ISSN 0001-4346, Mathematical Notes, 2020, Vol. 108, No. 2, pp. 179–187. © Pleiades Publishing, Ltd., 2020.

# Nonlinear Triple Product $A^*B + B^*A$ for Derivations on $*$-Algebras*

**Vahid Darvish[1]**, Mojtaba Nouri[2]***, and Mehran Razeghi[2]******

[1] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044 China*

[2] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, 47416-1468 Iran*

Received September 21, 2019; in final form, February 27, 2020; accepted March 18, 2020

**Abstract**—Let $\mathcal{A}$ be a prime $*$-algebra. In this paper, assuming that $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\Phi(A \diamond B \diamond C) = \Phi(A) \diamond B \diamond C + A \diamond \Phi(B) \diamond C + A \diamond B \diamond \Phi(C)$$

where $A \diamond B = A^*B + B^*A$ for all $A, B \in \mathcal{A}$, we prove that $\Phi$ is additive an $*$-derivation.

**DOI:** 10.1134/S0001434620070196

Keywords: *triple product derivation, prime $*$-algebra, additive map.*

## 1. INTRODUCTION

Let $\mathcal{R}$ be a $*$-algebra. For $A, B \in \mathcal{R}$, we write $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$ for the $*$-Jordan product and $*$-Lie product, respectively. These products play an important role in some research topics, and their study has recently attracted the attention of many authors (for example, see [1]–[5]).

Recall that a map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be an additive derivation if

$$\Phi(A + B) = \Phi(A) + \Phi(B) \quad \text{and} \quad \Phi(AB) = \Phi(A)B + A\Phi(B)$$

for all $A, B \in \mathcal{R}$. A map $\Phi$ is an additive $*$-derivation if it is an additive derivation and $\Phi(A^*) = \Phi(A)^*$. Derivations are very important maps both in theory and applications and have been studied intensively ([6]–[11]).

A *von Neumann algebra* $\mathcal{A}$ is a self-adjoint subalgebra of $B(H)$, the algebra of all bounded linear operators acting on a complex Hilbert space, which satisfies the double commutant property: $\mathcal{A}'' = \mathcal{A}$ where $\mathcal{A}' = \{T \in B(H), TA = AT\}$ for all $A \in \mathcal{A}$, and $\mathcal{A}'' = \{\mathcal{A}'\}'$. We denote by $\mathcal{Z}(\mathcal{A}) = \mathcal{A}' \cap \mathcal{A}$ the center of $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ is called a *factor* if its center is trivial, i.e., $\mathcal{Z}(\mathcal{A}) = \mathbb{C}I$. For $A \in \mathcal{A}$, recall that the *central carrier* of $A$, denoted by $\overline{A}$, is the smallest central projection $P$ such that $PA = A$. It is not difficult to see that $\overline{A}$ is the projection onto the closed subspace spanned by $\{BAx : B \in \mathcal{A}, x \in H\}$. If $A$ is self-adjoint, then the core of $A$, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{A}) : S = S^*, S \leq A\}$. If $A = P$ is a projection, it is clear that $\underline{P}$ is the largest central projection $Q$ satisfying $Q \leq P$. A projection $P$ is said to be *core-free* if $\underline{P} = 0$ (see [12]). It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$, [13, 14].

Recently, Yu and Zhang in [15] proved that every nonlinear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, in [16], Li, Lu, and Fang investigated nonlinear $\lambda$-Jordan $*$-derivations. They showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central Abelian projections and $\lambda$ is a nonzero scalar, then $\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a nonlinear $\lambda$-Jordan $*$-derivation if and only if $\Phi$ is an additive $*$-derivation.

*The article was submitted by the authors for the English version of the journal.

**E-mail: vahid.darvish@mail.com

***E-mail: mojtaba.nori2010@gmail.com

****E-mail: razeghi.mehran19@yahoo.com

On the other hand, many mathematicians have studied the $*$-Jordan product $A \bullet B = AB + BA^*$. In [17], F. Zhang proved that every nonlinear $*$-Jordan derivation map $\Phi : \mathcal{A} \to \mathcal{A}$ on a factor von Neumann algebra is an additive $*$-derivation.

