \cup \mathbb{D}^{n-1} \otimes S^1; 1, 1, 2) \neq \varnothing, \qquad MS(\mathbb{S}^n, 2, 2, 2) \neq \varnothing,$$
$$MS(\mathbb{S}^n, 1, 3, 2) \neq \varnothing, \qquad MS(\mathbb{S}^n, 3, 1, 2) \neq \varnothing, \qquad MS(M^n; 1, 2, 2) \neq \varnothing,$$
$$MS(M^n; 2, 1, 2) \neq \varnothing, \qquad MS(\mathbb{S}^n, 1, 1, 2) \neq \varnothing.$$

The article is organized as follows. Section 1 contains the main definitions and auxiliary propositions needed to prove the main results. All the main results are proved in Sec. 2.

1. AUXILIARY STATEMENTS

Let $f\colon M^n \to M^n$ be a diffeomorphism of a smooth closed n-dimensional manifold M^n ($n \geq 1$). Recall that a point $x \in M^n$ is said to be *nonwandering* if, for any neighborhood U of x and any positive integer N_0, there exists an $n_0 \in \mathbb{Z}$ such that

$$|n_0| \geq N_0 \qquad \text{and} \qquad f^{n_0}(U) \cap U \neq \varnothing.$$

The set of nonwandering points of f is denoted by $NW(f)$. Obviously, any periodic point is nonwandering. A periodic point $x_0 \in \mathrm{Per}(f)$, $f^q(x_0) = x_0$, is said to be *hyperbolic* if the derivative

$$Df^q(x_0)\colon T_{x_0}M^n \to T_{x_0}M^n,$$

viewed as a linear mapping of the tangent space to itself, has no eigenvalues with unit modulus. For a hyperbolic point x_0, there exists a stable manifold $W^s(x_0)$ and an unstable manifold $W^u(x_0)$, which are defined as the sets of points $y \in M^n$ such that

$$\varrho_M(f^{qk}x_0, f^{qk}y) \to 0 \qquad \text{as} \quad k \to +\infty \quad \text{and} \quad k \to -\infty, \quad \text{respectively,}$$

where ϱ_M is a metric on M^n. Note that the unstable manifold $W^u(x_0)$ is a stable manifold with respect to f^{-1}. It is well known that $W^s(x_0)$ and $W^u(x_0)$ are homeomorphic (in the intrinsic topology) to the

Euclidean spaces $\mathbb{R}^{\dim W^s(x_0)}$ and $\mathbb{R}^{\dim W^u(x_0)}$, respectively, and are injective immersions of the latter in M^n.

A diffeomorphism f is called a *Morse–Smale diffeomorphism* if $NW(f)$ is hyperbolic and consists of finitely many periodic points and the invariant manifolds $W^s(x)$ and $W^u(y)$ intersect transversally (whenever the intersection is nonempty) for any points $x, y \in NW(f)$.

Let $f\colon M^n \to M^n$ be a Morse–Smale diffeomorphism. A hyperbolic fixed point $p \in NW(f)$ is called a *node* if either $\dim W^s(p) = n$ (in this case, p is a *sink*) or $\dim W^u(p) = n$ (in this case, p is a *source*). A hyperbolic fixed point $\sigma \in NW(f)$ is called a *saddle* if its stable and unstable manifolds have nonzero topological dimension. If $\dim W^u(\sigma) = i$, then we will refer to each component of the set $W^u(\sigma) \setminus \{\sigma\}$ as an *i-dimensional unstable separatrix* and to each component of the set $W^s(\sigma) \setminus \{\sigma\}$ as an *$(n-i)$-dimensional stable separatrix*. A saddle $\sigma \in NW(f)$ is called a *saddle of codimension* 1 if one of its separatrices is one-dimensional. Since the removal of a point disconnects one-dimensional Euclidean space but does not disconnect Euclidean spaces of higher dimension, it follows that a one-dimensional stable or unstable manifold of a saddle periodic point consists of the saddle point itself and two one-dimensional separatrices, while an i-dimensional manifold for $i \geq 2$ consists of the saddle point and one i-dimensional separatrix.

