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# Mean Convergence of Periodic Pseudotrajectories and Invariant Measures of Dynamical Systems

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**Abstract**—A discrete dynamical system generated by a homeomorphism of a compact manifold is considered. A sequence $\omega_n$ of periodic $\varepsilon_n$-trajectories converges in the mean as $\varepsilon_n \to 0$ if, for any continuous function $\varphi$, the mean values on the period $\overline{\varphi}(\omega_n)$ converge as $n \to \infty$. It is shown that $\omega_n$ converges in the mean if and only if there exists an invariant measure $\mu$ such that $\overline{\varphi}(\omega_n)$ converges to $\int \varphi \, d\mu$. If a sequence $\omega_n$ converges in the mean and converges uniformly to a trajectory Tr, then the trajectory Tr is recurrent and its closure is a minimal strictly ergodic set.

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## 1. PERIODIC PSEUDOTRAJECTORIES

Consider the discrete system

$$x_{n+1} = f(x_n), \tag{1}$$

generated by a homeomorphism $f\colon M \to M$ on a compact manifold $M$. Recall that a two-sided point sequence $T = \{x(n),\, n \in \mathbb{Z}\}$ infinite in both directions is called a *trajectory* of the system if $f(x(n)) = x(n+1)$. A two-sided point sequence $\{x(n),\, n \in \mathbb{Z}\}$ is infinite in both directions is called an *$\varepsilon$-trajectory*, or a *pseudotrajectory*, if $\rho(f(x(n)), x(n+1)) < \varepsilon$ for any $n$ ($\rho$ denotes distance). If such a sequence $\{x(n)\}$ is periodic, then it is called a *periodic $\varepsilon$-trajectory*, and the points $x(n)$ are said to be *$\varepsilon$-periodic*.

It should be noted that the exact trajectory of a system is seldom known in practice, and in reality, we deal with $\varepsilon$-trajectories for sufficiently small positive $\varepsilon$. So all computer calculations are performed with an accuracy of $\varepsilon > 10^{-19}$, and since the number of calculations is large, $\varepsilon$ may take significant values, which affects the qualitative result.

A point $x$ is said to be *chain-recurrent* if $x$ is $\varepsilon$-periodic for any $\varepsilon > 0$. The set all chain-recurrent points is called the *chain-recurrent set* and denoted by $CR$. The chain-recurrent set $CR$ is invariant and closed, and it contains all types of reverse trajectories: periodic, almost periodic, nonwandering, homoclinic, and so on. If a chain-recurrent point is not periodic and $\dim M > 1$, then there exists an arbitrarily small perturbation $f$ in the $C^0$-topology for which this point is periodic [1]. One can say that chain-recurrent points generate periodic trajectories under $C^0$-perturbations. Therefore, in computer calculations, the chain-recurrent points look like periodic ones.

**Definition 1.** Two chain-recurrent points are said to be *equivalent* if they can be connected by a periodic $\varepsilon$-trajectory for any $\varepsilon > 0$. The chain-recurrent set is divided into equivalence classes $\Omega_i$, which will be called the *components* of the chain-recurrent set.

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We note that a component is not necessarily isolated from the other components. For example, the equilibrium $y = 0$ of the differential equation $y' = y^3 \sin(1/y)$ is not an isolated equilibrium. Let $f$ be the shift along a trajectory of this equation by unit time. Each equilibrium of the equation generates a fixed point of the diffeomorphism $f$, which is a component of the chain-recurrent set. The fixed point $y = 0$ is a component that is not isolated from the other components.

We denote the set of all $\varepsilon$-periodic points by $Q(\varepsilon)$. It was shown in the book [2] that the sets of $\varepsilon$-periodic points have the following properties:

- the $Q(\varepsilon)$ are open sets for $\varepsilon > 0$;
- if $\varepsilon_2 < \varepsilon_1$, then $Q(\varepsilon_2) \subset Q(\varepsilon_1)$.

Each chain-recurrent point is $\varepsilon$-periodic for any $\varepsilon > 0$, and hence the chain-recurrent set $CR$ can be obtained as the limit

$$CR = \lim_{\varepsilon \to 0} Q(\varepsilon) = \bigcap_{\varepsilon > 0} Q(\varepsilon).$$

Thus, the families $\{Q(\varepsilon),\ \varepsilon > 0\}$ of open sets are embedded in one another and give the chain-recurrent set in the limit; i.e., they form a fundamental system of neighborhoods of the chain-recurrent set. This implies that the chain-recurrent set is measurable for any Borel measure $\mu$ and

$$\mu(CR) = \lim_{\varepsilon \to 0} \mu(Q(\varepsilon)).$$

It is well known [3] that any invariant measure is zero outside the chain-recurrent set. Therefore, the chain-recurrent set is a set of full measure, i.e., $\mu(RC) = 1$ for any invariant measure.

**The limit set of a sequence of periodic pseudotrajectories** Let $\{\omega_n\}$ be a sequence of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$. On each pseudotrajectory $\omega_n$, we mark a point $x_n$. The sequence $\{x_n\}$ is contained in a compact set $M$. Hence there exists a subsequence $\{x_{n_k}\}$ that converges to $x^*$. The point $x^*$ is called a *limit* point, and the set of all limit points is called the *limit set* of the sequence $\{\omega_n\}$.

**Proposition 1** ([4]). *Let $\{\omega_n\}$ be a sequence of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$. Then the limit set of the sequence $\{\omega_n\}$ consists of chain-recurrent points.*

Consider a sequence $\{\omega_n\}$ of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$. Let us mark a point $x_n$ on each $\omega_n$. The fact that $M$ is a compact set implies that there exists a subsequence $x_{n_k}$ that converges to a certain limit point $x^*$. The point $x^*$ uniquely determines the component $\Omega$ that contains all limit points of the sequence $\{\omega_{n_k}\}$ of periodic pseudotrajectories. However, there exist sequences $\{\omega_n\}$ of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$, such that none of the sequences $\{x_n \in \omega_n\}$ has a limit. To construct such a sequence, it suffices to take periodic pseudotrajectories $\{\omega_n\}$ which have limit points in different components of the chain-recurrent set.

**Proposition 2** ([4]). *Let $\{\omega_n\}$ be a sequence of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$, and let a point $x_n$ be marked on each pseudotrajectory $\omega_n$ so that $x^* = \lim_{n\to\infty} x_n$. Suppose that the limit point $x^*$ lies in a component $\Omega$ of the chain-recurrent set. Then the sequence $\{\omega_n\}$ uniformly converges to $\Omega$, i.e., the distance between $\omega_n$ and $\Omega$ tends to zero*:

$$\rho(\omega_n, \Omega) = \max_i \{\rho(x_i, \Omega),\ x_i \in \omega_n\} \to 0 \qquad as \quad n \to \infty.$$

Let $\{\omega_n\}$ be a sequence of periodic $\varepsilon_n$-trajectories for which the conditions of the preceding assertion are satisfied. Without loss of generality, we assume that each marked point $x_n \in \omega_n$ is the zeroth element $x_n(0)$ of the periodic sequence $\omega_n$. It is easy to prove the following assertion.