In [18], we showed that $*$-Jordan derivation map on every factor von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is an additive $*$-derivation.

Quite recently, the authors of [19] discussed some bijective maps preserving the new product $A^*B + B^*A$ between von Neumann algebras with no central Abelian projections. In other words, they considered the map $\Phi$ that satisfies the following assumption:

$$\Phi(A^*B + B^*A) = \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A).$$

They showed that such a map is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.

We say that $\mathcal{A}$ is *prime*, i.e., if $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$, then $A = 0$ or $B = 0$.

In [20], we assumed that $\mathcal{A}$ is a prime $*$-algebra and the map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies the following condition:

$$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B)$$

where $A \diamond B = A^*B + B^*A$ for all $A, B \in \mathcal{A}$. We proved that, in this case, $\Phi$ is an additive $*$-derivation.

The authors of [21] introduced the concept of $*$-Lie triple derivations. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is a *nonlinear $*$-Lie triple derivation* if

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$, where $[A, B]_* = AB - BA^*$. They showed that if $\Phi$ preserves the above characterization of factor von Neumann algebras, then $\Phi$ is an additive $*$-derivation.

Motivated by the above results, we introduce the triple product $A \diamond B \diamond C := (A \diamond B) \diamond C$, where $A \diamond B = A^*B + B^*A$. In this paper, let $\mathcal{A}$ be a prime $*$-algebra, and let $\Phi : \mathcal{A} \to \mathcal{A}$ satisfy the following equality:

$$\Phi(A \diamond B \diamond C) = \Phi(A) \diamond B \diamond C + A \diamond \Phi(B) \diamond C + A \diamond B \diamond \Phi(C)$$

for all $A, B, C \in \mathcal{A}$. We prove that $\Phi$ is an additive $*$-derivation.

## 2. MAIN RESULTS

Our main theorem is as follows.

**Theorem 1.** *Let $\mathcal{A}$ be a prime $*$-algebra, and let $\Phi : \mathcal{A} \to \mathcal{A}$ satisfy the condition*

$$\Phi(A \diamond B \diamond C) = \Phi(A) \diamond B \diamond C + A \diamond \Phi(B) \diamond C + A \diamond B \diamond \Phi(C) \tag{2.1}$$

*for all $A, B, C \in \mathcal{A}$, then $\Phi$ is an additive $*$-derivation.*

**Proof.** Let $P_1$ be a nontrivial projection in $\mathcal{A}$, and let $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$; then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$, we can write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In what follows, when we write $A_{ij}$, this will indicate that $A_{ij} \in \mathcal{A}_{ij}$. In order to show additivity of $\Phi$ on $\mathcal{A}$, we apply the above partitions of $\mathcal{A}$ and establish some claims that imply that $\Phi$ is additive on each $\mathcal{A}_{ij}$ for $i, j = 1, 2$.

Thus, the above theorem is a consequence of the following claims.

**Claim 1.** $\Phi(0) = 0$.

This claim is easy to prove.

**Claim 2.** $\Phi(I/2) = 0$, $\Phi(-I/2) = 0$, *and* $\Phi(iI/2) = 0$.

To show that $\Phi(I/2)=0$, we write

$$\Phi\left(\frac{I}{2}\diamond\frac{I}{2}\diamond\frac{I}{2}\right)=\Phi\left(\frac{I}{2}\right)\diamond\frac{I}{2}\diamond\frac{I}{2}+\frac{I}{2}\diamond\Phi\left(\frac{I}{2}\right)\diamond\frac{I}{2}+\frac{I}{2}\diamond\frac{I}{2}\diamond\Phi\left(\frac{I}{2}\right).$$

Thus,

$$\Phi\left(\frac{I}{2}\right)=\frac{3}{2}\left(\Phi\left(\frac{I}{2}\right)+\Phi\left(\frac{I}{2}\right)^*\right). \tag{2.2}$$

From (2.2), we deduce that $\Phi(I/2)$ is self-adjoint. Therefore, we have the desired result.