Let $W^\tau(\sigma)$, where $\tau = u$ or s, be an i-dimensional invariant manifold, $i \geq 1$, of a saddle σ. If $i \geq 2$, then we denote the separatrix of σ lying in $W^\tau(\sigma)$ by $W^\tau_{\mathrm{sep}}(\sigma)$. If $i = 1$ (that is, the invariant manifold $W^\tau(\sigma)$ is one-dimensional), then $W^\tau_{\mathrm{sep}}(\sigma)$ denotes one of the two separatrices; where necessary, we will denote them by $W^\tau_{\mathrm{sep},1}(\sigma)$ and $W^\tau_{\mathrm{sep},2}(\sigma)$. We say that a separatrix $W^\tau_{\mathrm{sep}}(\sigma)$ *has no heteroclinic intersections* if it does not meet other separatrices. We need the following well-known statements (see [1]).

Proposition 1. *Let $W^{u(s)}_{\mathrm{sep}}(\sigma)$ be a d-dimensional separatrix of a saddle σ of a Morse–Smale diffeomorphism, and assume that $W^{u(s)}_{\mathrm{sep}}(\sigma)$ has no heteroclinic intersections. Then $W^{u(s)}_{\mathrm{sep}}(\sigma)$ lies in the attraction domain $W^s(p)$ (respectively, the repulsion domain $W^u(p)$) of exactly one sink (respectively, source) periodic point p.*

Moreover, if $d \geq 2$, then the topological closure $\operatorname{clos} W^{u(s)}_{\mathrm{sep}}(\sigma)$ of $W^{u(s)}_{\mathrm{sep}}(\sigma)$ coincides with

$$\operatorname{clos} W^{u(s)}_{\mathrm{sep}}(\sigma) = W^{u(s)}_{\mathrm{sep}}(\sigma) \cup \{p\}$$

and is a topologically embedded d-sphere.

If $d = 1$ and each one-dimensional separatrix $W^{u(s)}_{\mathrm{sep,j}}(\sigma)$, $j = 1, 2$, lies in the attraction domain $W^s(p_j)$ (respectively, the repulsion domain $W^u(p_j)$) of exactly one sink (respectively, source) periodic point p_j, then the topological closure $\operatorname{clos}(W^{u(s)}_{\mathrm{sep},1}(\sigma) \cup W^{u(s)}_{\mathrm{sep},2}(\sigma))$ is either a topologically embedded closed segment (for $p_1 \neq p_2$) or a topologically embedded circle (for $p_1 = p_2$).

Proposition 2. *Suppose that σ is a codimension-1 saddle of a Morse–Smale diffeomorphism $f\colon M^n \to M^n$, $n \geq 4$, and assume that neither of the one-dimensional separatrices $\mathrm{Sep}_1(\sigma)$ and $\mathrm{Sep}_2(\sigma)$ of σ has heteroclinic intersections.*

If $\operatorname{clos}(\mathrm{Sep}_1(\sigma) \cup \mathrm{Sep}_2(\sigma))$ is a topologically embedded circle S_0, then S_0 has a closed neighborhood T homeomorphic to $\mathbb{D}^{n-1} \otimes S^1$ and containing only two fixed points, the saddle σ and some node n_0.

Moreover, if n_0 is a sink, then T is forward invariant, and if n_0 is a source, then T is backward invariant.

Proposition 3. *Suppose that σ is a codimension-1 saddle of a Morse–Smale diffeomorphism $f\colon M^n \to M^n$, $n \geq 4$, and assume that neither of the one-dimensional separatrices $\mathrm{Sep}_1(\sigma)$ and $\mathrm{Sep}_2(\sigma)$ σ has heteroclinic intersections.*

If $\operatorname{clos}(\mathrm{Sep}_1(\sigma) \cup \mathrm{Sep}_2(\sigma))$ is a topologically embedded segment I, then I has a closed neighborhood B homeomorphic to an n-dimensional disk and containing only three fixed points, the saddle σ and two nodes n_1 and n_2.

Moreover, if n_1 and n_2 are sinks, then B is forward invariant, and if n_1 and n_2 are sources, then B is backward invariant.

The following proposition, which we give here for reference and which can be extracted from [7] and [8], holds for a Morse–Smale diffeomorphism $f\colon M^n \to M^n$, $n \geq 4$, with exactly one saddle (i.e., for $f \in MS(M^n; a, b, 1)$).