**Proposition 3.** *Let $\omega_n = \{x_n(k),\ k \in \mathbb{Z}\}$ be a sequence of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$, and let $\lim_{n\to\infty} x_n(0) = x_0$. Then the sequence $\omega_n = \{x_n(k)\}$ converges pointwise to the trajectory* $\mathrm{Tr}(x_0)$:

$$\lim_{n\to\infty} x_n(k) = f^k(x_0).$$

## 2. INVARIANT MEASURES AND PERIODIC PSEUDOTRAJECTORIES

**Definition 2.** We shall say that a sequence $\omega_n = \{x_n(k),\ k \in \mathbb{Z}\}$ of periodic $\varepsilon_n$-trajectories *converges in the mean* as $\varepsilon_n \to 0$ if, for any continuous function $\varphi\colon\ M \to \mathbb{R}$, the mean values over the period

$$\overline{\varphi}(\omega_n) = \frac{1}{p_n} \sum_{k=1}^{p_n} \varphi(x_n(k))$$

converge as $n \to \infty$, where $p_n$ is the period of the pseudotrajectory $\omega_n$.

**Theorem 1.** *Assume that a sequence $\omega_n$ of periodic $\varepsilon_n$-trajectories converges in the mean as $\varepsilon_n \to 0$. Then there exists an invariant measure $\mu$ such that, for any continuous function $\varphi$,*

$$\lim_{n\to\infty} \overline{\varphi}(\omega_n) = \int_M \varphi\, d\mu.$$

**Proof.** We shall define a functional $\Phi$ on the space $C^0 = \{\varphi\}$ of continuous functions by setting

$$\Phi(\varphi) = \lim_{n\to\infty} \overline{\varphi}(\omega_n).$$

The functional thus constructed is bounded, linear, and positive definite; hence, by the Riesz theorem [5], it can be represented as the integral

$$\Phi(\varphi) = \int \varphi\, d\mu,$$

where $\mu$ is a measure on the manifold $M$. To show the invariance of the constructed measure $\mu$, it is necessary to verify the relation $\Phi(\varphi) = \Phi(\varphi(f))$, which is equivalent to the relation

$$\lim_{n\to\infty} \overline{\varphi}(\omega_n) = \lim_{n\to\infty} \overline{\varphi}(f(\omega_n)),$$

where $\omega_n = \{x(1), x(2), \dots, x(p) = x(0)\}$ and $f(\omega_n) = \{y(1), y(2), \dots, y(p) = y(0)\}$, $y(i) = f(x(i))$. The sequence $f(\omega_n)$ is periodic. Let us show that $f(\omega_n)$ is an $\eta(\varepsilon)$-trajectory, where $\eta(\,\cdot\,)$ is the modulus of continuity of the mapping $f$. Indeed, since $\omega_n$ is a periodic $\varepsilon$-trajectory, we have $\rho(f(x(i)), x(i+1)) < \varepsilon$. We have

$$\begin{aligned}\rho(f(y(i)), y(i+1)) &= \rho(f(y(i)), f(x(i+1))) < \eta(\rho(y(i), x(i+1))) \\ &= \eta(\rho(f(x(i)), x(i+1))) < \eta(\varepsilon).\end{aligned}$$

Thus, $f(\omega_n)$ is a periodic $\eta(\varepsilon)$-trajectory. Next,

$$|\varphi(x(i+1)) - \varphi(y(i))| = |\varphi(x(i+1)) - \varphi(f(x(i)))| < \theta(\rho(x(i+1), f(x(i)))) < \theta(\varepsilon),$$

where $\theta(\,\cdot\,)$ is the modulus of continuity of the function $\varphi$. We have

$$\begin{aligned}\overline{\varphi}(\omega) &= \frac{1}{p}\sum_{i=1}^{p} \varphi(x(i)) = \frac{1}{p}\sum_{i=0}^{p-1} \varphi(x(i+1)) \\ &= \frac{1}{p}\sum_{i=0}^{p-1} \varphi(y(i)) + \frac{1}{p}\sum_{i=0}^{p-1} (\varphi(x(i+1)) - \varphi(y(i))) = \overline{\varphi}(f(\omega)) + E,\end{aligned}$$

where

$$|E| \le \frac{1}{p}\sum_{i=0}^{p-1} |\varphi(x(i+1)) - \varphi(y(i))| < \theta(\varepsilon).$$

If $\varepsilon_n \to 0$, then $\theta(\varepsilon_n) \to 0$ and, therefore, $\lim_{n\to\infty} \overline{\varphi}(\omega_n) = \lim_{n\to\infty} \overline{\varphi}(f(\omega_n))$. The proof of the theorem is complete. □

The further presentation is based on the notion of a symbolic image of a dynamical system [2], which combines the symbolic dynamics [6]–[8] and numerical methods [9]. Let $C = \{M(1), \dots, M(n)\}$ be a finite cover of the manifold $M$ by closed subsets; the set $M(i)$ will be called a *cell* with index $i$.

**Definition 3** (see [10]). The *symbolic image of a dynamical system for a cover $C$* is the directed graph $G$ with vertices $i$ corresponding to the cells $M(i)$, in which vertices $i$ and $j$ are connected by a directed edge (arc) $i \to j$ if and only if

$$f(M(i)) \cap M(j) \neq \varnothing.$$

Each symbolic image generates a symbolic dynamics, which reflects the dynamics of the system. Studying a symbolic image allows one to understand the global structure of trajectories of the system. A symbolic image depends on a cover $C$; a change of $C$ changes the symbolic image. The existence of an edge $i \to j$ guarantees the existence of a point $x$ in the cell $M(i)$ such that its image $f(x)$ lies in the cell $M(j)$. In other words, an edge $i \to j$ is a trace of the mapping $x \to f(x)$ in the sense that $x \in M(i)$ and $f(x) \in M(j)$. If there is no edge $i \to j$, then there are no points $x \in M(i)$ whose image $f(x)$ belongs to $M(j)$.

We shall consider covers $C$ in which the cells $M(i)$ are polyhedrons intersecting in boundary disks. Such covers always exist; this follows from the theorem on a triangulation of a compact manifold. We shall also assume that the cell-polyhedrons are the closures of their interiors. In numerical calculations [2], $M$ is a compact domain in $\mathbb{R}^d$, and the cells $M(i)$ are cubes or parallelepipeds. Let $d = \operatorname{diam}(C)$ be the largest diameter of a cell of a cover $C$. The number $d$ is called the *diameter of the cover $C$*. In the sections of this paper related to measure theory, we pass from a cover $C$ to a partition $C^*$ associating the boundary disks with one of the adjacent cells. In this case, $C^*$ is a measurable partition of the manifold $M$.

