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Riesz Potential with Integrable Density in Hölder-Variable Spaces

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Abstract—Boundedness conditions for spherical and spatial variable-order Riesz potential-type operators with integrable density in variable-exponent Hölder spaces are proved.

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1. INTRODUCTION

Consider a variable-order Riesz potential-type operator on a metric space (Ω, r), where Ω is the hypersurface in the multidimensional space $\mathbb{R}^n$, $n \geq 3$, of vectors with real coordinates:

$$(I_\Omega^{\alpha(\,\cdot\,)} f)(x) = \int_\Omega \frac{c(x,\sigma) f(\sigma)}{r^{n-\alpha(x)-1}(x,\sigma)}\, d\sigma, \qquad x \in \Omega; \tag{1.1}$$

the function $c(x,\sigma)$ is called the *characteristic* [1, p. 18].

For the set Ω, we first consider the hypersphere of radius 1 centered at the origin:

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}, \qquad |\mathbb{S}^{n-1}| = 2\pi^{(n-1)/2}\Gamma^{-1}\biggl(\frac{n-1}{2}\biggr),$$

which is metrized by the Euclidean metric denoted by the symbol r:

$$r := |x-y| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}, \qquad |x| = |x-0|.$$

Along with $(\mathbb{S}^{n-1}, r)$, we consider the metric space $(\dot{\mathbb{R}}^{n-1}, r_*)$ with metric (3.3), where the symbol $\dot{\mathbb{R}}^{n-1}$ denotes the one-point compactification of $\mathbb{R}^{n-1}$, which is a hyperplane of $\mathbb{R}^n$.

The purpose of the present paper is to confirm the form of the exponent $\lambda(x)$ in the definition of a variable-exponent Hölder space $H^{\lambda(\,\cdot\,)}$, which was found in [2] for a constant order α, and determine conditions on $\alpha(x)$ ensuring the boundedness of the operators $I_{\mathbb{S}^{n-1}}^{\alpha(\,\cdot\,)}$ with constant characteristic and $I_{\dot{\mathbb{R}}^{n-1}}^{\alpha(\,\cdot\,)}$ with characteristic of special form (3.6) under the mapping of a function from L^p and, in the spatial case, from L^p with weight (3.7) to $H^{\lambda(x)}$.

Let us recall that, in the case of a constant α, the Riesz potential on the sphere is a spherical convolution operator and, for its study in Hölder spaces, one can successfully apply an approach involving the Fourier–Laplace multiplier theory. Thus, an analog on the sphere of the result concerning the reflection of a potential-type operator from L^p to $H^{\alpha-n/p}$ (known for domains in $\mathbb{R}^n$ [3, p. 251]) was established in [2]. In the same paper, spaces with weights of the form $w(x) = |x-\sigma|^\mu$ and $w(x) = |x-e_n|^\mu |x+e_n|^\beta$ were studied for the general class of potential-type operators defined in terms of Fourier–Laplace multipliers.

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The paper [2] was preceded by [4], in which a similar result was obtained for a spherical potential with multiplier

$$\left\{\frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)}\right\}_{m=0}^{\infty},$$

while the spherical Riesz potential has the multiplier [5]

$$\left\{\frac{\Gamma(m+(n-1)/2-\alpha/2)}{\Gamma(m+(n-1)/2+\alpha/2)}\right\}_{m=0}^{\infty}.$$

The images of a complex-order spherical potential-type operator and of spherical convolutions with kernels depending on the inner product and having a multiplier over spherical harmonics with given asymptotics at infinity in unweighteded variable-exponent Hölder spaces and in the case of exponential weight were studied in [6]. Moreover, theorems on the action of these operators were used to construct isomorphisms of these spaces. The results of [6] were developed further in [7], where they were carried over to the case of a spatial potential by applying stereographic projection. The present paper also uses this approach. It was used earlier in [8] and has led to the proof of a homeomorphism realized by the spatial Riesz potential for generalized Hölder classes.

One will also be interested in the survey [9] dealing with the study of various potentials in generalized variable-exponent Hölder spaces and with applications of the results obtained.

