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The Relation "Commutator Equals Function" in Banach Algebras

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Abstract—The relation $xy - yx = h(y)$, where h is a holomorphic function, occurs naturally in the definitions of some quantum groups. To attach a rigorous meaning to the right-hand side of this equality, we assume that x and y are elements of a Banach algebra (or of an Arens–Michael algebra). We prove that the universal algebra generated by a commutation relation of this kind can be represented explicitly as an analytic Ore extension. An analysis of the structure of the algebra shows that the set of holomorphic functions of y degenerates, but at each zero of h, some local algebra of power series remains. Moreover, this local algebra depends only on the order of the zero. As an application, we prove a result about closed subalgebras of holomorphically finitely generated algebras.

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In quantum algebra, there arise commutation relations involving not only polynomials but also more general holomorphic functions. Similar relationships can be obtained by deforming the universal enveloping algebra of a semisimple Lie algebra (following Drinfeld and Jimbo; see the monograph [1, Definition 17.2.3]) or of the Lie algebra of the group of affine transformations of the line; see the paper [2] by Aizawa and Sato. In the latter case, the condition has the simplest form:

$$[x, y] = h(y), \tag{1}$$

where h is the hyperbolic sine [2, formula (3.1)]. (Here $[x, y] := xy - yx$.) We consider a general relation of this form, assuming that h is a function holomorphic in some domain.

Certainly, such equalities make no sense for arbitrary algebras[1]. To overcome this difficulty, quantum algebras over the ring of formal series in the quantization parameter are traditionally introduced. The alternative analytical approach of considering these relations in algebras for which the holomorphic functional calculus theorem ensures the existence of $h(y)$ (in particular, in Banach algebras) seems to be more natural. This point of view supposes a specialization of the quantization parameter to a complex number. Further, one must consider the universal algebra generated by those elements x and y for which the required relation holds, assuming additionally that these elements are contained in a Banach algebra with unit and h is holomorphic in some neighborhood of the spectrum of y. Before investigating the properties of the resulting "analytical form" of the quantum group, it is necessary to study the question of whether this form is nontrivial and how rich is its structure.

The main objective of this paper is to answer this question by providing a complete description of the universal algebra generated by elements x and y satisfying (1) for an arbitrary function h. Such problems do not always have a solution in the class of Banach algebras without additional conditions on the norms or the spectrum of elements; however, a solution can be found among algebras approximated by Banach algebras, namely, among the Arens–Michael algebras. (Recall that a complete locally convex

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[1] Here and in what follows, by "algebra" we mean "associative algebra with unit over the field $\mathbb{C}$ of complex numbers."

algebra is called an *Arens–Michael algebra* if its topology is generated by a family of submultiplicative seminorms, i.e., of seminorms $\|\cdot\|$ satisfying the condition $\|ab\| \le \|a\|\|b\|$ for any elements a and b.)

Certainly, particular cases of (1) have been studied from different points of view, at least for polynomials. For example, it has been established that if h is a nonconstant polynomial, then the problem of classification (up to similarity) of pairs of finite-dimensional operators satisfying this condition is very difficult (it is wild according to Ostrovskii and Samoilenko [3, Theorem 1]). Some spectral properties of a pair of elements of a Banach algebra satisfying (1) were studied by Turovskii [4] (cited in [5]); see also Shulman's survey [5, Sec. 3]. However, as far as the author knows, the universal Arens–Michael algebra generated by this relation has previously been identified only for constant or linear functions h. Recall that a classical result states that if h is a nonzero constant, then a nontrivial realization of relation (1) in a Banach algebra is impossible. The case of $h(y) = y$ was treated by Pirkovskii [6].

Although (1) does not imply upper bounds for the norms of powers of x, it rather rigidly determines the asymptotic behavior of $\|(y-\lambda)^n\|$ as $n \to \infty$, where λ is some zero of the function h. For example, it follows from $[x, y] = y$ that

$$n\|y^n\| \le 2\|x\|\|y^n\| \qquad \text{for all} \quad n \in \mathbb{N},$$

and thus $\|y^n\| = 0$ beginning with some n; cf. [6, Example 5.1]. This observation was used in [6, Proposition 5.2] in describing the corresponding universal algebra.

The relation $[x, y] = y^2$ also implies a restriction on growth: arguing by induction, we can show that

$$\|y^n\| = O\biggl(\frac{\|x\|^n}{n!}\biggr), \qquad n \to \infty.$$

A classical example is a pair of operators on $L^2[0,1]$, the operator T of multiplication by the independent variable and the indefinite integration operator

$$Vf(x) = \int_0^x f(t)\,dt$$

(which is a special case of the Volterra operator). It can readily be seen that $[T, V] = V^2$. There is a vast literature, devoted to calculation and estimation of the norms of powers of V; of special mention are the papers [7]–[10]. In particular, it has been proved that $\lim_{n\to\infty} n!\,\|V^n\| = 1/2$; see [8, Theorem 5.4] and [9, Remark 3].

A description of a universal algebra in the general case is given in Theorem 5 (see below). As expected from the above examples, it depends only on the zeros of h and their orders. Moreover, to every zero, there corresponds a subalgebra that consists of (not necessarily convergent) power series and is local.

Note also that (1) admits the following natural generalization.

Question 1. Let U be a domain in $\mathbb{C}$, let h be a nonzero function holomorphic on U, and let α be a continuous endomorphism of the algebra of functions holomorphic on U. What is the Arens–Michael universal algebra for the relation

$$xy - \alpha(y)x = h(y)?$$

This question remains open in the general case. The special case of the quantum Weyl algebra with $h = 1$ was described by Pirkovskii [6, Corollary 5.19].