**Definition 4.** A two-sided sequence $\sigma = \{i(k),\ k \in \mathbb{Z}\}$ of vertices of a graph $G$, which is infinite in both directions, is called a *path* (or an *admissible path*) if, for each $k$, the graph $G$ contains the arc $i(k) \to i(k+1)$.

A vertex of a symbolic image is said to be *recurrent* if a periodic path passes through this vertex. The set of recurrent vertices is denoted by $RV$. Two recurrent vertices $i$ and $j$ are said to be *equivalent* if there exists a periodic path through $i$ and $j$. The set $RV$ of recurrent vertices is partitioned into equivalence classes $H_k$. In graph theory, the equivalence classes $H_k$ of recurrent vertices are called the *strongly connected components*.

Let $V$ be the vertex set of a graph $G$. The symbolic image of $G$ can be treated as a multivalued mapping $G\colon V \to V$ between vertices, where the image $G(i)$ of a vertex is the set of those vertices $j$ which are the endvertices of arcs $i \to j$, i.e., $G(i) = \{j : i \to j\}$. Given a symbolic image of the dynamical system under consideration, there exists a natural multivalued mapping $h\colon M \to V$ from the set $M$ to the vertex set $V$ of the symbolic image, which takes each point $x$ to the set of vertices $i$ such that $x \in M(i)$: $h(x) = \{i : x \in M(i)\}$. It follows from the definition of symbolic image that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ V & \xrightarrow{G} & V \end{array} \tag{2}$$

is commutative in the sense that

$$h(f(x)) \subset G(h(x)). \tag{3}$$

Indeed, let $i \in h(x)$, and let $j \in h(f(x))$. Then $M(j) \cap f(M(i)) \neq \varnothing$ and there exists an arc $i \to j$; this means that $j \in G(i)$ or $h(f(x)) \in G(h(x))$. Therefore, $h(f(x)) \subset G(h(x))$. We cannot guarantee that $h(f(x)) = G(h(x))$. But inclusion (3) is sufficient for the mapping $h$ to transform the trajectories of the system into admissible paths of the symbolic image:

$$h(Tr) = \{i(n) : f^n(x) \in M(i(n))\} = \sigma.$$

In this case, we say that the path $\sigma$ is the *trace of the trajectory* $\mathrm{Tr}$ on the symbolic image $G$. The trace $\sigma$ can be treated as a coding of the trajectory $\mathrm{Tr}$.

If there is a path $\sigma = \{i(n),\ n \in \mathbb{Z}\}$ on the symbolic image $G$, then the sequence of points $\omega = \{x(n) : x(n) \in M(i(n))\}$ is a pseudotrajectory. In this case, we say that the pseudotrajectory $\omega$ is the *trace of the path* $\sigma$. It is clear that the trace of a path is determined not uniquely. There is a natural link between admissible paths on a symbolic image and pseudotrajectories of a dynamical system. The following theorem describes the dependence between pseudotrajectories and admissible paths on $G$ and their relationship with the parameters of the symbolic image.

**Theorem 2** ([2]). 1. *Let a sequence* $\sigma = \{i(k),\, k \in \mathbb{Z}\}$ *be an admissible path on the symbolic image* $G$*. Then there exists a point sequence* $\omega = \{x(k) : x(k) \in M(i(k))\}$ *which is an* $\varepsilon$*-trajectory for any* $\varepsilon > d$*. In particular, if the sequence* $\sigma = \{i(1), i(2), \dots, i(p) = i(0)\}$ *is periodic, then the* $\varepsilon$*-trajectory* $\omega = \{x(1), x(2), \dots, x(p) = x(0)\}$ *is periodic.*

2. *Let a sequence* $\sigma = \{i(k),\, k \in \mathbb{Z}\}$ *be an admissible path on the symbolic image* $G$*, and let* $x(k) \in M(i(k))$*. Then the sequence* $\omega = \{x(k)\}$ *is an* $\varepsilon$*-trajectory for any* $\varepsilon > d + \eta(d)$*, where* $\eta(\cdot)$ *is the modulus of continuity of the mapping* $f$*. In particular, if the sequence* $\sigma = \{i(1), i(2), \dots, i(p) = i(0)\}$ *is periodic, then the* $\varepsilon$*-trajectory* $\omega = \{x(1), x(2), \dots, x(p) = x_0\}$ *is periodic.*

3. *There exists a positive number* $r$ *such that if a point sequence* $\omega = \{x(k),\, k \in \mathbb{Z}\}$ *is an* $\varepsilon$*-trajectory,* $\varepsilon < r$*, and* $x(k) \in M(i(k))$*, then the sequence* $\sigma = \{i(k)\}$ *is an admissible path on the symbolic image* $G$*. In particular, if an* $\varepsilon$*-trajectory* $\omega = \{x(1), x(2), \dots, x(p) = x(0)\}$ *is periodic, then* $\sigma = \{i(1), i(2), \dots, i(p) = i(0)\}$ *is a periodic path on* $G$*.*

Thus, a symbolic image is a coding of the pseudotrajectories of a dynamical system.

**Definition 5.** Let $G$ be a directed graph. A *flow* on $G$ is defined to be a distribution $\{m_{ij}\}$ on the arcs $\{i \to j\}$ such that

- $m_{ij} \geq 0$;
- $\sum_{ij} m_{ij} = 1$;
- for any vertex $i$, $\sum_k m_{ki} = \sum_j m_{ij}$.

The last property can be called the *invariance of the flow* and interpreted as the Kirchhoff law, which says that, for each vertex, the incoming flow is equal to the outgoing flow. Given a flow $\{m_{ij}\}$ on a graph $G$, we can define the measure of the vertex $i$ as

$$m_i = \sum_k m_{ki} = \sum_j m_{ij}.$$

In this case, we obtain $\sum_i m_i = m(G) = 1$. Each invariant measure $\mu$ generates a flow on the symbolic image as follows. We construct a measurable partition $C^* = \{M^*(i)\}$, associating the boundary disks with one of the adjacent cells. Then to each arc $i \to j$ of the symbolic image $G$ we assign the measure

$$m_{ij} = \mu(M^*(i) \cap f^{-1}(M^*(j))) = \mu(f(M^*(i)) \cap M^*(j)), \tag{4}$$

where the last relation is a consequence of the invariance of the measure $\mu$; for details, see [11].