It is important to note that, for the study of variable-order potential-type operators, the theory of multipliers is no longer meaningful. Indeed, in these cases, such operators are not spherical convolution operators and, therefore, multipliers cannot be calculated.

Zygmund-type estimates were used in [10] to study the images of a spherical potential (as well as those of a hypersingular operator) of variable order $\alpha(x)$; the use of this method resulted in finding boundedness conditions for a mapping from the generalized variable-exponent Hölder space with characteristic $\omega(x,h)$ to the same space, but with "better" characteristic $\omega_\alpha(x,h)=h^{\operatorname{Re}\alpha(x)}\omega(x,h)$ and weight $\alpha(x)$. These results served as a basis for proving the boundedness of spatial operators in [11]. Earlier, Zygmund-type estimates constituted an apparatus for studying fractional integro-differential operators of variable order in the one-dimensional case [12], [13].

In the present paper, the majorant of the local modulus of continuity of a potential was constructed by using a special partition of the domain of integration and a subsequent application of inequalities of classical integral calculus and functional analysis.

It should also be noted that issues on Hölder spaces with variable exponent, namely, the action of fractional integro-differential operators on such spaces, were studied further in the papers [14], [15] of Ginzburg and Karapetyants, the paper [16] of Ross and Samko, and papers of other authors. We also refer the reader to the papers [17]–[19], which made a significant contribution to the development of the theory of integro-differential operators of variable order.

1.1. Definitions of Function Spaces

Let us recall the definitions of function spaces considered in what follows.

Definition 1. Let $w(x)$ be a weight on Ω. By the *symbol* $L^p(\Omega,w)$ we shall denote the function space defined by the norm

$$\|f\|_{L^p(\Omega,w)}=\left\{\int_\Omega w(x)|f(x)|^p\,dx\right\}^{1/p},\qquad 1\le p<\infty.$$

We understand the classical unweighted space $L^p(\Omega)$ as the particular case of this definition for $w(x)\equiv 1$.

Below we recall Hölder's inequality, which is widely used in the proof of the main result.

Lemma 1. *Let* $1 \le p \le \infty$, $p' = p/(p-1)$, *and let* $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$. *Then*

$$\int_\Omega |f(x)g(x)|\,dx \le \|f\|_{L^p(\Omega)}\|g\|_{L^{p'}(\Omega)}. \tag{1.2}$$

Let us proceed to the definitions of Hölder function spaces. The set of continuous functions on Ω will be denoted by the symbol $\mathrm{C}(\Omega)$.

Definition 2. By $\mathrm{Lip}(\Omega)$ we denote the space of functions $f \in \mathrm{C}(\Omega)$ satisfying the Lipschitz condition

$$\forall\, x, y \in \Omega \qquad |f(x) - f(y)| \le C|x-y|, \quad 0 < C < \infty.$$

We note that any function in $\mathrm{Lip}(\Omega)$ obviously satisfies the weak Lipschitz condition

$$f \in \mathrm{Lip}(\Omega) \qquad \Longrightarrow \qquad \forall\, x, y \in \Omega \quad |f(x) - f(y)| \le \frac{C}{\ln(1/|x-y|)}, \quad 0 < C < \infty. \tag{1.3}$$

We define a variable-exponent Hölder space on the basis of the notion of the local modulus of continuity of a function.

Definition 3. Let ρ be a metric on Ω. By the *local modulus of continuity* of a function f we mean the functional

$$M_\rho(f, x, t) = \sup_{y \in \Omega: \rho(x,y) \le t} |f(x) - f(y)|$$

defined for all $t > 0$ and $x \in \Omega$.

Definition 4. Let (Ω, ρ) be a metric space. By $H^{\lambda(\,\cdot\,)}(\Omega)$ we denote the set of functions $f \in \mathrm{C}(\Omega)$ satisfying the generalized Hölder condition with variable characteristic and having the form

$$\forall\, x \in \Omega,\ t < 1 \qquad M_\rho(f, x, t) \le Ct^{\lambda(x)}, \quad 0 < C < \infty.$$