Proposition 4. *Let $f\colon M^n \to M^n$ be a Morse–Smale diffeomorphism of a closed n-dimensional manifold M^n, $n \geq 4$, and assume that the nonwandering set $NW(f)$ consists of a sinks $\omega_1, \dots, \omega_a$, b sources $\alpha_1, \dots, \alpha_b$, and one saddle σ. Then one of the following cases takes place*:

(1) $a + b = 3$ *(i.e., either $a = 1$ and $b = 2$ or $a = 2$ and $b = 1$) and the manifold M^n is the sphere $\mathbb{S}^n$; moreover, the unstable Morse index of σ is $n - 1$ for $a = 1$ and $b = 2$ and 1 for $a = 2$ and $b = 1$;*

(2) $a = b = 1$, *and the dimension n of the manifold takes one of the values $n \in \{4, 8, 16\}$; moreover, M^n is the disjoint union of an open ball $\mathbb{B}^n$ and an $n/2$-dimensional sphere $S^{n/2}$ topologically embedded in M^n, the embedding of $S^{n/2}$ in M^n is locally flat if $n \in \{8, 16\}$, and the saddle σ has $n/2$-dimensional separatrices.*

2. PROOFS OF THE MAIN RESULTS

In this section, the main results of the article are proved. We use the following operation of cutting a manifold M^n along a submanifold of codimension 1. Let $N^{n-1} \subset M^n$ be an $(n-1)$-dimensional submanifold. By *cutting M^n along N^{n-1}* we mean removing a sufficiently small neighborhood U of N^{n-1} homeomorphic to $N^{n-1} \times (0; 1)$ from M^n. The resulting (possibly disconnected) manifold $\operatorname{clos}(M^n \setminus U)$ has two additional boundary components, each of which is homeomorphic to N^{n-1}; see [9] for a rigorous justification of the possibility of this operation.

Proof of Theorem 1. First, consider the case in which the codimension-1 separatrices of both saddles do not meet the one-dimensional separatrices. Let $\operatorname{Sep}_1(\sigma_i)$ and $\operatorname{Sep}_2(\sigma_i)$ be the one-dimensional separatrices of the saddle σ_i, $i = 1, 2$. By assumption,

$$(\operatorname{Sep}_1(\sigma_1) \cup \operatorname{Sep}_2(\sigma_1)) \cap (W^s(\sigma_2) \cup W^u(\sigma_2)) = \varnothing,$$
$$(\operatorname{Sep}_1(\sigma_2) \cup \operatorname{Sep}_2(\sigma_2)) \cap (W^s(\sigma_1) \cup W^u(\sigma_1)) = \varnothing.$$

By Proposition 1, the following three cases are possible:

(a) $\operatorname{clos}(\operatorname{Sep}_1(\sigma_1) \cup \operatorname{Sep}_2(\sigma_1)) = S_1$ and $\operatorname{clos}(\operatorname{Sep}_1(\sigma_2) \cup \operatorname{Sep}_2(\sigma_2)) = S_2$ are circles;

(b) $\operatorname{clos}(\operatorname{Sep}_1(\sigma_1) \cup \operatorname{Sep}_2(\sigma_1))$ and $\operatorname{clos}(\operatorname{Sep}_1(\sigma_2) \cup \operatorname{Sep}_2(\sigma_2))$ are segments;

(c) $\operatorname{clos}(\operatorname{Sep}_1(\sigma_1) \cup \operatorname{Sep}_2(\sigma_1))$ is a circle and $\operatorname{clos}(\operatorname{Sep}_1(\sigma_2) \cup \operatorname{Sep}_2(\sigma_2))$ is a segment.

In case (a), the circle S_i, $i = 1, 2$, has a neighborhood T_i homeomorphic to $\mathbb{D}^{n-1} \otimes S^1$ by Proposition 2. Let us show that one of the T_i is forward invariant and the other is backward invariant.

Assume the contrary. To be definite, we assume that both T_1 and T_2 are backward invariant; i.e., $f^{-1}(T_i) \subset T_i$, $i = 1, 2$. Since the Morse–Smale diffeomorphism f has only two stable separatrices $\operatorname{Sep}_1(\sigma_i)$ and $\operatorname{Sep}_2(\sigma_i)$, $i = 1, 2$, lying in $T_1 \cup T_2$, it follows that there exist no sources in $M^n \setminus (T_1 \cup T_2)$. The set $M^n \setminus (T_1 \cup T_2)$ is connected and hence contains exactly one sink. Therefore, f has three nodes. Since the unstable manifolds $W^u(\sigma_1)$ and $W^u(\sigma_2)$ do not meet, it follows by [10] that the number of nodes of f must be even. The resulting contradiction proves that one of the T_i is forward invariant and the other is backward invariant.