Now consider the reverse process. Assume that a flow $m = \{m_{ij}\}$ is defined on a symbolic image $G$. Then one can define a measure $\mu$ on $M$ by setting

$$\mu(A) = \sum_i m_i \frac{v(A \cap M(i))}{v(M(i))} \tag{5}$$

for any measurable set $A$, where $v$ is the Lebesgue measure; it is assumed that the Lebesgue measure of each cell is $v(M(i)) \neq 0$. In this case, the measure of the cell $M(i)$ coincides with the measure of the vertex $i$: $\mu(M(i)) = m_i$. Generally speaking, the measure $\mu$ is not invariant for $f$. But it was shown in [11] that this measure is an approximation to an invariant measure in the sense that $\mu$ converges in the weak topology to an invariant measure as the maximal diameter of cells tends to zero.

**Proposition 4** ([11]). *Assume that there exists a periodic path $\eta$ of period $p$ on the graph $G$. Then there is a flow $m$ on $G$ such that*

$$m_{ij} = \frac{k_{ij}}{p},$$

*where $k_{ij}$ is the number of passages of the path $\eta$ through the arc $i \to j$.*

The above-described flow is called a *periodic flow*. A periodic path $\eta = \{i(1), i(2), \dots, i(p) = i(0)\}$ will be called a *simple path*, or a *cycle*, if its vertices $\{i(1), i(2), \dots, i(p)\}$ are distinct. In this case, there exists a periodic flow $m_{ij} = 1/p$ for an arc $i \to j$ in $\eta$; otherwise, $m_{ij} = 0$. Such a flow is said to be *simple*.

Consider a directed graph $G$ and the set $\mathscr{M}(G)$ of all flows on $G$. Let $m^1 = \{m^1_{ij}\}$, and let $m^2 = \{m^2_{ij}\}$ lie in $\mathscr{M}(G)$. In the space of flows, the convex sum

$$m = \alpha m^1 + (1-\alpha) m^2 = \{\alpha m^1_{ij} + (1-\alpha) m^2_{ij}\}, \qquad 0 \le \alpha \le 1,$$

is defined. A distance on $\mathscr{M}(G)$ is defined as

$$\rho(m^1, m^2) = \max_{ij}\{|m^1_{ij} - m^2_{ij}|\}.$$

Thus, the set $\mathscr{M}(G)$ of flows forms a convex compact set in the metric of $\rho$, and the simple flows are its extreme points. Let $H$ be the class of equivalent recurrent vertices. Then the set $\mathscr{M}(H)$ of flows concentrated on $H$ is also a convex compact set.

**Theorem 3** ([12]). *Let $H$ be a class of equivalent recurrent vertices. Then the periodic flows are dense in $\mathscr{M}(H)$ in the metric of $\rho$.*

If $\omega_n$ is a sequence of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$, and, on each pseudotrajectory $\omega_n$, a point $x_n$ is marked so that $x^* = \lim_{n\to\infty} x_n$, then, by Proposition 2, the limit set of the sequence $\omega_n$ lies in a component $\Omega$ of the chain-recurrent set: $x^* \in \Omega$. The component $\Omega$ is uniquely determined by the point $x^*$. In this case, the sequence $\omega_n$ converges uniformly to $\Omega$.

**Theorem 4.** *Let $\omega_n = \{x_n(1), x_n(2), \dots, x_n(p_n) = x_n(0)\}$ be a sequence of periodic $\varepsilon_n$-trajectories, and let $\varepsilon_n$ tend to zero. Then there exists a subsequence $\omega_{n_k}$ and an invariant measure $\mu$ such that $\omega_{n_k}$ converges in the mean and the mean values of any continuous function $\varphi$ on $\omega_{n_k}$ converge to its mean with respect to the measure $\mu$:*

$$\overline{\varphi}(\omega_{n_k}) = \frac{1}{p_{n_k}} \sum_{j=1}^{p_{n_k}} \varphi(x_{n_k}(j)) \to \int \varphi\, d\mu$$

*as $k \to \infty$, where each $p_{n_k}$ is the period of the pseudotrajectory $\omega_{n_k}$. Moreover, the support of the measure $\mu$ lies in the component $\Omega$ of the chain-recurrent set.*

**Proof.** We assume that the points $x_n(0)$ converge to a point $x^*$ as $n \to \infty$. If this is not true, we can pass to a subsequence $\omega_{n_k}$ for which the points $x_{n_k}(0)$ converge.

Let $C = \{M(i)\}$ be a closed cover by cells that are polyhedrons intersecting in boundary disks, and let $G$ be the symbolic image of the mapping $f$ for the cover $C$. By Theorem 2 on tracing, if $\varepsilon_n > 0$ is sufficiently small, then the periodic pseudotrajectory $\omega_n$ is traced on $G$ by a periodic path $\eta$ of period $p$. By Proposition 4, the periodic path $\eta$ determines the flow

$$m = \left\{ m_{ij} = \frac{k_{ij}}{p} \right\},$$

where $k_{ij}$ is the number of passages of the periodic path $\eta$ through the arc $i \to j$. The flow $m$ generates a measure $\mu$ such that the measure of any measurable set $A$ is given by the formula

$$\mu(A) = \sum_i m_i \frac{v(A \cap M(i))}{v(M(i))},$$

where $v$ is the Lebesgue measure and $m_i = \sum_j m_{ij}$. The measure of the cell $M(i)$ is calculated as

$$\mu(M(i)) = \sum_j m_{ij} = \sum_j \frac{k_{ij}}{p} = \frac{k_i}{p},$$

where $k_i$ is the number of passages of the periodic path $\eta$ through the vertex $i$. Now consider the sequence of subdivisions $C_k$ of the cover $C$ with partition diameters $d_k$ converging to zero. Let $G_k$ be the corresponding sequence of symbolic images. We use the technique described above to construct a subsequence of periodic $\varepsilon_k$-trajectories, a sequence of flows $m_k$ on $G_k$, and a sequence of measures $\mu_k$ on $M$. By Theorem 3 in [11], there is a subsequence of measures $\mu_{k_t}$ which converges to the invariant measure $\mu$ in the weak topology. This means that, for any function $\varphi$,

$$\int \varphi\, d\mu_{k_t} \to \int \varphi\, d\mu$$

as $t \to \infty$. In the process of constructing the desired sequence, we distinguished subsequences two times: the first subsequence $\omega_{n_k}$ was obtained when constructing the sequence of flows $m_k$ on $G_k$, and the second subsequence was extracted from $\omega_{n_k}$ to construct a converging sequence of measures $\mu_{k_t}$ on $M$. In what follows, to avoid difficulties in the notation, we assume that all subsequences coincide with the initial sequence.