1.2. Preliminaries

Let us recall the Catalan formula, which is a particular case of the Funk–Hecke formula [1, p. 20]:

$$\int_{\mathbb{S}^{n-1}} f(x \cdot \sigma)\,d\sigma = |\mathbb{S}^{n-2}| \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2}\,dt, \qquad x \in \mathbb{S}^{n-1}, \tag{1.4}$$

where $x \cdot \sigma := x_1 \cdot \sigma_1 + \cdots + x_n \cdot \sigma_n$ is inner product in $\mathbb{R}^n$. As in the proof of (1.4), we can calculate and estimate integrals of the form

$$J(a, b, x) = \int_{a < |x-\sigma| < b} g(|x-\sigma|, x)\,d\sigma, \qquad x \in \mathbb{S}^{n-1}, \tag{1.5}$$

where $0 \le a < b \le 2$; these integrals turn out to be independent of the variable x by virtue of the well-known representation

$$|x - \sigma| = \sqrt{2}\sqrt{1 - x \cdot \sigma}$$

(see also (1.4)). In particular, the following statement is valid.

Lemma 2. *Let* $n \ge 2$. *Then*

$$J(a, b, x) = 2^{3-n}|\mathbb{S}^{n-2}| \int_a^b g(u, x)u^{n-2}(4-u^2)^{(n-3)/2}\,du \tag{1.6}$$

and, for $n \ge 3$,

$$J(a, b, x) \le |\mathbb{S}^{n-2}| \int_a^b g(u, x)u^{n-2}\,du. \tag{1.7}$$

Lemma 2 is a direct analog of formula (1.4). It has already been used in the theory of spherical potential-type operators and hypersingular integrals. In particular, it was used in [6] in the study of such operators of constant order on Hölder spaces with variable exponent.

Along with this lemma, we shall also need the following corollary, which can be verified directly as well.

Corollary 1. *Let* $0 \le a < b \le 2$. *Then*

$$\int_{a<|x-\sigma|<b} g(|x-\sigma|, y)\, d\sigma = \int_{a<|y-\sigma|<b} g(|y-\sigma|, y)\, d\sigma, \qquad x, y \in \mathbb{S}^{n-1}. \tag{1.8}$$

Another widely used result is as follows.

Lemma 3. *Let* $x, y \in \mathbb{S}^{n-1}$, $|x-\sigma| \ge 2|x-y|$, *and let* $\gamma \ge -1$. *Then*

$$\big||x-\sigma|^{-\gamma} - |y-\sigma|^{-\gamma}\big| \le 2^{\gamma+1}|\gamma| \frac{|x-y|}{|x-\sigma|^{\gamma+1}}\,. \tag{1.9}$$

This inequality, together with its proof, can be found, for example, in [6], [10].

We shall also need the formulas

$$u^a - u^b = (a-b)u^a \ln u \int_0^1 u^{s(b-a)}\, ds, \tag{1.10}$$

$$|t^{\alpha(x)-\alpha(y)} - 1| \le 2^{1+\alpha_+} \frac{|\alpha(x)-\alpha(y)|}{t^{\alpha(y)-\alpha(x)+1}}\,, \tag{1.11}$$

where α_+ is the supremum of the function α on the set under consideration. Both formulas are consequences of the known representation

$$f(a) - f(b) = (a-b)\int_0^1 f'(a + s(b-a))\, ds.$$

2. THE RIESZ POTENTIAL ON THE SPHERE

Let a function $\alpha\colon \mathbb{S}^{n-1} \to \mathbb{R}$ satisfy the conditions

$$\forall\, x \in \mathbb{S}^{n-1} \quad 0 \le \alpha(x) \le 1, \qquad |\{x \in \mathbb{R}^n : \alpha(x) = 0\}| = 0.$$

For the spherical Riesz potential of variable order with density f,

$$(I^{\alpha(\cdot)}_{\mathbb{S}^{n-1}} f)(x) = \int_{\mathbb{S}^{n-1}} \frac{f(y)}{|x-y|^{n-1-\alpha(x)}}\, dy, \qquad x \in \mathbb{S}^{n-1}, \tag{2.1}$$

we prove the following boundedness theorem.