Theorem 5 has an application to the theory of holomorphically finitely generated (or, briefly, HFG) algebras. This class of algebras was first considered by Pirkovsky (see [11] and [12]) and is of interest from the point of view of noncommutative geometry, since the commutative HFG algebras are Stein algebras, i.e., algebras of holomorphic functions, or, more precisely, of global sections of the structure sheaf on some Stein space (in the case of finite dimension of the embedding).

At present, the study of noncommutative HFG algebras remains in an embryonic state. In particular, it is not clear how wide is the class of their closed subalgebras. However, at least, it is known that a closed subalgebra of a Stein algebra need not be a Stein algebra, which means that the class of HFG algebras is not stable with respect to the passage to closed subalgebras either.

When studying our main problem, a family of local algebras $\mathscr{A}_s$, $s \in [0, \infty]$, of power series arises. We show below that, for rational values of s, such an algebra can be embedded as a closed subalgebra in some HFG algebra (Theorem 14), although it is not HFG for $s \ne 0$ (Proposition 13). The following questions remain open.

Question 2. Is $\mathscr{A}_s$ a closed subalgebra of an HFG algebra for an arbitrary irrational positive s?

Question 3. Is $\mathscr{A}_s$ a closed subalgebra of a Stein algebra for $s \in (0, +\infty]$?

1. THE UNIVERSAL ALGEBRA AND FORMULATION OF THE MAIN THEOREM

Let us begin with the definition of a family of local algebras that plays an important role in our reasoning. For every $s \geq 0$, consider the following completion of the algebra of polynomials in a formal variable y:

$$\mathscr{A}_s := \left\{ a = \sum_{n=0}^{\infty} a_n y^n : \|a\|_{r,s} := \sum_{n=0}^{\infty} |a_n| \frac{r^n}{n!^s} < \infty \text{ for all } r > 0 \right\}. \tag{2}$$

It can readily be seen that the restrictions of the seminorms $\|\cdot\|_{r,s}$ to the algebra of polynomials in y are submultiplicative (cf. the proof of part (A) of Proposition 7 below), and thus $\mathscr{A}_s$ is an Arens–Michael algebra. Let us denote the algebra $\mathbb{C}[[y]]$ of all formal power series in y by $\mathscr{A}_\infty$. This is also an Arens–Michael algebra with respect to the topology generated by the system of submultiplicative seminorms

$$\|a\|_{m,\infty} := \sum_{n=0}^{m} |a_n|, \qquad m \in \mathbb{Z}_+.$$

For a given domain U in $\mathbb{C}$, we denote the algebra of all holomorphic functions on U by $\mathscr{O}(U)$ and choose a nonzero $h \in \mathscr{O}(U)$. Let $\{\lambda_j : j \in J\}$ be the set of all zeros of the function h (without repetitions), and let $s_j := 1/(k_j - 1)$, where k_j stands for the order of λ_j. For every j, we take the algebra $\mathscr{A}_{s_j}$ and set

$$\mathscr{A} := \prod_{j \in J} \mathscr{A}_{s_j}. \tag{3}$$

Below we show that the desired universal algebra is an analytic Ore extension of the algebra $\mathscr{A}$. This analytic version of a classical notion was suggested by Pirkovskii in [6, Sec. 4.1]. Let us recall the necessary definitions and facts. Let d be a derivation of some algebra R. A seminorm $\|\cdot\|$ on R is said to be *d-stable* if there is a $C > 0$ such that $\|d(r)\| \leq C\|r\|$ for all $r \in R$ [6, Definition 4.1]. A derivation d of an Arens–Michael algebra R is said to be *m-localizable* if the topology on R is generated by a family of d-stable submultiplicative seminorms [6, Definition 4.4].

Proposition 4 [6, Propositions 4.4 and 4.6 and Remark 4.6]**.** *Let R be an Arens–Michael algebra, and let d be its m-localizable derivation. Then there exists an Arens–Michael algebra E, an $x \in E$, and a continuous homomorphism $\eta\colon R \to E$ such that $[x, \eta(r)] = \eta\, d(r)$ for all $r \in R$ which has the following universal property. If B is an Arens–Michael algebra, $\breve{x} \in B$, and $\nu\colon R \to B$ is a continuous homomorphism such that $[\breve{x}, \nu(r)] = \nu\, d(r)$ for all $r \in R$, then there is a unique continuous homomorphism $\tau\colon E \to B$ such that $\nu = \tau\eta$ and $\tau(x) = \breve{x}$.*

The universal algebra E is denoted by $\mathscr{O}(\mathbb{C}, R; d)$ and called an *analytic Ore extension* of the algebra R [6, Definition 4.3]. (Note that this notion still makes sense in the more general case in which d is an α-derivation for some endomorphism α of the algebra R.) The underlying locally convex space of the algebra $\mathscr{O}(\mathbb{C}, R; d)$ is the complete projective tensor product $R \mathbin{\widehat{\otimes}} \mathscr{O}(\mathbb{C})$, the homomorphism $\eta\colon R \to \mathscr{O}(\mathbb{C}, R; d)$ is defined by the condition $r \mapsto r \otimes 1$, and $x = 1 \otimes z$, where z is the identity function on $\mathbb{C}$ [6, Proposition 4.3].