To complete the proof, we must show that

$$\left|\overline{\varphi}(\omega_n) - \int \varphi\, d\mu_n\right| \to 0$$

as $n \to \infty$, where $\omega_n = \{x(1), x(2), \dots, x(p) = x(0)\}$. Taking into account the fact that the Lebesgue measure of boundary disks is zero, we can write

$$\int \varphi\, d\mu_n = \sum_i \int_{M(i)} \varphi\, d\mu_n = \sum_i \varphi(x_i)\mu_n(M(i)) = \sum_i \varphi(x_i)\frac{k_i}{p},$$

where $x_i$ is a point in $M(i)$ determined by the mean value theorem. The number $k_i$ coincides with the number of passages of the pseudotrajectory $\omega_n$ through the cell $M(i)$. The mean value

$$\overline{\varphi}(\omega_n) = \frac{1}{p}\sum_{j=1}^{p} \varphi(x(j))$$

can be written as follows. We collect all terms corresponding to the points $x(j) \in M(i)$ and then sum over $i$. We obtain

$$\overline{\varphi}(\omega_n) = \frac{1}{p}\sum_i \sum_{x_j \in M(i)} \varphi(x(j)).$$

The number of terms in the last sum is the number of passages of the pseudotrajectory $\omega_n$ through the cell $M(i)$. Then we can write

$$\left|\overline{\varphi}(\omega_n) - \int \varphi\, d\mu_n\right| = \frac{1}{p}\sum_i \sum_{x(j) \in M(i)} |\varphi(x(j)) - \varphi(x_i)| < \theta(d_n),$$

where $x_i$ and $x(j)$ lie in the cell $M(i)$, $\theta(\,\cdot\,)$ is the modulus of continuity of the function $\varphi$, $d_n$ is the diameter of the cover $C_n$, and $d_n \to 0$. Therefore, the limit of the mean over the period coincides with the mean with respect to the measure.

By Proposition 2, the limit set of the sequence $\omega_n$ lies in a component of the chain-recurrent set. By the choice of the subsequence, the limit point $x^* = \lim_{n\to\infty} x_n(0)$ lies in the component $\Omega$. The sequence $\{\omega_n\}$ converges to $\Omega$ uniformly. The multivalued mapping $h\colon M \to V$ takes each point $x$ to the set of all vertices $i$ such that $x \in M(i)$. By Proposition 7 in [4], the image $h(x)$ of a chain-recurrent point consists of recurrent vertices, and there exists a unique class $H(\Omega)$ of equivalent recurrent vertices such that $h(\Omega) \subset H(\Omega)$. By the theorem on the localization of the chain-recurrent set [2], the union

$U = \{\bigcup M(i),\, i \in H(\Omega)\}$ is a closed neighborhood of the component $\Omega$, and if the diameter $d$ of the cover tends to zero, then $U$ converges to $\Omega$. By construction, the support of the measure $\mu_k$ lies in $U_k = \{\bigcup M(i),\, i \in H(\Omega),\, M(i) \in C_k\}$, and hence the support of the measure $\mu$ lies in the component $\Omega = \bigcap_k U_k$. The proof of the theorem is complete. □

Theorem 4 states that a sequence of periodic pseudotrajectories determines a certain invariant measure. The next theorem states the converse: an invariant measure generates a sequence of periodic pseudotrajectories that converges in the mean to the average with respect to the given measure.

**Theorem 5.** *If the support of an invariant measure $\mu$ lies in one component of the chain-recurrent set, then there exists a sequence of periodic $\varepsilon_n$-trajectories $\omega_n$, $\varepsilon_n \to 0$, such that, for any continuous function $\varphi$, the mean over the period $\overline{\varphi}(\omega_n)$ converges to $\int_M \varphi\, d\mu$.*

**Proof.** Let $\Omega$ be a component of the chain-recurrent set, and let $\mu$ be an invariant measure whose support lies in $\Omega$. Assume that $C = \{M(i)\}$ is a cover of the manifold $M$ whose cells are closed polyhedrons that intersect in boundary disks and that $d$ is the diameter of the cover. Consider the symbolic image $G$ of the mapping $f$ with respect to the cover $C$. For each vertex $i$, we define a number $b[i]$ as follows. We fix a point $x_i$ in each cell $M(i)$ and put $b[i] = \varphi(x_i)$. It is clear that such a labeling depends on the choice of points $x_i \in M(i)$. If $x_i^*$ is another point in $M(i)$, then

$$|\varphi(x_i) - \varphi(x_i^*)| < \theta(d),$$

where $\theta(\,\cdot\,)$ is the modulus of continuity of the function $\varphi$.

We use the cover $C = \{M(i)\}$ to construct a measurable partition $C^* = \{M^*(i)\}$ consisting of polyhedrons, in which boundary disks belong to only one of neighboring cells. Setting

$$m_{ij} = \mu(f(M^*(i)) \cap M^*(j)) = \mu(M^*(i) \cap f^{-1}M^*(j)),$$

we construct a flow on the symbolic image; for details, see [11]. The mapping $h\colon M \to V$ takes each point $x$ to the set of all vertices $i$ such that $x \in M(i)$. The image $h(x)$ of a chain-recurrent point consists of recurrent vertices, and the class $H(\Omega)$ of equivalent recurrent vertices is such that $h(\Omega) \subset H(\Omega)$. The union $U = \{\bigcup M(i),\, i \in H(\Omega)\}$ is a closed neighborhood of the component $\Omega$, and $U$ converges to $\Omega$ as $d \to 0$. Since the measure $\mu$ is concentrated on $\Omega \subset \{\bigcup M(i),\, i \in H(\Omega)\}$, it follows that the flow $m$ is concentrated on $H(\Omega)$. In this case, the quantity

$$\mu(M^*(i)) = \sum_j m_{ij} = \sum_k m_{ki} = m_i$$

determines the measure of the vertex $i$. The *average of the labeling over the flow* $m_{ij}$ is the number

$$\sum_{ij} m_{ij} b[i] = \sum_i m_i b[i] = \sum_i \mu(M^*(i))\varphi(x_i).$$

We obtain an integral sum which depends on the choice of the points $x_i$ in $M(i)$. If $\{x_i^* \in M(i)\}$ is another set of points, then we have the inequality

$$\left|\sum_i \varphi(x_i)\mu(M^*(i)) - \sum_i \varphi(x_i^*)\mu(M^*(i))\right| \le \theta(d) \sum_i \mu(M^*(i)) = \theta(d).$$

By the mean value theorem, we have

$$\int_M \varphi(x)\, d\mu = \sum_i \int_{M^*(i)} \varphi(x)\, d\mu = \sum_i \varphi(x_i^*)\mu(M^*(i)),$$

where the mean point $x_i^*$ lies in the closed cell $M(i)$. As a result, we obtain the inequality

$$\left|\sum_i m_i b[i] - \int_M \varphi(x)\, d\mu\right| \le \theta(d). \tag{6}$$

Choosing a cover $C$ of sufficiently small diameter $d$, we can obtain a sufficiently small difference between the average over the flow and the average with respect to the measure.