To be definite, assume that T_1 is backward invariant and T_2 is forward invariant. Then T_1 contains a source $\alpha \in S_1 \subset T_1$ and T_2 contains a sink $\omega \in S_2 \subset T_2$. Let us show that the set $M^n \setminus (T_1 \cup T_2)$ contains no fixed points of f.

Assume the contrary. Without loss of generality, we can assume that there exists a sink $\omega_0 \in M^n \setminus (T_1 \cup T_2)$. Note that since the invariant manifold $W^u(\sigma_1)$ is simply connected, it follows that its limit set $\mathrm{Lim}(W^u(\sigma_1))$ is connected. By assumption, f has only two unstable manifolds. This, together with the inclusions $W^u(\sigma_2) \subset T_2$, $\omega_0 \in M^n \setminus (T_1 \cup T_2)$, implies the inclusion $\omega_0 \in \mathrm{Lim}(W^u(\sigma_1))$. Since $\mathrm{Lim}(W^u(\sigma_1))$ is connected, it follows that $\mathrm{Lim}(W^u(\sigma_1))$ cannot contain $W^u(\sigma_2) \cup \{\omega\} \subset T_2$. Consequently,

$$W^u(\sigma_1) \cap W^s(\sigma_2) = \varnothing.$$

By Proposition 1, the union $W^u(\sigma_1) \cup \{\omega_0\} = S_0^{n-1}$ is a topologically embedded $(n-1)$-dimensional sphere, which we denote by S_0^{n-1}. Since its codimension is strictly greater than 2, it follows that this embedding is locally flat [11], [12]. Removing it does not disconnect the manifold M^n, because the circle $\mathrm{clos}(\mathrm{Sep}_1(\sigma_1) \cup \mathrm{Sep}_2(\sigma_1))$ meets S_0^{n-1} at exactly one point and the intersection is transversal. Therefore, cutting M^n along S_0^{n-1} gives a connected manifold $\widehat{M}^n$ with two boundary components M_1 and M_2, each of which is homeomorphic to S_0^{n-1}. We glue M_1 and M_2 with n-dimensional balls B_1^n and B_2^n, respectively.

Then we obtain a closed manifold $\widetilde{M}^n$. Since the sphere S_0^{n-1} had a backward invariant neighborhood in the original manifold M^n, it follows that f extends to $\widetilde{M}^n$ as a Morse–Smale diffeomorphism $\widetilde{f}\colon \widetilde{M}^n \to \widetilde{M}^n$ with two sinks $\omega_i \in B_i^n$, $i = 1, 2$. Then the nonwandering set of $\widetilde{f}$ consists of the saddle σ_2, the source α, and at least three sinks ω and ω_i, $i = 1, 2$, which contradicts Proposition 4.

Thus,

$$NW(f) = \{\alpha\} \cup \{\omega\} \cup \{\sigma_1\} \cup \{\sigma_2\} \in T_1 \cup T_2.$$

Moreover, the set $S_1 \subset T_1$ is repulsive, and the set $S_2 \subset T_2$ is attracting. Since ∂T_1 is compact, it follows that there exists a $k \in \mathbb{N}$ such that $f^k(\partial T_1) \subset T_2$. Consequently, $f \in MS(M^n; 1, 1, 2)$, and the manifold M^n is homeomorphic to the union of two copies of $\mathbb{D}^{n-1} \otimes S^1$.