We denote by y_j the corresponding formal variable in $\mathscr{A}_{s_j}$ and by y the sequence $(y_j + \lambda_j\colon j \in J)$ in $\mathscr{A}$. It can readily be proved that $y_j \in \mathscr{A}_{s_j}$ is quasinilpotent (since $s_j > 0$). Thus, each of the algebras $\mathscr{A}_{s_j}$ is local, and hence the spectrum of y coincides with $\{\lambda_j : j \in J\}$. Since the spectrum is contained in U and $\mathscr{A}$ is an Arens–Michael algebra, it follows that the holomorphic functional calculus for y is well defined [13, Chap. VI, Theorem 3.2]; in particular, there is an $h(y) \in \mathscr{A}$.

It can readily be seen that

$$\delta_j := h(y_j + \lambda_j)\,\frac{d}{dy_j} \tag{4}$$

is a derivation of $\mathscr{A}_{s_j}$, i.e.,

$$\delta_j(ab) = a\delta_j(b) + \delta_j(a)b \qquad \text{for any} \quad a, b \in \mathscr{A}_{s_j}.$$

Moreover, the map $\delta\colon \mathscr{A} \to \mathscr{A}$, being the product of all δ_j, is also a derivation.

We are now ready to state the main result of the paper, which is the following theorem on the universal algebra.

Theorem 5. *Suppose that U is a domain in $\mathbb{C}$ and h is a nonzero holomorphic function on U. Let $\{\lambda_j : j \in J\}$ be the set of all distinct zeros of h, and let $s_j := 1/(k_j - 1)$, where k_j is the order of λ_j.*

(A) *Let an Arens–Michael algebra $\mathscr{A}$, its derivation δ, and $y \in \mathscr{A}$ be defined as above. Then δ is m-localizable, and hence $\mathscr{O}(\mathbb{C}, \mathscr{A}; \delta)$ is well defined and is an Arens–Michael algebra; moreover, the spectrum of y is contained in U and the relation $[x, y] = h(y)$ holds.*

(B) *If an Arens–Michael algebra (in particular, a Banach algebra) B contains elements $\breve{x}$ and $\breve{y}$ such that the spectrum of $\breve{y}$ is contained in U and $[\breve{x}, \breve{y}] = h(\breve{y})$, then there is a unique continuous homomorphism*

$$\tau\colon \mathscr{O}(\mathbb{C}, \mathscr{A}; \delta) \to B$$

for which $\tau(x) = \breve{x}$ and $\tau(y) = \breve{y}$.

In particular, if h has no zeros, then the universal algebra is isomorphic to $\{0\}$. Thus, the theorem includes the classical result on the relation $[x, y] = 1$ as a special case. On the other hand, deforming the function $h(y) = y$ into $h(y) = \sinh \hbar y/\sinh \hbar$ as in [2] (here the quantization parameter is $\hbar \in \mathbb{C} \setminus \{0\}$), we see that the universal algebra contains infinitely many copies of $\mathbb{C}[[y]]$. This follows from the theorem, because the set of zeros of the hyperbolic sine is infinite and all zeros are of order 1. Since the universal algebra in the classical case contains only one such copy, this visually demonstrates the effect of quantization.

Remark 6. If h is identically equal to zero in a domain U, then, obviously, the universal algebra also exists and is topologically isomorphic to $\mathscr{O}(U) \mathbin{\widehat{\otimes}} \mathscr{O}(\mathbb{C})$. For the case in which $U = \mathbb{C}$, this well agrees with our notation, since $\mathscr{O}(\mathbb{C}) \cong \mathscr{A}_0$ (if we assume that we have here a "zero of infinite order").

We will first prove part (A) of Theorem 5 and then part (B). The idea of the proof of part (A) is to construct a family of δ_0-stable submultiplicative seminorms on $\mathscr{O}(U)$, where the derivation δ_0 is given by the formula $\delta_0(f) = hf'$. To establish the validity of part (B), we will prove that this family is sufficient for describing the topology of the universal algebra.

2. PROOF OF PART (A) OF THEOREM 5

On $\mathscr{O}(U)$, we consider the seminorms (cf. the definition of the algebras $\mathscr{A}_s$ in (2))

$$\|f\|_{\lambda,r,s} := \sum_{n=0}^{\infty} |f^{(n)}(\lambda)|\frac{r^n}{n!^{s+1}}, \qquad \|f\|_{\lambda,m,\infty} := \sum_{n=0}^{m} \frac{|f^{(n)}(\lambda)|}{n!} \tag{5}$$

(here $\lambda \in U$, $r > 0$, $s > 0$, and $m \in \mathbb{Z}_+$) and the standard seminorms

$$|f|_K := \sup\{|f(z)| : z \in K\}$$

(here K stands for a compact subset of U).

The proofs of both parts of Theorem 5 use the following two assertions.

Proposition 7. *Consider the derivation of the algebra $\mathscr{O}(U)$ given by the formula $\delta_0(f)=hf'$.*

(A) *Each of the seminorms $\|\cdot\|_{\lambda,r,s}$ and $\|\cdot\|_{\lambda,m,\infty}$ ($\lambda\in U$, $r>0$, $s>0$, and $m\in\mathbb{Z}_+$) is submultiplicative and, up to a constant, is dominated by the seminorm $|\cdot|_D$ for some closed disk $D\subset U$ of sufficiently small radius centered at λ; hence these seminorms are continuous.*

(B) *If λ is a zero of the function h, then $\|\cdot\|_{\lambda,m,\infty}$ is δ_0-stable for all $m\in\mathbb{Z}_+$.*

(C) *If λ is a zero of order $k>1$ of the function h, then $\|\cdot\|_{\lambda,r,s}$ is δ_0-stable for all $r>0$ and $s\ge 1/(k-1)$.*

Proof. Let us prove (A). If s is finite, then the submultiplicativity of every seminorm of the form $\|\cdot\|_{\lambda,r,s}$ follows from the Leibniz formula and the inequality

$$\frac{r^{l+n}}{(l+n)!^s}\le\frac{r^l}{l!^s}\frac{r^n}{n!^s}\,,\qquad l,n\in\mathbb{Z}_+.$$

In the case of $\|\cdot\|_{\lambda,m,\infty}$, the proof of submultiplicativity is straightforward.