By Theorem 3, the periodic flows are dense in the space of all flows. This means that, for any $\delta > 0$, there exists a periodic flow $m^* = \{m^*_{ij}\}$ for which $\sum_{ij} |m_{ij} - m^*_{ij}| < \delta$. The flow $m^*$ is generated by a periodic path $\gamma^* = \{i(k),\ k = 1, 2, \dots, p\}$, so that $m^*_{ij} = k_{ij}/p$, where $k_{ij}$ is the number of passages of the path $\gamma^*$ through the arc $i \to j$. We have

$$\left|\sum_i m_i b[i] - \sum_i m^*_i b[i]\right| = \left|\sum_i b[i] \sum_j (m_{ij} - m^*_{ij})\right| \le K \sum_{ij} |m_{ij} - m^*_{ij}| < K\delta,$$

where $K = \max |\varphi|$. The average over the periodic flow $m^*$ has the form

$$\sum_i m^*_i b[i] = \sum_{ij} m^*_{ij} b[i] = \frac{1}{p} \sum_{ij} k_{ij} b[i] = \frac{1}{p} \sum_i k_i b[i] = \frac{1}{p} \sum_{k=1}^{p} \varphi(x_{i(k)}),$$

where $k_i = \sum_j k_{ij}$ is the number of passages of the periodic path $\gamma^*$ through the vertex $i$, $\sum_i k_i = p$, and $x_{i(k)} \in M(i(k))$, $k = 1, 2, \dots, p$. We see that the average of the labeling $\{b[i]\}$ over the flow $m^*$ coincides with the average of this labeling over the periodic path $\gamma^*$

$$b(\gamma^*) = \frac{1}{p} \sum_i k_i b[i].$$

By Theorem 2 on tracing, the periodic path $\gamma^*$ on the symbolic image generates the periodic pseudotrajectory $\omega = \{x(k),\ k = 1, 2, \dots, p\}$, so that $x(k) \in M(i(k))$ and $\omega$ is a $\varepsilon_1$-trajectory for any $\varepsilon_1 > d$. Let $\varepsilon_1 = (3/2)d$; then $\varepsilon_1$ converges to zero together with $d$. Since the points $x_{i(k)}$ and $x(k)$ lie in $M(i_k)$, we have

$$\left|\frac{1}{p} \sum_{k=1}^{p} \varphi(x_{i(k)}) - \frac{1}{p} \sum_{k=1}^{p} \varphi(x(k))\right| < \theta(d).$$

Summing the obtained inequalities, we obtain

$$\begin{aligned} \left|\int_M \varphi\, d\mu - \frac{1}{p} \sum_{k=1}^{p} \varphi(x(k))\right| &\le \left|\int_M \varphi\, d\mu - \sum_i m_i b[i]\right| + \left|\sum_{ij} m_{ij} b[i] - \sum_{ij} m^*_{ij} b[i]\right| \\ &\quad + \left|\sum_{ij} m^*_{ij} b[i] - \frac{1}{p} \sum_k \varphi(x(k))\right| \\ &< 2\theta(d) + K\delta. \end{aligned}$$

Let $\varepsilon$ be an arbitrary positive number; then we can choose $d > 0$ and $\delta > 0$ so that $2\theta(d) + K\delta < \varepsilon$. In this case, the difference between the average of the function $\varphi$ over the periodic pseudotrajectory $\omega$ and the average of this function with respect to the measure $\mu$ is less than $\varepsilon$.

Now consider a sequence of subdivisions $C_k$ of the cover $C$ with partition diameters $d_k$ converging to zero. Let $G_k$ be the corresponding sequence of symbolic images. The above-described procedure allows us to construct a sequence of periodic $\varepsilon_n$-trajectories $\{\omega_n,\ \varepsilon_n \to 0\}$ such that the mean values $\overline{\varphi}(\omega_n)$ on them converge to the average $\int_M \varphi\, d\mu$ with respect to the measure. The proof of the theorem is complete. □

## 3. RECURRENT TRAJECTORIES AND PERIODIC PSEUDOTRAJECTORIES

Let $\{\omega_n\}$ be a sequence of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$. On each pseudotrajectory $\omega_n$, we mark a point $x_n$ such that $x^* = \lim_{n\to\infty} x_n$, and the limit point $x^*$ lies in a component $\Omega$ of the chain-recurrent set. Then the sequence $\{\omega_n\}$ uniformly converges to the component $\Omega$ (by Proposition 2), and the pseudotrajectories $\{\omega_n\}$ converge poinwise to the trajectory $\mathrm{Tr}(x^*)$ (by Proposition 3).

**Definition 6.** A sequence $\omega_n = \{x_n(k),\ k \in \mathbb{Z}\}$ of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$, *converges uniformly* to a trajectory $\mathrm{Tr} = \{y(k) = f^k(y_0),\ k \in \mathbb{Z}\}$ if $\sup_k \rho(x_n(k), y(k))$ converges to zero as $n \to \infty$.

A recurrent trajectory is defined by different authors in different ways. Thus, Definition 3.3.2 of a recurrent trajectory in [3] coincides with the definition in [5, p. 363] of a trajectory stable in the sense of Poisson. Here we shall use the definition of a recurrent trajectory in the sense of Birkhoff.

**Definition 7.** A trajectory $K$ is said to be *recurrent* if, for each $\varepsilon > 0$, there exists an integer $p > 0$ such that the $\varepsilon$-neighborhood of any segment of this trajectory of length $p$ contains the whole trajectory $K$.

An invariant closed set $B$ is said to be *minimal* if it has no invariant closed proper subsets.

**Theorem 6.** *If a sequence $\omega_n$ of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$, converges uniformly to a trajectory* Tr*, then this trajectory is recurrent. The limit set of the sequence $\omega_n$ coincides with the closure of the trajectory* Tr *and is a minimal set.*

**Proof.** The pseudotrajectory $\omega_n = \{x_n(k),\ k \in \mathbb{Z}\}$ is $p_n$-periodic, i.e., $x_n(k + p_n) = x_n(k)$ for any $k \in \mathbb{Z}$. We fix $n$ and put $r_n = \sup_k \rho(x_n(k), y(k))$. Then, by the proposition on uniform convergence, we have $r_n \to 0$ as $n \to \infty$. Let $B(x, r)$ be the ball of radius $r$ centered at a point $x$. Since the pseudotrajectory $\omega_n$ is $p_n$-periodic, it follows that the union of $p_n$ balls of the form

$$U_n = \left\{ \bigcup_{k=1}^{p_n} B(x_n(k), r_n),\ x_n(k) \in \omega_n \right\}$$

contains the whole trajectory $\mathrm{Tr} = \{y(k),\ k \in \mathbb{Z}\}$. In this case, $x_n(k) = x_n(k + zp_n)$, $z \in \mathbb{Z}$, and

$$\rho(y(k + zp_n), x_n(k)) = \rho(y(k + zp_n), x_n(k + zp_n)) \le r_n \qquad \text{for} \quad z \in \mathbb{Z}.$$