In case (b), we denote the segments $\mathrm{clos}(\mathrm{Sep}_1(\sigma_1) \cup \mathrm{Sep}_2(\sigma_1))$ and $\mathrm{clos}(\mathrm{Sep}_1(\sigma_2) \cup \mathrm{Sep}_2(\sigma_2))$ by I_1 and I_2, respectively. First, consider the case in which I_1 is a repelling set and I_2 is an attracting set. Then there exist neighborhoods $U_i \supset I_i$, $i = 1, 2$, such that $U_1 \subset f(U_1)$ and $f(U_2) \subset U_2$. Therefore, one can modify f in $U_1 \cup U_2$ so as to obtain a Morse–Smale diffeomorphism $\widetilde{f}\colon M^n \to M^n$ that has one source $\alpha_0 \in U_1$ and one sink $\omega_0 \in U_2$ and coincides with f outside $U_1 \cup U_2$. Thus, $\widetilde{f}$ is a Morse–Smale diffeomorphism without saddle periodic points. Consequently, $\widetilde{f} \in MS(\mathbb{S}^n, 1, 1, 0)$ and $f \in MS(\mathbb{S}^n, 2, 2, 2)$.

Now assume that I_1 and I_2 are repelling sets. Using the above saddle removal method and Proposition 4, one can show that $\widetilde{f} \in MS(\mathbb{S}^n, 1, 2, 1) \cup MS(\mathbb{S}^n, 2, 1, 1)$ and hence

$$f \in MS(\mathbb{S}^n, 1, 3, 2) \cup MS(\mathbb{S}^n, 3, 1, 2).$$

Let us prove that case (c) cannot be realized. Assume the contrary. Removing the saddle σ_2 lying in the segment $\mathrm{clos}(\mathrm{Sep}_1(\sigma_2) \cup \mathrm{Sep}_2(\sigma_2))$, we obtain a Morse–Smale diffeomorphism $\widetilde{f} \in MS(M^n; 1, 1, 1)$. By Proposition 4, both separatrices of the remaining saddle σ_1 must be $n/2$-dimensional. This contradicts the assumption that σ_1 is a saddle of codimension 1.

Now consider the case in which the codimension-1 separatrix of one of the saddles, say σ_1, meets the one-dimensional separatrix of the other saddle σ_2. Let us show that then $f \in MS(\mathbb{S}^n, 2, 2, 2)$. To be definite, assume that the one-dimensional separatrices of the saddles σ_i, $i = 1, 2$, are stable separatrices. Since periodic points do not form cycles in the Smale graph, it follows that

$$W^u(\sigma_2) \cap W^s(\sigma_1) = \varnothing.$$

This, together with Proposition 1, implies that $\mathrm{clos}(W^u(\sigma_2)) = S^{n-1}$ is a topologically embedded $(n-1)$-dimensional sphere containing the sink ω. Since $n \geq 4$, it follows that the embedding of S^{n-1} is locally flat [11], [12]. Cutting M^n along S^{n-1}, we obtain a manifold $\widehat{M}^n$ with two boundary components M_1 and M_2, each of which is homeomorphic to $\mathbb{S}^{n-1}$. By gluing n-dimensional balls B_1^n and B_2^n to M_1 and M_2, respectively, we obtain a closed manifold $\widetilde{M}^n$. Since S^{n-1} is an attracting set, it follows that f extends to $\widetilde{M}^n$ as a Morse–Smale diffeomorphism $\widetilde{f}\colon \widetilde{M}^n \to \widetilde{M}^n$ with sinks $\omega_i \in B_i^n$, $i = 1, 2$. Note that $\widetilde{f}$ has only one saddle σ_1 and does not have the fixed points σ_2 and ω.

If $\widetilde{M}^n$ is connected, then the nonwandering set $NW(\widetilde{f}\,)$ contains only one saddle σ_1 with unstable Morse index $(n-1)$ and at least two sinks ω_i, $i = 1, 2$. This contradicts Proposition 4. If the manifold $\widetilde{M}^n$ is disconnected, then one of its components $\widetilde{M}_2^n$ does not contain saddles. Therefore, $\widetilde{M}_2^n = \mathbb{S}^n$, and the nonwandering set $NW(\widetilde{f}\,) \cap \widetilde{M}_2^n$ in $\widetilde{M}_2^n$ consists of a sink and a source. It follows that $M^n = \widetilde{M}_1^n \sharp \mathbb{S}^n$ is homeomorphic to $\widetilde{M}_1^n$ and $NW(\widetilde{f}\,)$ contains only one source. By Proposition 4, $\widetilde{f} \in MS(\mathbb{S}^n; 1, 2, 1)$. Therefore, $f \in MS(\mathbb{S}^n, 2, 2, 2)$. □ □

Proof of Theorem 2. To be definite, we assume that f has a saddle σ of codimension 1 with stable one-dimensional separatrices and a saddle σ_0 that is not a saddle of codimension 1. Since the invariant manifolds of saddle periodic points of a Morse–Smale diffeomorphism must intersect transversally, it follows that the one-dimensional separatrices of σ do not have heteroclinic intersections. By Proposition 3, the topological closure $\operatorname{clos} W^s(\sigma)$ is a segment I with sources α_1 and α_2 at the endpoints.