The second part of the assertion follows from the Cauchy inequalities for the coefficients of the Taylor series and from the fact that the topology on $\mathscr{O}(U)$ is generated by the family $\{|\cdot|_K\}$ of seminorms, where K ranges over all compact subsets of U.

Let us prove (B). Suppose that $h(\lambda)=0$ and choose an $m\in\mathbb{Z}_+$. Since

$$\delta_0(f)^{(m)}=(hf')^{(m)}=\sum_{p=0}^{m}\binom{m}{p}h^{(p)}f^{(m-p+1)},\qquad f\in\mathscr{O}(U),\tag{6}$$

it follows that the number $\delta_0(f)^{(m)}(\lambda)$ is a linear function in $f(\lambda),f'(\lambda),\dots,f^{(m)}(\lambda)$ with coefficients independent of f. This readily implies that $\|\cdot\|_{\lambda,m,\infty}$ is δ_0-stable for every $m\in\mathbb{Z}_+$.

Let us prove (C). Suppose that λ is a zero of order $k>1$ of the function h. Choose an $r>0$ and an $s\ge 1/(k-1)$. By the Cauchy inequalities, there are positive numbers M and R such that $|h^{(p)}(\lambda)/p!|\le M/R^p$ for all $p\in\mathbb{Z}_+$. Substituting (6) into (5) and taking into account the fact that

$$h(\lambda)=h'(\lambda)=\dots=h^{(k-1)}(\lambda)=0,$$

we obtain

$$\|\delta_0(f)\|_{\lambda,r,s}=\sum_{n=k}^{\infty}\biggl|\sum_{p=k}^{n}\binom{n}{p}h^{(p)}(\lambda)f^{(n-p+1)}(\lambda)\biggr|\frac{r^n}{n!^{s+1}}\le\sum_{n=k}^{\infty}\sum_{p=k}^{n}\frac{M}{R^p}\frac{|f^{(n-p+1)}(\lambda)|}{(n-p)!}\frac{r^n}{n!^s}$$

for every $f\in\mathscr{O}(U)$. Making the changes $p=m+k$ and $n=q+m+k-1$, we can write

$$\begin{aligned}&\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\frac{M}{R^{m+k}}\frac{r^{q+m+k-1}}{(q+m+k-1)!^s}\frac{|f^{(q)}(\lambda)|}{(q-1)!}\\&\qquad=\sum_{q=1}^{\infty}\frac{qr^{q+k-1}}{(q+k-1)!^s}\frac{|f^{(q)}(\lambda)|}{q!}\sum_{m=0}^{\infty}\frac{M}{R^{m+k}}\frac{r^m(q+k-1)!^s}{(q+m+k-1)!^s}\,.\end{aligned}$$

Applying the inequality

$$\frac{(q+k-1)!^s}{R^{m+k}(q+m+k-1)!^s}\le\frac{1}{R^k m!^s}$$

to the terms of the sum over m, we arrive at

$$\|\delta_0(f)\|_{\lambda,r,s}\le\frac{C_rM}{R^k}\sum_{q=1}^{\infty}\frac{qr^{q+k-1}}{(q+k-1)!^s}\frac{|f^{(q)}(\lambda)|}{q!}\,,$$

where $C_r := \sum_{m=0}^{\infty} r^m/m!^s$ (note that $C_r < \infty$, because $s > 0$). Further, the condition $s \ge 1/(k-1)$ implies the inequality

$$q!\, q^{1/s} \le (q+k-1)!.$$

Thus, we have obtained the final bound

$$\|\delta_0(f)\|_{\lambda,r,s} \le \frac{C_r M}{R^k} \sum_{q=1}^{\infty} |f^{(q)}(\lambda)| \frac{r^{q+k-1}}{q!^{s+1}} = \frac{C_r M r^{k-1}}{R^k} \|f\|_{\lambda,r,s}.$$

This completes the proof of the proposition. □

Further, we consider the following family of seminorms on $\mathscr{O}(U)$:

$$\{\|\cdot\|_{\lambda_j,r_j,s_j} : j \in I\}, \tag{7}$$

where, as above, $\{\lambda_j : j \in J\}$ is the set of all zeros of the function h, k_j is the order of λ_j, $s_j := 1/(k_j - 1)$, and the domain of variation of the parameters is given by the following rule: if $\lambda_j > 1$, then $r_j \in \mathbb{R}_+$, and if $\lambda_j = 1$, then $s_j = \infty$ and $r_j \in \mathbb{N}$.

Proposition 8. *The homomorphism*

$$\mu\colon \mathscr{O}(U) \to \mathscr{A} : f \mapsto \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda_j)}{n!} y_j^n\right)_j$$

is well defined and continuous, and this is a completion homomorphism with respect to (7). *Moreover,* $\mu\delta_0 = \delta\mu$.

Proof. The definition of $\mathscr{A}$ (see (2) and (3)) implies that μ is well defined and continuous and that the topologies coincide. It remains to prove that every $(\alpha_j : j \in J) \in \mathscr{A}$ can be approximated by elements of the form $\mu(f)$, where $f \in \mathscr{O}(U)$. Moreover, we may assume that (α_j) is a finite sequence of polynomials.