Therefore, the distance $\rho(y(k + zp_n), y_k)$ is less that $2r_n$ for $z \in \mathbb{Z}$, and the ball of radius $2r_n$ centered at $y_k$ contains all points of the form $y(k + zp_n)$, $z \in \mathbb{Z}$. Thus, the union of $p_n$ balls of the form

$$U_n^*(k_0) = \left\{ \bigcup_z B(y(k_0 + z), 2r_n),\ 0 < z \le p_n \right\}$$

contains the whole trajectory Tr for any number $k_0 \in \mathbb{Z}$. We shall show that the trajectory Tr is recurrent. We fix $\varepsilon > 0$. Since $r_n \to 0$, we can find $r_n < \varepsilon/2$, which determines the number $n$ and the period $p_n$ of the pseudotrajectory $\omega_n$. It follows from the above construction that the $2r_n$-neighborhood of any segment of length $p_n$ of the trajectory Tr contains the whole trajectory Tr. Therefore, Tr is a recurrent trajectory. Since $r_n \to 0$ as $n \to \infty$, we see that the limit set of the sequence $\{\omega_n\}$ coincides with the closure of the trajectory Tr. The Birkhoff theorem (see p. 404 in [5]) states that the closure of a recurrent trajectory is a compact minimal set. The proof of the theorem is complete. □

Assume that a semitrajectory $T = \{x(k) = f^k(x(0)),\ k \in \mathbb{Z}^+\}$ has a limit point $x^*$, i.e., for some sequence, $x(k_m) \to x*$ as $m \to \infty$. Thus, for any $\varepsilon > 0$, there exist points $x(k_1)$ and $x(k_2)$ such that the distance between them is $\rho(x_{k_1}, x_{k_2}) < \varepsilon$ and $k_2 \ne k_1$. Let $k_2 > k_1$; then $\rho(f^{k_2-k_1}(f^{k_1}(x_0)), f^{k_1}(x_0)) < \varepsilon$. Therefore,

$$\omega = \{y(0) = f^{k_1}(x(0)),\ y(1) = f^{k_1+1}(x(0)), \dots, y(p-1) = f^{k_2-1}(x(0)),\ y(p) = y(0)\}$$

is a $p$-periodic $\varepsilon$-trajectory. In this way, we construct a sequence $\omega_n$ of periodic $\varepsilon_n$-trajectories $\varepsilon_n \to 0$. On each pseudotrajectory $\omega_n$, the point $y_n = f^{k_n}(x(0))$ is marked, and $y_n = f^{k_n}(x(0)) \to x^*$ as $n \to \infty$. By Theorem 4, there exists a subsequence $\omega_{n_m}$ and an invariant measure $\mu$ such that the mean value over $\omega_{n_m}$ of any continuous function converges to its average with respect to the measure $\mu$:

$$\overline{\varphi}(\omega_{n_m}) = \frac{1}{p_m} \sum_{k=0}^{p_m-1} \varphi(y(k)) \to \int_M \varphi\, d\mu$$

as $m \to \infty$. The support of the measure $\mu$ lies in the $\omega$-limit set of the trajectory $T$.

Now we consider a recurrent trajectory $\mathrm{Tr} = \{x(k) = f^k(x(0)),\ k \in \mathbb{Z}\}$. In this case, the marked and limit points coincide with the initial point $y_n = x(0) = x^*$, $k_1 = 0$, and $y(k) = f^k(x^*)$. This implies that, for the recurrent trajectory Tr, there exists a sequence of integers $p_n \to \infty$ and an invariant measure $\mu$ such that

$$\lim_{n\to\infty} \frac{1}{p_n} \sum_{k=0}^{p_n-1} \varphi(f^k(x^*)) = \int_M \varphi\, d\mu. \tag{7}$$

Moreover, the support of $\mu$ coincides with the closure $\overline{\mathrm{Tr}} = \Omega$ of the recurrent trajectory, which is a minimal set.

The following question arises: How is the invariant measure related to the subsequence $p_n \to \infty$? Here we need results obtained in [13]. A point $x \in \Omega$ is said to be *quasiregular* if, for any continuous function $\varphi$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) \tag{8}$$

exists. An example of a minimal set with an irregular point is given in [13]. This means that the limit of the subsequence (7) always exists, but the general limit (8) does not necessarily exist. The following assertion holds.

**Theorem 7** ([13]). *Each minimal set is either an ergodic set or contains at least one nonquasiregular point, i.e., a point $x \in \Omega$ for which there exists a continuous function $\varphi\colon M \to \mathbb{R}$ such that the limit*

$$\overline{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(f^k(x))$$

*does not exist.*

In the latter case, the closure $\overline{\mathrm{Tr}} = \Omega$ of the recurrent trajectory is the support of more than one ergodic measure.

The following theorem was proved in [13, Proposition 5.5].

**Theorem 8.** *For a compact dynamical system $(\Omega, f)$, the following conditions on a point $p \in \Omega$ are equivalent*:

- *for each continuous function $\varphi$, the sequence*

$$\frac{1}{m} \sum_{k=0}^{m-1} \varphi(f^{k+s}(p))$$

  *converges uniformly in $s \in \mathbb{Z}$ as $m \to \infty$*;

- *the subsystem $(\overline{\mathrm{Tr}(p)}, f)$ is strictly ergodic (admits a unique invariant measure).*

**Theorem 9.** *If a sequence $\omega_n$ of periodic $\varepsilon_n$-trajectories converges in the mean as $\varepsilon_n \to 0$ and converges uniformly to a trajectory* Tr *as $n \to \infty$, then the closure of the trajectory* Tr *is a minimal strictly ergodic set.*

**Proof.** Theorem 6 states that, under our conditions, the trajectory Tr is recurrent and the closure $\overline{\mathrm{Tr}} = \Omega$ is a minimal set. It follows from Theorem 8 that, to complete the proof, it is necessary to show that the sequence

$$\frac{1}{m} \sum_{k=0}^{m-1} \varphi(f^{k+s}(p))$$

converges uniformly in $s \in \mathbb{Z}$ as $m \to \infty$, where $p \in \mathrm{Tr}$.