Theorem 1. *Let* $n \ge 3$, $1 < p < \infty$, $1/p + 1/p' = 1$, *and let the following conditions hold*:

1) $\alpha \in \mathrm{Lip}(\mathbb{S}^{n-1})$;

2) $\alpha_- := \inf_{x\in\mathbb{S}^{n-1}} \alpha(x) > (n-1)/p$;

3) $\alpha_+ := \sup_{x\in\mathbb{S}^{n-1}} \alpha(x) < (n-1)/p + 1$.

Then the operator $I^{\alpha(\cdot)}_{\mathbb{S}^{n-1}}$ *is bounded as an operator from* $L^p(\mathbb{S}^{n-1})$ *to* $H^{\lambda(\cdot)}(\mathbb{S}^{n-1})$, *where* $\lambda(x) = \alpha(x) - (n-1)/p$.

Proof. In what follows, we assume that $0 < |x-y| \le h < 1$ and, to be definite, that $\alpha(y) \ge \alpha(x)$. We have the following representation:

$$\begin{aligned}
&|(I^{\alpha(\,\cdot\,)}_{\mathbb{S}^{n-1}} f)(x) - (I^{\alpha(\,\cdot\,)}_{\mathbb{S}^{n-1}} f)(y)| \\
&\quad = \left| \int_{\mathbb{S}^{n-1}} \frac{f(\sigma)\,d\sigma}{|x-\sigma|^{n-1-\alpha(x)}} - \int_{\mathbb{S}^{n-1}} \frac{f(\sigma)\,d\sigma}{|y-\sigma|^{n-1-\alpha(y)}} \right| \\
&\quad \le \left| \int_{\mathbb{S}^{n-1}_{<}} \frac{f(\sigma)\,d\sigma}{|x-\sigma|^{n-1-\alpha(x)}} + \int_{\mathbb{S}^{n-1}_{\ge}} \frac{f(\sigma)\,d\sigma}{|x-\sigma|^{n-1-\alpha(y)}} - \int_{\mathbb{S}^{n-1}_{\ge}} \frac{f(\sigma)\,d\sigma}{|x-\sigma|^{n-1-\alpha(y)}} \right. \\
&\qquad \left. + \int_{\mathbb{S}^{n-1}_{\ge}} \frac{f(\sigma)\,d\sigma}{|x-\sigma|^{n-1-\alpha(y)}} - \int_{\mathbb{S}^{n-1}_{<}} \frac{f(\sigma)\,d\sigma}{|y-\sigma|^{n-1-\alpha(y)}} - \int_{\mathbb{S}^{n-1}_{\ge}} \frac{f(\sigma)\,d\sigma}{|y-\sigma|^{n-1-\alpha(y)}} \right| \\
&\quad \le I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where the following notation is used:

$$\begin{gathered}
\mathbb{S}^{n-1}_{<} := \{\sigma \in \mathbb{S}^{n-1} : 0 < |x-\sigma| \le h\}, \qquad \mathbb{S}^{n-1}_{\ge} := \{\sigma \in \mathbb{S}^{n-1} : 0 < h < |x-\sigma|\}, \\
I_1 := \int_{\mathbb{S}^{n-1}_{<}} |f(\sigma)|\,|x-\sigma|^{\alpha(x)-n+1}\,d\sigma, \qquad I_2 := \int_{\mathbb{S}^{n-1}_{<}} |f(\sigma)|\,|y-\sigma|^{\alpha(y)-n+1}\,d\sigma, \\
I_3 := \int_{\mathbb{S}^{n-1}_{\ge}} |f(\sigma)|\,\big||x-\sigma|^{\alpha(x)-n+1} - |x-\sigma|^{\alpha(y)-n+1}\big|\,d\sigma, \\
I_4 := \int_{\mathbb{S}^{n-1}_{\ge}} |f(\sigma)|\,\big||x-\sigma|^{\alpha(y)-n+1} - |y-\sigma|^{\alpha(y)-n+1}\big|\,d\sigma.
\end{gathered}$$