To prove that $\Phi(I/2)=0$, we write

$$\Phi\left(\frac{I}{2}\diamond\frac{I}{2}\diamond-\frac{I}{2}\right)=\frac{I}{2}\diamond\frac{I}{2}\diamond\Phi\left(-\frac{I}{2}\right).$$

It follows that

$$\Phi\left(-\frac{I}{2}\right)=\frac{1}{2}\left(\Phi\left(-\frac{I}{2}\right)+\Phi\left(-\frac{I}{2}\right)^*\right). \tag{2.3}$$

Then

$$\Phi\left(-\frac{I}{2}\right)^*=\Phi\left(-\frac{I}{2}\right). \tag{2.4}$$

On the other hand, we have

$$\Phi\left(\frac{I}{2}\diamond-\frac{I}{2}\diamond-\frac{I}{2}\right)=\frac{I}{2}\diamond\Phi\left(-\frac{I}{2}\right)\diamond\left(-\frac{I}{2}\right)+\frac{I}{2}\diamond-\frac{I}{2}\diamond\Phi\left(-\frac{I}{2}\right)$$

It follows that

$$\Phi\left(-\frac{I}{2}\right)^*=-\Phi\left(\frac{-I}{2}\right) \tag{2.5}$$

Then, from (2.4) and (2.5), we obtain $\Phi(-\frac{I}{2})=0$. To show that $\Phi(i\frac{I}{2})=0$, we write

$$\Phi\left(i\frac{I}{2}\diamond i\frac{I}{2}\diamond\frac{I}{2}\right)=\Phi\left(i\frac{I}{2}\right)\diamond i\frac{I}{2}\diamond\frac{I}{2}+i\frac{I}{2}\diamond\Phi\left(i\frac{I}{2}\right)\diamond\frac{I}{2}.$$

Thus,

$$\Phi\left(i\frac{I}{2}\right)^*-\Phi\left(i\frac{I}{2}\right)=0. \tag{2.6}$$

Also, we have

$$\Phi\left(\frac{I}{2}\diamond\frac{I}{2}\diamond i\frac{I}{2}\right)=\frac{I}{2}\diamond\frac{I}{2}\diamond\Phi\left(i\frac{I}{2}\right).$$

Thus,

$$\Phi\left(i\frac{I}{2}\right)^*+\Phi\left(i\frac{I}{2}\right)=0. \tag{2.7}$$

From (2.6) and (2.7), we obtain $\Phi(i\frac{I}{2})=0$.

**Claim 3.** *Suppose that, for each $A\in\mathcal{A}$,*

1. $\Phi(-iA)=-i\Phi(A)$.
2. $\Phi(iA)=i\Phi(A)$.

It is easy to see that

$$\Phi\Big(-iA\diamond\frac{I}{2}\diamond\frac{I}{2}\Big)=\Phi\Big(A\diamond i\frac{I}{2}\diamond\frac{I}{2}\Big).$$

Thus,

$$\Phi(-iA)\diamond\frac{I}{2}\diamond\frac{I}{2}=\Phi(A)\diamond i\frac{I}{2}\diamond\frac{I}{2}.$$

It follows that

$$\Phi(-iA)^*+\Phi(-iA)=i\Phi(A)^*-i\Phi(A). \tag{2.8}$$

On the other hand, one can check that

$$\Phi\Big(-iA\diamond i\frac{I}{2}\diamond\frac{I}{2}\Big)=\Phi\Big(-\frac{I}{2}\diamond A\diamond\frac{I}{2}\Big).$$

Thus,

$$\Phi(-iA)\diamond i\frac{I}{2}\diamond\frac{I}{2}=-\frac{I}{2}\diamond\Phi(A)\diamond\frac{I}{2}.$$

It follows that

$$i\Phi(-iA)^*-i\Phi(-iA)=-\Phi(A)-\Phi(A)^*. \tag{2.9}$$

Equivalently, we have

$$-\Phi(-iA)^*+\Phi(-iA)=-i\Phi(A)-i\Phi(A)^*. \tag{2.10}$$

By adding equeations (2.8) and (2.10), we obtain

$$\Phi(-iA)=-i\Phi(A).$$

Similarly, we can show that $\Phi(iA)=i\Phi(A)$.