Further, I has a closed backward invariant neighborhood B homeomorphic to an n-dimensional disk; i.e., $\operatorname{int} f(B) \subset B$. Therefore, one can modify f in B so as to obtain a Morse–Smale diffeomorphism $\widetilde{f}\colon M^n \to M^n$ that has one source α_0 in B and coincides with f outside B. In other words, $\widetilde{f}$ has one source less and one saddle less than f. Since $\widetilde{f}$ has exactly one saddle of codimension greater than 1, it follows by Proposition 4 that the dimension of M^n can only take one of the values $n \in \{4, 8, 16\}$, that M^n is the disjoint union of an open ball $\mathbb{B}^n$ and an $n/2$-dimensional sphere $S^{n/2}$, and that if $n \in \{8, 16\}$, then the embedding of $S^{n/2}$ in M^n is locally flat; in addition, the saddle σ_0 has $n/2$-dimensional separatrices. Moreover, since $\widetilde{f} \in MS(M^n; 1, 1, 1)$, it follows that $f \in MS(M^n; 1, 2, 2)$.

Clearly, if we assume that the saddle σ of codimension 1 has unstable one-dimensional separatrices, then we obtain $f \in MS(M^n; 2, 1, 2)$. □

Proof of Theorem 3. It is well known that if a Morse–Smale diffeomorphism does not contain saddles of codimension 1, then it is a polar diffeomorphism; i.e., $a = b = 1$ [1], [6]. It remains to prove that the manifold M^n is simply connected. Consider a mapping $\gamma\colon S^1 \to M^n$ representing an element of the fundamental group $\pi_1(M^n)$. Without loss of generality, we can assume γ to be a smooth embedding [13]. Further, by deforming γ, one can ensure that the image $\gamma(S^1)$ contains no fixed points of f.

Since the invariant manifolds of the saddles are the images of smooth immersions of Euclidean spaces and there are finitely many of them, we can ensure by successive deformations of γ that $\gamma(S^1)$ intersects all the invariant manifolds of the saddles transversally. Indeed, first, we can ensure the transversality of the intersection with separatrices that do not have heteroclinic intersections, because, outside some neighborhoods of fixed points, these separatrices are embeddings of compact domains of Euclidean spaces [13]. Outside some neighborhoods of these separatrices, the remaining separatrices (with heteroclinic intersections) are embeddings of compact domains of Euclidean spaces as well. Thus, we can assume that $\gamma(S^1)$ intersects all the invariant manifolds of the saddles transversally.

By assumption, the codimension of all invariant manifolds of the saddles is greater than 1. Since $\gamma(S^1)$ is a circle, we see that the transversality of the intersection of $\gamma(S^1)$ with the invariant manifolds of the saddles means that $\gamma(S^1)$ belongs to the attraction or repulsion domain of a sink or source, respectively. Since such a domain is homeomorphic to an open ball, it follows that the curve $\gamma(S^1)$ is contractible to a point. Consequently, the manifold M^n is simply connected. □

Remark 2. An analysis of the proof of Theorem 3 shows that its statement remains true for an arbitrary number of saddles. We thank the referee for bringing this to our attention.

Remark 3. For the reader's convenience, we present a list of closed two- and three-dimensional manifolds admitting Morse–Smale diffeomorphisms with one and two saddles. The only two-dimensional manifolds that admit Morse–Smale diffeomorphisms with one saddle are the sphere and the projective plane. The only 3-manifold admitting Morse–Smale diffeomorphisms with one saddle is the sphere. The only two-dimensional manifolds that admit Morse–Smale diffeomorphisms with two saddles are the sphere, the torus, the Klein bottle, and the projective plane. The 3-manifolds admitting Morse–Smale diffeomorphisms with two saddles are lenses, the sphere, and the direct and skew products of the two-dimensional sphere and the circle.

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