Estimate of I_1. We first use Hölder's inequality (1.2) and then inequality (1.7), obtaining

$$\begin{aligned}
I_1 &:= \int_{\mathbb{S}^{n-1}_{<}} \left| \frac{f(\sigma)}{(x-\sigma)^{n-\alpha(x)-1}} \right| d\sigma \le \|f\|_{L^p(\mathbb{S}^{n-1}_{<})} \left\{ \int_{\mathbb{S}^{n-1}_{<}} |x-\sigma|^{(\alpha(x)-n+1)p'}\,d\sigma \right\}^{1/p'} \\
&\le \|f\|_{L^p(\mathbb{S}^{n-1}_{<})} \left\{ |\mathbb{S}^{n-2}| \int_0^h u^{(\alpha(x)-n)p'+n-1}\,du \right\}^{1/p'}.
\end{aligned}$$

The last integral converges if and only if $\forall x \in \mathbb{S}^{n-1}_{<} \alpha(x) > (n-1)/p$, which corresponds to condition 2) in the statement of the theorem.

Thus,

$$I_1 \le |\mathbb{S}^{n-2}|^{1/p'} \|f\|_{L^p} \frac{h^{\alpha(x)-(n-1)/p}}{\{(\alpha(x)-n+1)p'+n-1\}^{1/p'}} \le C_\alpha \|f\|_{L^p(\mathbb{S}^{n-1})} |x-y|^{\alpha(x)-(n-1)/p},$$

where

$$0 < C_\alpha := |\mathbb{S}^{n-2}|^{1/p'} \left(\alpha_- - \frac{n-1}{p} \right)^{-1/p'} < \infty.$$

Estimate of I_2. We note that $\{\sigma : |x-\sigma| < h\} \subset \{\sigma : |y-\sigma| < 2h\}$. Therefore, again using Hölder's inequality (1.2) and estimate (1.7), we can write

$$\begin{aligned}
I_2 &\le \|f\|_{L^p(\mathbb{S}^{n-1}_{<})} \left\{ \int_{|x-\sigma|<h} |y-\sigma|^{p'(\alpha(y)-n+1)} \right\}^{1/p'} \\
&\le \|f\|_{L^p(\mathbb{S}^{n-1}_{<})} \left\{ \int_{|y-\sigma|<2h} |y-\sigma|^{p'(\alpha(y)-n+1)} \right\}^{1/p'} \le C|x-y|^{\alpha(y)-(n-1)/p}.
\end{aligned}$$

Since the function α satisfies condition 3) of the theorem, it will obviously satisfy the weak Lipschitz condition (1.3), and hence

$$\begin{gathered}
\forall\, x, y, \sigma \in \mathbb{S}^{n-1}\colon |x-\sigma| \le |x-y|, \\
|x-\sigma|^{\alpha(y)-\alpha(x)} \le |x-y|^{|\alpha(x)-\alpha(y)|} \le |x-y|^{C/\ln(1/|x-y|)} = e^{-C} =: C.
\end{gathered}$$

Therefore,

$$I_2 \le C\|f\|_{L^p(\mathbb{S}^{n-1})}|x-y|^{\alpha(x)-(n-1)/p}, \qquad 0<C<\infty.$$

Estimate of I_3**.** Let $x,y,\sigma \in \mathbb{S}^{n-1} : |x-y|<|x-\sigma|$. Using inequalities (1.10) and (1.11), we obtain

$$\begin{aligned}\big||x-\sigma|^{\alpha(x)-n+1}-|x-\sigma|^{\alpha(y)-n+1}\big| &= |x-\sigma|^{\alpha(x)-n+1}\big||x-\sigma|^{\alpha(y)-\alpha(x)}-1\big| \\ &\le Ch|x-\sigma|^{\alpha(y)-n}.\end{aligned}$$

Now, by virtue of (1.3) and (1.8), applying Hölder's inequality (1.2), we obtain

$$\begin{aligned}I_3 &\le Ch\|f\|_{L^p(\mathbb{S}^{n-1}_{\ge})}\bigg\{\int_{h<|y-\sigma|<2}|y-\sigma|^{p'(\alpha(y)-n)}\,d\sigma\bigg\}^{1/p'} \\ &\le Ch\|f\|_{L^p(\mathbb{S}^{n-1})}\bigg\{\int_h^2 u^{p'(\alpha(y)-n)+n-2}\,du\bigg\}^{1/p'}.\end{aligned}$$