Let us choose an r_j for every j such that $\alpha_j \ne 0$. It follows from the finiteness of the set of these j and from part (A) of Proposition 7 that there are a $C > 0$ and a compact subset K of U which is a union of closed disks D_j centered at λ_j such that $\|f\|_{\lambda_j,r_j,s_j} \le C\|f\|_K$ for all $f \in \mathscr{O}(U)$. Reducing the radii if necessary, we may assume that the disks D_j are pairwise disjoint. We define a function g on K as follows: for every $z \in D_j$, g coincides with the polynomial α_j in which the substitution $y_j = z - \lambda_j$ is made. Since K is compact, $\mathbb{C} \setminus K$ is connected, and g is continuous on K and holomorphic on its interior, it follows that we can apply Mergelyan's theorem, which claims that g can be approximated by polynomials uniformly on K (see, e.g., [14, Theorem 20.5]). Hence g is approximated by polynomials with respect to the topology given by (7).

The equality $\mu\delta_0 = \delta\mu$ is obtained by a straightforward calculation using (6). This completes the proof of the proposition. □

We can now complete the proof of the first part of the main theorem.

Proof of Theorem 5. Part (A). Recall that δ is the product of all derivations δ_j defined in (4). Therefore, to prove that δ is m-localizable, it suffices to prove this for every δ_j. By Proposition 8 and parts (B) and (C) of Proposition 7, every seminorm in the defining system for $\mathscr{A}_{s_j}$ is δ_j-stable, as desired.

The fact that the spectrum of y consists of those zeros of the function h that belong to U, and hence $h(y)$ is well defined, was already noted above. The relation $[x,y] = h(y)$ follows from the construction of the algebra $\mathscr{O}(\mathbb{C}, \mathscr{A}; \delta)$ (see Proposition 4). □

3. PROOF OF PART (B) OF THEOREM 5

We set $\delta_0(f) = hf'$, as above. We need the following assertion.

Proposition 9. *Every continuous submultiplicative δ_0-stable seminorm on $\mathscr{O}(U)$ is dominated by the family* (7) *of seminorms.*

To prove this proposition, we use two lemmas. Below we denote by $\mathrm{Sp}_B b$ the spectrum of the element b of the algebra B.

Lemma 10. *Let d be a continuous derivation of a commutative Banach algebra B, let $b \in B$, and let $h \in \mathscr{O}(V)$, where V is a domain in $\mathbb{C}$ containing $\mathrm{Sp}_B b$. If $d(b) = h(b)$, then $\mathrm{Sp}_B b$ consists of zeros of h and is finite.*

Proof. The Singer–Wermer theorem [15, Theorem 7.2.10] claims that the range of a continuous derivation of a commutative Banach algebra is contained in the Jacobson radical. In particular, it follows from $h(b) = d(b)$ that $h(b)$ belongs to the radical. Since B is commutative, every element of the radical is topologically nilpotent [15, Theorem 2.1.34]; hence the spectrum $\mathrm{Sp}_B h(b)$ is $\{0\}$. Since $\mathrm{Sp}_B b \subset V$, it follows from the spectral mapping theorem [15, Theorem 2.2.23] that $\mathrm{Sp}_B b$ is contained in the set of zeros of h. Since this set does not have limit points in V and $\mathrm{Sp}_B b$ is compact, it is finite. This completes the proof of the lemma. □

Let $\|\cdot\|$ be a continuous submultiplicative δ_0-stable seminorm on $\mathscr{O}(U)$. We denote the completion of $\mathscr{O}(U)$ with respect to $\|\cdot\|$ by B and the image of the identity function under the completion homomorphism $\mathscr{O}(U) \to B$ by $\breve{y}$. Obviously, δ_0 extends to a continuous derivation of B. We denote the norm and the derivation extended to B by the same symbols.

Choose a $\lambda \in \mathrm{Sp}_B \breve{y}$. By Lemma 10, the number λ is a zero of h. Let k denote the order of this zero. Then there is a $g \in \mathscr{O}(U)$ such that

$$g(\lambda) \neq 0 \qquad \text{and} \qquad h(z) = (z-\lambda)^k g(z)$$

for all $z \in U$. Let V be an open neighborhood of λ contained in U and such that $g(z) \neq 0$ for $z \in V$. Then V contains no other points of $\mathrm{Sp}_B \breve{y}$. Since $\mathrm{Sp}_B \breve{y}$ is finite, it follows from the holomorphic functional calculus theorem that the characteristic function χ of the set V can be applied to $\breve{y}$. For brevity, we will write g and χ instead of $g(\breve{y})$ and $\chi(\breve{y})$, respectively.

Lemma 11. (A) *If $k = 1$, then $\|(y-\lambda)^n \chi\| = 0$ for all sufficiently large n.*

(B) *If $k \geq 2$, then there are $K > 0$ and $r > 0$ such that*

$$\|(y-\lambda)^n \chi\| \leq K \frac{r^n}{\sqrt[k-1]{(n+k-1)!}}, \qquad n \geq k-1. \tag{8}$$

Proof. Without loss of generality, we can assume that $\lambda = 0$. Let us first estimate the norm of $\breve{y}^n g\chi$. Let $C > 0$ be such that $\|\delta_0(b)\| \leq C\|b\|$ for all $b \in B$. Further, we note that, first, χ is an idempotent and, therefore, it follows from $\delta_0(\chi) = 2\delta_0(\chi)\chi$ and $(1-2\chi)^2 = 1$ that $\delta_0(\chi) = 0$. Thus, $\delta_0(b\chi) = \delta_0(b)\chi$ for every $b \in B$. Second, since the function g has no zeros in V, there is a $w \in B$ such that $g\chi w = \chi$. Therefore, for every $n \in \mathbb{N}$, we obtain the following equality from $h(z) = z^k g(z)$:

$$\delta_0(\breve{y}^n \chi) = \delta_0(\breve{y}^n)\chi = n\breve{y}^{n+k-1} g\chi.$$

Hence

$$\|\breve{y}^{n+k-1} g\chi\| \leq n^{-1} C \|\breve{y}^n \chi\| \leq n^{-1} C \|\breve{y}^n g\chi\|\, \|w\|. \tag{9}$$

Let us prove (A). Suppose that $k=1$. Then $n+k-1=n$ and, for sufficiently large n, we obtain $\|y^n g\chi\|=0$, whence $\|y^n\chi\| \le \|y^n g\chi\|\,\|w\|=0$.