**Claim 4.** *For each $A_{11}\in\mathcal{A}_{11}$, $A_{12}\in\mathcal{A}_{12}$, the following equality holds*:

$$\Phi(A_{11}+A_{12})=\Phi(A_{11})+\Phi(A_{12}).$$

Setting $T=\Phi(A_{11}+A_{12})-\Phi(A_{11})-\Phi(A_{12})$ let us prove that $T=0$. We have

$$\begin{aligned}\Phi(A_{11}+A_{12})\diamond C_{21}\diamond I+(A_{11}+A_{12})\diamond\Phi(C_{21})\diamond I+(A_{11}+A_{12})\diamond C_{21}\diamond\Phi(I)\\
&=\Phi(A_{11}+A_{12}\diamond C_{21}\diamond I)\\
&=\Phi(A_{11}\diamond C_{21}\diamond I)+\Phi(A_{12}\diamond C_{21}\diamond I)\\
&=\Phi(A_{11})\diamond C_{21}\diamond I+A_{11}\diamond\Phi(C_{21})\diamond I+A_{11}\diamond C_{21}\diamond\Phi(I)+\Phi(A_{12})\diamond C_{21}\diamond I\\
&\quad+A_{12}\diamond\Phi(C_{21})\diamond I+A_{12}\diamond C_{21}\diamond\Phi(I)\\
&=(\Phi(A_{11})+\Phi(A_{12}))\diamond C_{21}\diamond I+(A_{11}+A_{12})\diamond\Phi(C_{21})\diamond I+(A_{11}+A_{12})\diamond C_{21}\diamond\Phi(I).\end{aligned}$$

Since $T_{11}+T_{12}+T_{21}+T_{22}$, it follows that

$$T_{22}^*C_{21}+T_{21}^*C_{21}+C_{21}^*T_{22}+C_{21}^*T_{21}=0.$$

So $T_{22}=T_{21}=0$. Similarly, we have

$$\begin{aligned}\Phi(A_{11}+A_{12})\diamond C_{12}\diamond P_1+(A_{11}+A_{12})\diamond\Phi(C_{12})\diamond P_1+(A_{11}+A_{12})\diamond C_{12}\diamond\Phi(P_1)\\
&=\Phi((A_{11}+A_{12})\diamond C_{12}\diamond P_1)\\
&=\Phi(A_{11}\diamond C_{12}\diamond P_1)+\Phi(A_{12}\diamond C_{12}\diamond P_1)\\
&=(\Phi(A_{11})+\Phi(A_{12}))\diamond C_{12}\diamond P_1+(A_{11}+A_{12})\diamond\Phi(C_{12})\diamond P_1+(A_{11}+A_{12})\diamond C_{12}\diamond\Phi(P_1).\end{aligned}$$

Therefore, $T\diamond C_{12}\diamond P_1=0$. So $T_{11}^*C_{12}+C_{12}^*T_{11}=0$. It follows that $T_{11}^*C_{12}=0$. Hence, for all $C\in\mathcal{A}$, we have $T_{11}^*CP_2=0$. Since $\mathcal{A}$ is prime, it follows that $T_{11}=0$. Similarly, we can show that $T_{12}=0$ by applying $P_2$ instead of $P_1$ in the above.