Evaluating the last integral, we arrive at the following estimate:

$$\begin{aligned}h\bigg\{\frac{2^{p'(\alpha(y)-n)+n-1}-h^{p'(\alpha(y)-n)+n-1}}{p'(\alpha(y)-n)+n-1}\bigg\}^{1/p'} &\le Ch\{h^{p'(\alpha(y)-n)+n-1}-2^{p'(\alpha(y)-n)+n-1}\}^{1/p'} \\ &\le Ch\{h^{\alpha(y)-n+(n-1)/p'}+2^{\alpha(y)-n+(n-1)/p'}\} \\ &= Ch^{\alpha(y)-(n-1)/p}\{(2h)^{\alpha(y)-n/p-1/p'}+1\} \\ &\le Ch^{\alpha(y)-(n-1)/p},\end{aligned}$$

where

$$0<C<\frac{1}{1-n-p'(\alpha_- - n)} =: \kappa, \qquad \kappa>1.$$

Therefore, just as in the final argument in estimating I_2,

$$I_3 \le C\|f\|_{L^p(\mathbb{S}^{n-1})}|x-y|^{\alpha(x)-(n-1)/p}.$$

Estimate of I_4**.** Successively using estimates (1.2) and (1.9), we reduce estimating I_4 to estimating I_3:

$$\begin{aligned}I_4 &:= \int_{\mathbb{S}^{n-1}_{\ge}}|f(\sigma)|\big||x-\sigma|^{\alpha(y)-n+1}-|y-\sigma|^{\alpha(y)-n+1}\big|\,d\sigma \\ &\le C\|f\|_{L^p(\mathbb{S}^{n-1})}\bigg\{\int_{\mathbb{S}^{n-1}_{\ge}}\big||x-\sigma|^{\alpha(y)-n+1}-|y-\sigma|^{\alpha(y)-n+1}\big|^{p'}\,d\sigma\bigg\}^{1/p'} \\ &\le Ch\|f\|_{L^p(\mathbb{S}^{n-1})}\bigg\{\int_{h\le|x-\sigma|\le 2}|x-\sigma|^{p'(\alpha(y)-n)}\bigg\}^{1/p'} \\ &\le C\|f\|_{L^p(\mathbb{S}^{n-1})}|x-y|^{\alpha(x)-(n-1)/p}.\end{aligned}$$

Conclusion. Thus, due to the arbitrary choice of $|x-y| \le h$, the summation of the majorants of each of the summands I_i, $i = 1, \dots, 4$ leads to the estimate

$$M_r(I^{\alpha(\cdot)}_{\mathbb{S}^{n-1}} f, x, h) \le C\|f\|_{L^p(\mathbb{S}^{n-1})}|x-y|^{\alpha(x)-(n-1)/p}.$$

By definition 4, this means that the operator $I^{\alpha(\cdot)}_{\mathbb{S}^{n-1}}$ is bounded as an operator from L^p to $H^{\lambda(\cdot)}$. □

3. THE SPATIAL POTENTIAL

Let us recall that by the *stereographic projection* we mean the transformation of Euclidean space $\dot{\mathbb{R}}^{n-1}$ into the unit sphere $\mathbb{S}^{n-1}$ defined by the formulas [20, p. 35]

$$\xi_k = \frac{2x_k}{|x|^2+1}, \qquad k = 1, 2, \dots, n-1, \tag{3.1}$$

$$\xi_n = \frac{|x|^2-1}{|x|^2+1}, \qquad \xi \in \mathbb{S}^{n-1}, \quad x \in \dot{\mathbb{R}}^{n-1}. \tag{3.2}$$

This transformation yields the following formulas relating the metrics on $\mathbb{S}^{n-1}$ and $\dot{\mathbb{R}}^{n-1}$ to the volume elements $d\sigma$ and dy [20, pp. 36–37]:

$$|\xi - \sigma| = \frac{2|x-y|}{\sqrt{|x|^2+1}\sqrt{|y|^2+1}} = r_*, \tag{3.3}$$

$$d\sigma = 2^{n-1}(|y|^2+1)^{1-n}\,dy, \tag{3.4}$$

where the points x and y of the space $\dot{\mathbb{R}}^{n-1}$ are, respectively, the preimages of the points ξ and σ of the sphere $\mathbb{S}^{n-1}$ under the stereographic projection.