Let us prove (B). Suppose that $k\ge 2$. Let $m\in\{0,\dots,k-1\}$, and let $j\in\mathbb{N}$. Applying inequality (9) j times, we obtain

$$\|\breve{y}^{m+j(k-1)}\chi\| \le \|\breve{y}^{m+j(k-1)}g\chi\|\,\|w\| \le \frac{C^j\|w\|^{j+1}\|\breve{y}^m g\chi\|}{m(m+k-1)\cdots(m+(j-1)(k-1))}.$$

Every positive integer n which is not less than $k-1$ can be written in the form $m+j(k-1)$ with the above conditions on m and j. Since

$$m^{k-1}(m+k-1)^{k-1}\cdots(m+(j-1)(k-1))^{k-1} \ge (m+(j-1)(k-1))!,$$

there are $K>0$ and $r>0$ for which (8) holds. This completes the proof of the lemma. □

Proof of Proposition 9. We use the notation introduced before Lemma 11. It follows from Lemma 10 that $\mathrm{Sp}_B\breve{y}=\{\lambda_1,\dots,\lambda_l\}$, where $\lambda_1,\dots,\lambda_l$ are pairwise distinct zeros of h. For every $j\in\{1,\dots,l\}$, there is a function g_j in $\mathscr{O}(U)$ such that

$$g_j(\lambda_j)\ne 0 \qquad\text{and}\qquad h(z)=(z-\lambda_j)^{k_j}g_j(z),$$

where k_j is the order of λ_j as a zero of h. Let us choose, for every j, a neighborhood V_j of the point λ_j such that $g_j(z)\ne 0$ in V_j and $V_j\subset U$. It can be assumed that the neighborhoods $V_1,\dots,V_l$ are pairwise disjoint. Let χ_j be the characteristic function of V_j. As above, we write χ_j instead of $\chi_j(\breve{y})$.

For an arbitrary f in $\mathscr{O}(U)$, we write out the Taylor expansion in a neighborhood of the point λ_j and use the bounds from Lemma 11. For some K_j with j for which $r_j>0$, we have

$$\|f(\breve{y})\chi_j\| \le K_j\|f\|_{\lambda_j,r_j,s_j},$$

where $s_j:=1/(k_j-1)$ (if the order of λ_j is equal to 1, then $s_j=\infty$ and $r_j\in\mathbb{N}$). It follows readily from the representation of holomorphic functional calculus in the form of Cauchy integral that $\sum_j\chi_j=1$. Hence

$$\|f\| \le \sum_j K_j\|f(\breve{y})\chi_j\|,$$

and we obtain the desired assertion. □

We can now complete the proof of the second part of the main theorem.

Proof of Theorem 5. Part (B). Suppose that B is an Arens–Michael algebra, $\breve{x},\breve{y}\in B$, the spectrum of $\breve{y}$ is contained in U, and $[\breve{x},\breve{y}]=h(\breve{y})$. Then, for every open $V\subset U$ containing the spectrum, there is a holomorphic functional calculus $\mathscr{O}(V)\to B$ for $\breve{y}$.

(1) First, we show that

$$[\breve{x},f(\breve{y})]=h(\breve{y})f'(\breve{y}) \tag{10}$$

for every $f\in\mathscr{O}(U)$.

Suppose first that B is a Banach algebra. It can readily be seen that (10) holds if f is a polynomial.

Let B_0 denote the closed subalgebra of B generated by $\breve{y}$. Since B_0 is commutative, we can apply Lemma 10 to the continuous derivation

$$B_0\to B_0\colon b\mapsto[\breve{x},b].$$

Thus, $\mathrm{Sp}_{B_0}\breve{y}$ is finite and, therefore, so is $\mathrm{Sp}_B\breve{y}$. Let V be a finite union of open disks of finite radius with pairwise disjoint closures such that $\mathrm{Sp}_B\breve{y}\subset V\subset U$. Since the holomorphic functional calculus is continuous, it follows that there is a compact subset K of V such that $|\cdot|_K$ dominates the norm on B up to constant. We may assume that K is a finite union of closed disks.

Since K is a compact set contained in U, it follows that every function in $\mathscr{O}(V)$ is continuous on K and holomorphic in its interior. Moreover, the complement of K is connected; therefore, by Mergelyan's theorem, this function can be approximated by polynomials uniformly on K. This implies that (10) holds for al arbitrary $f \in \mathscr{O}(V)$. The uniqueness of holomorphic functional calculus implies the equality (10) for an arbitrary $f \in \mathscr{O}(U)$.

For the general case, in which B is an Arens–Michael algebra, we consider an arbitrary continuous submultiplicative seminorm $\|\cdot\|$ on B. It follows from what was proved above that the desired equality holds in the completion with respect to $\|\cdot\|$; in particular,

$$\|[\breve{x}, f(\breve{y})] - h(\breve{y})f'(\breve{y})\| = 0.$$

Since $\|\cdot\|$ is arbitrary, it follows that (10) holds in B.