Let $\omega_n = \{x_n(k),\ k \in \mathbb{Z}\}$ be a sequence of periodic $\varepsilon_n$-trajectories, $\varepsilon_n \to 0$, and let $p_n$ be the period of $\omega_n$. The mean value of the function $\varphi$ is

$$\overline{\varphi}(\omega_n) = \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \varphi(x_n(k)) = \frac{1}{p_n} \sum_{k=0}^{p_n-1} \varphi(x_n(k)).$$

Without loss of generality, we assume that $x_n(0)$ converges to a point $p$. Then $y(0) = p$, the trajectory is $\mathrm{Tr} = \{y(k) = f^k(p),\ k \in \mathbb{Z}\}$, and $r_n = \sup_k \rho(x_n(k), y(k))$ converges to zero as $n \to \infty$. The difference of mean values satisfies the relation

$$\left|\frac{1}{m} \sum_{k=0}^{m-1} \varphi(f^{k+s}(p)) - \frac{1}{m} \sum_{k=0}^{m-1} \varphi(x_n(k+s))\right| \le \frac{1}{m} \sum_{k=0}^{m-1} |\varphi(f^{k+s}(p)) - \varphi(x_n(k+s))| \le \theta(r_n),$$

where $\theta(\,\cdot\,)$ is the modulus of continuity of the function $\varphi$ and the numbers $m$ and $s$ are arbitrary. By assumption, the sequence $\omega_n$ converges in the mean; thus, by Theorem 1,

$$\lim_{n\to\infty} \overline{\varphi}(\omega_n) = \int_M \varphi\, d\mu,$$

where $\mu$ is an invariant measure. Let us show that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \varphi(f^{k+s}(p)) = \int_M \varphi\, d\mu$$

as $m \to \infty$, and this convergence is uniform in $s$. We have

$$\begin{aligned}\left|\frac{1}{m} \sum_{k=0}^{m-1} \varphi(f^{k+s}(p)) - \int_M \varphi\, d\mu\right| &\le \left|\frac{1}{m} \sum_{k=0}^{m-1} \varphi(f^{k+s}(p)) - \frac{1}{m} \sum_{k=0}^{m-1} \varphi(x_n(k+s))\right| \\ &\quad + \left|\frac{1}{m} \sum_{k=0}^{m-1} \varphi(x_n(k+s)) - \overline{\varphi}(\omega_n)\right| + \left|\overline{\varphi}(\omega_n) - \int_M \varphi\, d\mu\right| \\ &\le \theta(r_n) + \left|\frac{1}{m} \sum_{k=0}^{m-1} \varphi(x_n(k+s)) - \frac{1}{p_n} \sum_{k=0}^{p_n-1} \varphi(x_n(k))\right| + \delta_n,\end{aligned}$$

where $\delta_n \to 0$ as $n \to \infty$. Choosing an appropriate $n$, we can make the first and last terms sufficiently small for any $m$ and $s$. Let us show that if $m$ is sufficiently large, then the quantity

$$\left|\frac{1}{m} \sum_{k=0}^{m-1} \varphi(x_n(k+s)) - \frac{1}{p_n} \sum_{k=0}^{p_n-1} \varphi(x_n(k))\right|$$

is sufficiently small uniformly in $s$ for fixed $n$. Since the sequence $\omega_n$ is $p_n$-periodic, we have

$$\sum_{k=0}^{p_n-1} \varphi(x_n(k+s)) = p_n \overline{\varphi}(\omega_n)$$

for any $s$. Let $m = tp_n + r$, where $t$ is the integer part of the fraction $m/p_n$ and the remainder $r$ satisfies the inequality $0 \le r < p_n$. Then

$$\begin{aligned}\frac{1}{m} \sum_{k=0}^{m-1} \varphi(x_n(k+s)) &= \frac{1}{m}\Biggl(\sum_{k=0}^{p_n-1} \varphi(x_n(k+s)) + \sum_{k=0}^{p_n-1} \varphi(x_n(k+p_n+s)) + \cdots \\ &\quad + \sum_{k=0}^{p_n-1} \varphi(x_n(k+(t-1)p_n+s)) + \sum_{k=0}^{r-1} \varphi(x_n(k+tp_n+s))\Biggr)\end{aligned}$$

$$= \frac{1}{m}\left(tp_n\overline{\varphi}(\omega_n) + \sum_{k=0}^{r-1}\varphi(x_n(k+tp_n+s))\right) = \overline{\varphi}(\omega_n) + \alpha(t),$$

where

$$\alpha(t) = \frac{-r}{tp_n+r}\,\overline{\varphi}(\omega_n) + \frac{1}{tp_n+r}\sum_{k=0}^{r-1}\varphi(x_n(k+tp_n+s))).$$

Taking into account the boundedness of the function $\varphi$ ($|\varphi| < C$) and the inequality $r < p_n$, we obtain the estimate $|\alpha(t)| \le 2C/(t-1)$. Thus, we have shown that, for any $\varepsilon > 0$, one can choose $n$ for which $\theta(r_n) + \delta_n \le 2\varepsilon/3$ and then choose $t^*$ so that $2C/(t^*-1) \le \varepsilon/3$. Then

$$\left|\frac{1}{m}\sum_{k=0}^{m-1}\varphi(f^{k+s}(p)) - \int_M \varphi\, d\mu\right| \le \varepsilon$$

for $m > t^*p_n$ uniformly in $s$. In this case, by Theorem 6, the limit set of the sequence $\omega_n$ coincides with the closure of the recurrent trajectory Tr and is a minimal set with unique ergodic measure $\mu$. The proof of the theorem is complete. □

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## REFERENCES


1. M. Shub, *Asterisque*. Vol. 56: *Stabilité globale de systems dynamiques* (Soc. Math. France, Paris, 1978).
2. G. Osipenko, *Lectures Notes in Math*. Vol. 1889: *Dynamical Systems, Graphs, and Algorithms* (Springer, Berlin, 2007).
3. A. Katok and B. Hasselblat, *Introduction to the Modern Theory of Dynamical Systems* (Cambridge Univ. Press, Cambridge, 1995).
4. G. S. Osipenko, "The spectrum of the averaging of a function over pseudotrajectories of a dynamical system," Sb. Math. **209** (8), 1211–1233 (2018).
5. V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations* in *Princeton Math. Ser.* (Princeton Univ. Press, Princeton, NJ, 1960), Vol. 22.
6. V. M. Alekseev, *Symbolic dynamics. The Eleventh Mathematical School* (Izd. Inst. Matem. AH USSR, Kiev, 1976) [in Russian].
7. D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding* (Cambridge Univ. Press, Cambridge, 1995).
8. C. Robinson, *Dynamical Systems. Stability, Symbolic Dynamics and Chaos* (CRC Press, Boca Raton, FL, 1995).
9. C. S. Hsu, *Cell-to-Cell Mapping. A Method of Global Analysis for Nonlinear Systems* (Springer, New York, 1987).
10. G. S. Osipenko, "On the symbolic image of a dynamical system," in *Boundary Problems* (Perm, 1983), pp. 101–105 [in Russian].
11. G. Osipenko, "Symbolic images and invariant measures of dynamical systems," Ergodic Theory Dynam. Systems **30** (4), 1217–1237 (2010).
12. G. S. Osipenko, "Lyapunov exponents and invariant measures on a projective bundle," Math. Notes **101** (4), 666–676 (2017).
13. J. C. Oxtoby, "Ergodic sets," Bull. Amer. Math. Soc. **58** (2), 116–136 (1953).