**Claim 5.** *For each* $A_{11}\in\mathcal{A}_{11}, A_{12}\in\mathcal{A}_{12}, A_{21}\in\mathcal{A}_{21}$, *and* $A_{22}\in\mathcal{A}_{22}$,

1. $\Phi(A_{11}+A_{12}+A_{21})=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})$.
2. $\Phi(A_{12}+A_{21}+A_{22})=\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22})$.

*Then*

$$T=\Phi(A_{11}+A_{12}+A_{21})-\Phi(A_{11})-\Phi(A_{12})-\Phi(A_{21})=0.$$

From Claim 4, we obtain

$$\begin{aligned}\Phi(A_{11}+A_{12}+A_{21})\diamond C_{21}\diamond I+(A_{11}+A_{12}+A_{21})\diamond\Phi(C_{21})\diamond I+(A_{11}+A_{12}+A_{21})\diamond C_{21}\diamond\Phi(I)\\ =\Phi(A_{11}+A_{21}+A_{12}\diamond C_{21}\diamond I)\\ =\Phi(A_{11}\diamond C_{21}\diamond I)+\Phi(A_{21}\diamond C_{21}\diamond I)+\Phi(A_{12}\diamond C_{21}\diamond I)\\ =(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21}))\diamond C_{21}\diamond I+(A_{11}+A_{12}+A_{21})\diamond\Phi(C_{21})\diamond I\\ +(A_{11}+A_{12}+A_{21})\diamond C_{21}\diamond\Phi(I).\end{aligned}$$

It follows that $T\diamond C_{21}\diamond I=0$. Since $T=T_{11}+T_{12}+T_{21}+T_{22}$, we have

$$T_{22}^*C_{21}+T_{21}^*C_{21}+C_{21}^*T_{22}+C_{21}^*T_{21}=0.$$

Therefore, $T_{22}=T_{21}=0$. From Claim 4, we obtain

$$\begin{aligned}\Phi(A_{11}+A_{12}+A_{21})\diamond P_1\diamond P_1+(A_{11}+A_{12}+A_{21})\diamond\Phi(P_1)\diamond P_1+(A_{11}+A_{12}+A_{21})\diamond P_1\diamond\Phi(P_1)\\ =\Phi((A_{11}+A_{12}+A_{21})\diamond P_1\diamond P_1)\\ =\Phi(A_{11}\diamond P_1\diamond P_1)+\Phi(A_{12}\diamond P_1\diamond P_1)+\Phi(A_{21}\diamond P_1\diamond P_1)\\ =(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21}))\diamond P_1\diamond P_1+(A_{11}+A_{12}+A_{21})\diamond\Phi(P_1)\diamond P_1\\ +(A_{11}+A_{12}+A_{21})\diamond P_1\diamond\Phi(P_1).\end{aligned}$$

So $T\diamond P_1\diamond P_1=0$ Then $2T_{11}+2T_{11}^*+T_{12}+T_{12}^*=0$. Therefore,

$$T_{12}=0, T_{11}+T_{11}^*=0. \tag{2.11}$$

Using Claim 3 and Claim 4, we obtain

$$\begin{aligned}\Phi(A_{11}+A_{12}+A_{21})\diamond iP_1\diamond I+(A_{11}+A_{12}+A_{21})\diamond\Phi(iP_1)\diamond I+(A_{11}+A_{12}+A_{21})\diamond iP_1\diamond\Phi(I)\\ =\Phi(A_{11}+A_{12}\diamond iP_1\diamond I)+\Phi(A_{21}\diamond iP_1\diamond I)\\ =\Phi(A_{11}\diamond iP_1\diamond I)+\Phi(A_{12}\diamond iP_1\diamond I)+\Phi(A_{21}\diamond iP_1\diamond I)\\ =\Phi(A_{11})\diamond iP_1\diamond I+A_{11}\diamond\Phi(iP_1)\diamond I+A_{11}\diamond iP_1\diamond\Phi(I)\\ +\Phi(A_{12})\diamond iP_1\diamond I+A_{12}\diamond\Phi(iP_1)\diamond I+A_{12}\diamond iP_1\diamond\Phi(I)\\ +\Phi(A_{21})\diamond iP_1\diamond I+A_{21}\diamond\Phi(iP_1)\diamond I+A_{21}\diamond iP_1\diamond\Phi(I)\\ =(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21}))\diamond iP_1\diamond I+(A_{11}+A_{12}+A_{21})\diamond\Phi(iP_1)\diamond I\\ +(A_{11}+A_{12}+A_{21})\diamond iP_1\diamond\Phi(I).\end{aligned}$$