(2) Further, we show that there is a continuous homomorphism $\nu\colon \mathscr{A} \to B$ such that $f(\breve{y}) = \nu\mu(f)$ for all $f \in \mathscr{O}(U)$ (here $\mu\colon \mathscr{O}(U) \to \mathscr{A}$ stands for the completion homomorphism of Proposition 8). Recall that $\delta_0(f) = hf'$ for $f \in \mathscr{O}(U)$. It follows from (10) that

$$\delta_0(f)(\breve{y}) = h(\breve{y})f'(\breve{y}) = [\breve{x}, f(\breve{y})], \qquad f \in \mathscr{O}(U). \tag{11}$$

If $\|\cdot\|$ is a continuous submultiplicative seminorm on B, then $\|f\|_1 := \|f(\breve{y})\|$ defines a continuous submultiplicative seminorm on $\mathscr{O}(U)$. It follows from (11) that

$$\|\delta_0(f)\|_1 = \|\delta_0(f)(\breve{y})\| = \|[\breve{x}, f(\breve{y})]\| \le 2\|\breve{x}\|\,\|f(\breve{y})\| = 2\|\breve{x}\|\|f\|_1, \qquad f \in \mathscr{O}(U).$$

Thus, $\|\cdot\|_1$ is δ_0-stable. It follows from Proposition 9 that $\|\cdot\|_1$ is dominated by the family (7) of seminorms. By Proposition 8, the algebra $\mathscr{A}$ is the completion with respect to (7); hence there is a continuous homomorphism $\nu\colon \mathscr{A} \to B$ such that $f(\breve{y}) = \nu\mu(f)$ for all $f \in \mathscr{O}(U)$.

(3) In conclusion, we show that $[\breve{x}, \nu(a)] = \nu\delta(a)$ for all $a \in \mathscr{A}$. Since the image of μ is dense, it suffices to prove the equality for the case in which $a = \mu(f)$, where $f \in \mathscr{O}(U)$. Using (11) and Proposition 8, we obtain

$$[\breve{x}, \nu(a)] = [\breve{x}, f(\breve{y})] = \delta_0(f)(\breve{y}) = \nu\mu\delta_0(f) = \nu\delta\mu(f) = \nu\delta(a).$$

By Proposition 4, there is a unique continuous homomorphism

$$\tau\colon \mathscr{O}(\mathbb{C}, \mathscr{A}; \delta) \to B$$

such that $\tau(x) = \breve{x}$ and $\nu = \tau\eta$. It follows from the last equality that $\tau(y) = \breve{y}$. This completes the proof of Theorem 5. □

4. EMBEDDING OF ALGEBRAS OF POWER SERIES IN HFG ALGEBRAS

A Fréchet–Arens–Michael algebra is said to be *holomorphically finitely generated*, or an HFG *algebra* for short, if it is the quotient of the algebra of free entire functions with finitely many generators by some closed two-sided ideal (up to topological isomorphism) [12, Definition 3.16, Proposition 3.20].

For the sake of the completeness of our presentation, we recall that the *algebra of free entire functions* with generators $\zeta_1, \dots, \zeta_m$ [16], [17] is the set of series

$$\left\{a = \sum_{\alpha \in W_m} c_\alpha \zeta_\alpha : \|a\|_\rho = \sum_{\alpha \in W_m} |c_\alpha| \rho^{|\alpha|} < \infty \text{ for all } \rho > 0\right\}$$

(with complex coefficients) equipped with the multiplication extending the concatenation operation on the semigroup W_m of words in the alphabet $\{1, \dots, m\}$ (for a given $\alpha \in W_m$, the corresponding monomial is denoted by ζ_α). It can readily be seen that this algebra is a Fréchet–Arens–Michael algebra.

Proposition 12. *For every domain U in $\mathbb{C}$ and any $h \in \mathscr{O}(U)$, the algebra $\mathscr{O}(\mathbb{C}, \mathscr{A}; \delta)$ considered in Theorem 5 is holomorphically finitely generated.*

Proof. Although this assertion can be derived from [12, Proposition 6.2], we present a detailed proof.

Let C denote the free product (or, which is the same thing, the coproduct) of $\mathscr{O}(\mathbb{C})$ and $\mathscr{O}(U)$ in the category of (unital) Arens–Michael algebras [12, Sec. 4]. We denote the elements of C corresponding to the identity functions on $\mathbb{C}$ and U by X and Y, respectively, and consider the the closed two-sided ideal I of C generated by the element $[X, Y] - h(Y)$. Since $\mathscr{O}(\mathbb{C})$ and $\mathscr{O}(U)$ are Stein algebras and the finiteness condition on the dimension of the embedding is satisfied, it follows that these two algebras are HFG [12, Theorem 3.22]. The property of being an HFG algebra is preserved under the formation of free products [12, Corollary 4.7] and passage to quotients by closed ideals [12, Proposition 3.18]; hence C/I is also an HFG algebra.