The application of the stereographic projection allows us to obtain, on the basis of Theorem 1, boundedness conditions for a spatial potential-type operator under the mapping of a function from the space L^p with weight that naturally arises from formula (3.4) under the change of coordinates.

Theorem 2. *Let $n \ge 3$, $1 < p < \infty$, $1/p + 1/p' = 1$, and let the following conditions hold*:

1) $\alpha \in \mathrm{Lip}(\dot{\mathbb{R}}^{n-1})$;

2) $\inf_{x\in\dot{\mathbb{R}}^{n-1}} \alpha(x) > (n-1)/p$;

3) $\sup_{x\in\dot{\mathbb{R}}^{n-1}} \alpha(x) < (n-1)/p + 1$.

Then the operator

$$(I^{\alpha(\cdot)}_{\dot{\mathbb{R}}^{n-1}} f)(x) = \int_{\dot{\mathbb{R}}^{n-1}} \frac{c_0(x,y) f(y)}{|x-y|^{n-\alpha(x)-1}}\,dy \tag{3.5}$$

with characteristic

$$c_0(x,y) = 2^{\alpha(x)}(1+|x|^2)^{(n-\alpha(x)-1)/2}(1+|y|^2)^{(1-\alpha(x)-n)/2} \tag{3.6}$$

is bounded as an operator from $L^p(\dot{\mathbb{R}}^{n-1}, w_0)$ to $H^{\lambda(\cdot)}(\dot{\mathbb{R}}^{n-1})$, where $\lambda(x) = \alpha(x) - (n-1)/p$ and the weight is

$$w_0^{1/p}(x) = 2^{n-1}(|x|^2+1)^{1-n}. \tag{3.7}$$

Proof. Let Π be the vector function implementing the change of coordinates by formulas (3.1) and (3.2), and let Π^{-1} be its inverse function. By virtue of the bijectiveness of Π, the stereographic projection preserves its Lipschitz continuity, as well as the supremum and infimum of the function α.

Let the functions with tilde be the components of the superposition of the original functions with Π^{-1}. Applying formula (3.4), we can easily verify the following equality:

$$\|\widetilde{f}\|_{L^p(\mathbb{S}^{n-1})} = \|f\|_{L^p(\dot{\mathbb{R}}^{n-1},w_0)} = \|w_0^{1/p} f\|_{L^p(\dot{\mathbb{R}}^{n-1})}.$$

For the potential, the following representation holds:

$$\int_{\mathbb{S}^{n-1}} \frac{\widetilde{f}(\sigma)}{|\xi-\sigma|^{n-\widetilde{\alpha}(\xi)-1}}\,d\sigma = 2^{\alpha(x)}(1+|x|^2)^{(n-\alpha(x)-1)/2} \times \int_{\dot{\mathbb{R}}^{n-1}} \frac{f(y)}{|x-y|^{n-\alpha(x)-1}(1+|y|^2)^{(n-1+\alpha(x))/2}}\,dy = \int_{\dot{\mathbb{R}}^{n-1}} \frac{c_0(x,y)f(y)}{|x-y|^{n-\alpha(x)-1}}\,dy.$$

Therefore, the local moduli of continuity of the corresponding operators coincide:

$$M_r(I^{\alpha(\,\cdot\,)}_{\dot{\mathbb{R}}^{n-1}} f,t,x) = M_{r_*}(I^{\widetilde{\alpha}(\,\cdot\,)}_{\mathbb{S}^{n-1}} \widetilde{f},t,\xi),$$

and hence their seminorms in the spaces $H^{\widetilde{\lambda}(\,\cdot\,)}(\mathbb{S}^{n-1})$ and $H^{\lambda(\,\cdot\,)}(\dot{\mathbb{R}}^{n-1})$ are also equal.

Thus, by virtue of the bijectiveness of the stereographic projection, the conditions of Theorem 2 are equivalent to the conditions of Theorem 1 for $I^{\widetilde{\alpha}(\,\cdot\,)}_{\mathbb{S}^{n-1}}$, which concludes the proof of the theorem. □

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