Thus, $T\diamond iP_1\diamond I=0$. We obtain

$$T_{11}-T_{11}^*=0. \tag{2.12}$$

Relations (2.11) and (2.12) imply $T_{11}=0$. Similarly, we can show that

$$\Phi(A_{12}+A_{21}+A_{22})=\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}).$$

**Claim 6.** *For each* $A_{11}\in\mathcal{A}_{11}, A_{12}\in\mathcal{A}_{12}, A_{21}\in\mathcal{A}_{21}$, *and* $A_{22}\in\mathcal{A}_{22}$,

$$\Phi(A_{11}+A_{12}+A_{21}+A_{22})=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}).$$

Then

$$T = \Phi(A_{11} + A_{12} + A_{21} + A_{22}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) - \Phi(A_{22}) = 0.$$

From Claim 5, we obtain

$$\begin{aligned}
\Phi(A_{11} &+ A_{12} + A_{21} + A_{22}) \diamond C_{12} \diamond I + (A_{11} + A_{12} + A_{21} + A_{22}) \diamond \Phi(C_{12}) \diamond I \\
&\quad + (A_{11} + A_{12} + A_{21} + A_{22}) \diamond C_{12} \diamond \Phi(I) \\
&= \Phi((A_{11} + A_{12} + A_{21} + A_{22}) \diamond C_{12} \diamond I) \\
&= \Phi((A_{11} + A_{12} + A_{21}) \diamond C_{12} \diamond I) + \Phi(A_{22} \diamond C_{12} \diamond I) \\
&= \Phi(A_{11} \diamond C_{12} \diamond I) + \Phi(A_{12} \diamond C_{12} \diamond I) + \Phi(A_{21} \diamond C_{12} \diamond I) + \Phi(A_{22} \diamond C_{12} \diamond I) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})) \diamond C_{12} \diamond I \\
&\quad + (A_{11} + A_{12} + A_{21} + A_{22}) \diamond \Phi(C_{12}) \diamond I \\
&\quad + (A_{11} + A_{12} + A_{21} + A_{22}) \diamond C_{12} \diamond \Phi(I).
\end{aligned}$$

Thus, $T \diamond C_{12} \diamond I = 0$. It follows that

$$C_{12}^* T_{11} + C_{12}^* T_{12} + T_{11}^* C_{12} + T_{12}^* C_{12} = 0.$$

Therefore, $T_{11} = T_{12} = 0$. Similarly, by applying $C_{21}$ instead of $C_{12}$ in the above, we obtain $T_{21} = T_{22} = 0$.

**Claim 7.** *For each $A_{ij}, B_{ij} \in \mathcal{A}_i$ such that $i \neq j$, the following equality holds:*

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

It is easy to show that

$$(P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} = A_{ij} + B_{ij} + A_{ij}^* + B_{ij}^*.$$

Thus, we can write

$$\begin{aligned}
\Phi(A_{ij} + B_{ij}) + \Phi(A_{ij}^* + B_{ij}^*) &= \Phi\left((P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2}\right) \\
&= \Phi(P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} + (P_i + A_{ij}^*) \diamond \Phi(P_j + B_{ij}) \diamond \frac{I}{2} \\
&\quad + (P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \Phi\left(\frac{I}{2}\right) \\
&= (\Phi(P_i) + \Phi(A_{ij}^*)) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} + (P_i + A_{ij}^*) \diamond (\Phi(P_j) \\
&\quad + \Phi(B_{ij})) \diamond \frac{I}{2} + (P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \Phi\left(\frac{I}{2}\right) \\
&= \Phi\left(P_i \diamond B_{ij} \diamond \frac{I}{2}\right) + \Phi\left(A_{ij}^* \diamond P_j \diamond \frac{I}{2}\right) \\
&= \Phi(B_{ij}) + \Phi(B_{ij}^*) + \Phi(A_{ij}) + \Phi(A_{ij}^*).
\end{aligned}$$