We claim that C/I is topologically isomorphic to $\mathscr{O}(\mathbb{C}, \mathscr{A}; \delta)$. To see this, it suffices to prove that the universal property of Theorem 5 holds. Note first that the element $Y + I$ is the image of the identity function on U under the composition $\mathscr{O}(U) \to C \to C/I$ of homomorphisms, and thus its spectrum also belongs to U. Suppose further that the Arens–Michael algebra B contains elements $\breve{x}$ and $\breve{y}$ such that the spectrum $\mathrm{Sp}_B \breve{y}$ is contained in U and $[\breve{x}, \breve{y}] = h(\breve{y})$. Since C is a free product, it follows that the correspondence $X \mapsto \breve{x}$, $Y \mapsto \breve{y}$ uniquely determines a homomorphism $C \to B$, which takes I to 0. Hence we obtain a continuous homomorphism $C/I \to B$ satisfying the desired conditions. □

Proposition 13. *For any $s \in (0, \infty]$, the algebra $\mathscr{A}_s$ is not HFG.*

Proof. (The idea of the argument below was suggested by Pirkovskii in the case of $s = \infty$; it is also applicable for the other values of s.) Suppose that $\mathscr{A}_s$ is an HFG algebra. Since $\mathscr{A}_s$ is commutative, it is a Stein algebra [12, Theorem 3.22]. Since $s > 0$, it is easy to see that the ideal generated by y is maximal and coincides with the Jacobson ideal. Thus, $\mathscr{A}_s$ is local. Therefore, the Gel'fand spectrum of the algebra consists of a single point. Hence $\mathscr{A}_s$ coincides with the algebra of germs of holomorphic functions at this unique point of the spectrum.

On the other hand, every algebra of germs of holomorphic functions is a (DF)-space (see the case of manifolds in the original paper by Grothendieck [18, pp. 97–98 (Russian transl.)] or in Mallios' monograph [13, pp. 136–137]; the proof in the general case is similar). The topology on the space $\mathscr{A}_s$ is generated by a countable family of seminorms and hence is metrizable. However, every metrizable (DF)-space is normable [19, Observation 8.3.6].

If $s \in (0, \infty)$, then $\mathscr{A}_s$ is not normable, since it is isomorphic as a locally convex space to the space $\mathscr{O}(\mathbb{C})$ of entire functions, which is well known to be nonnormable (one can also apply Kolmogorov's normability criterion [20, II.2.1] directly to $\mathscr{A}_s$). The space $\mathscr{A}_\infty$ cannot be normable, because it admits no continuous norm at all. Thus, we arrive at a contradiction. □

Note that $\mathscr{A}_0 \cong \mathscr{O}(\mathbb{C})$ and, therefore, this is an HFG algebra.

Theorem 14. *Let s be a rational positive number or ∞. Then $\mathscr{A}_s$ is isomorphic to a closed subalgebra of some HFG algebra.*

Let us denote by S the set of all positive real numbers s such that $\mathscr{A}_s$ is isomorphic to a closed subalgebra of some HFG algebra and prove an auxiliary lemma.

Lemma 15. *If $s, t \in S$, then $s + t \in S$.*

Proof. Suppose that $\mathscr{A}_s$ and $\mathscr{A}_t$ are isomorphic to closed subalgebras of HFG algebras B and C, respectively. By the Grothendieck–Pietsch criterion [21, Theorem 28.15], $\mathscr{A}_s$ and $\mathscr{A}_t$ are nuclear Fréchet spaces. Hence the homomorphism $\mathscr{A}_s \mathbin{\widehat{\otimes}} \mathscr{A}_t \to B \mathbin{\widehat{\otimes}} C$ is topologically injective (see, e.g., [22, Theorem A1.6]). Since the class of HFG algebras is stable with respect to projective tensor products [12], it follows that $\mathscr{A}_s \mathbin{\widehat{\otimes}} \mathscr{A}_t$ is isomorphic to a closed subalgebra of the HFG algebra $B \mathbin{\widehat{\otimes}} C$. Thus, it suffices to prove that the diagonal embedding $\mathscr{A}_{s+t} \to \mathscr{A}_s \mathbin{\widehat{\otimes}} \mathscr{A}_t \colon y^n \mapsto y^n \otimes y^n$ determines a well-defined topologically injective homomorphism of Fréchet algebras.

Note that $\mathscr{A}_s$ is the Köthe space $\lambda(P_s)$ corresponding to the Köthe set $P_s := \{r^n n!^{-s} : r > 0\}$ (a similar fact holds for $\mathscr{A}_t$). As noted by Pirkovskii [23, Proposition 3.3], Pietsch's results [24] readily imply

$$\lambda(P_s) \mathbin{\widehat{\otimes}} \lambda(P_t) \cong \lambda(P_s \times P_t),$$

where

$$P_s \times P_t = \{r^n q^m n!^{-s} m!^{-t} : r, q > 0\}.$$

Moreover, we may assume that $r = q$; thus, the natural diagonal embedding $\lambda(P_{s+t}) \to \lambda(P_s \times P_t)$ is a well-defined topologically injective continuous homomorphism. □

Proof of Theorem 14. Let $k \in \mathbb{N}$. By Theorem 5, $\mathscr{O}(\mathbb{C}, \mathscr{A}_{1/k}; \delta)$ is the universal Arens–Michael algebra generated by the elements x and y satisfying the relation $[x, y] = y^{k+1}$. The homomorphism $\eta\colon \mathscr{A}_{1/k} \to \mathscr{O}(\mathbb{C}, \mathscr{A}_{1/k}; \delta)$ has the form $a \mapsto a \otimes 1$ and hence is topologically injective. Therefore, $\mathscr{A}_{1/k}$ is a closed subalgebra of $\mathscr{O}(\mathbb{C}, \mathscr{A}_{1/k}; \delta)$, and the latter is an HFG algebra by Proposition 12. Thus, $1/k \in S$. It follows from Lemma 15 that all positive rational numbers belong to S.

If $s = \infty$, then it suffices to apply Theorem 5 to the relation $[x, y] = y$. This completes the proof of the theorem. □

For positive integer values of s, the assertion of Theorem 14 can be obtained also in another way, by using the author's results in [25].

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