Thus, we have shown that

$$\Phi(A_{ij} + B_{ij}) + \Phi(A_{ij}^* + B_{ij}^*) = \Phi(A_{ij}) + \Phi(B_{ij}) + \Phi(A_{ij}^*) + \Phi(B_{ij}^*). \tag{2.13}$$

By an easy computation, we obtain

$$(P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} = iA_{ij} + iB_{ij} - iA_{ij}^* - iB_{ij}^*.$$

Then, we have

$$\begin{aligned}\Phi(iA_{ij}+iB_{ij})+\Phi(-iA^*_{ij}-iB^*_{ij}) &= \Phi\left((P_i+A^*_{ij})\diamond(iP_j+iB_{ij})\diamond\frac{I}{2}\right)\\ &= \Phi(P_i+A^*_{ij})\diamond(iP_j+iB_{ij})\diamond\frac{I}{2}+(P_i+A^*_{ij})\diamond\Phi(iP_j+iB_{ij})\diamond\frac{I}{2}\\ &\quad+(P_i+A^*_{ij})\diamond(iP_j+iB_{ij})\diamond\Phi\left(\frac{I}{2}\right)\\ &= (\Phi(P_i)+\Phi(A^*_{ij}))\diamond(iP_j+iB_{ij})\diamond\frac{I}{2}+(P_i+A^*_{ij})\diamond(\Phi(iP_j)\\ &\quad+\Phi(iB_{ij}))\diamond\frac{I}{2}+(P_i+A^*_{ij})\diamond(iP_j+iB_{ij})\diamond\Phi\left(\frac{I}{2}\right)\\ &= \Phi\left(P_i\diamond iB_{ij}\diamond\frac{I}{2}\right)+\Phi\left(A^*_{ij}\diamond iP_j\diamond\frac{I}{2}\right)\\ &= \Phi(iB_{ij})+\Phi(-iB^*_{ij})+\Phi(iA_{ij})+\Phi(-iA^*_{ij}).\end{aligned}$$

We have shown that

$$\Phi(iA_{ij}+iB_{ij})+\Phi(-iA^*_{ij}-iB^*_{ij})=\Phi(iA_{ij})+\Phi(iB_{ij})+\Phi(-iA^*_{ij})+\Phi(-iB^*_{ij}).$$

From Claim 3 and the above equation, we have

$$\Phi(A_{ij}+B_{ij})-\Phi(A^*_{ij}+B^*_{ij})=\Phi(B_{ij})-\Phi(B^*_{ij})+\Phi(A_{ij})-\Phi(A^*_{ij}). \tag{2.14}$$

By adding equations (2.13) and (2.14), we obtain

$$\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij}).$$

**Claim 8.** *For each $A_{ii},B_{ii}\in\mathcal{A}_{ii}$ such that $1\le i\le 2$, the following equality holds*:

$$\Phi(A_{ii}+B_{ii})=\Phi(A_{ii})+\Phi(B_{ii}).$$

Let us show that

$$T=\Phi(A_{ii}+B_{ii})-\Phi(A_{ii})-\Phi(B_{ii})=0.$$

We have

$$\begin{aligned}&\Phi(A_{ii}+B_{ii})\diamond P_j\diamond I+(A_{ii}+B_{ii})\diamond\Phi(P_j)\diamond I+(A_{ii}+B_{ii})\diamond P_j\diamond\Phi(I)\\ &\quad=\Phi((A_{ii}+B_{ii})\diamond P_j\diamond I)\\ &\quad=\Phi(A_{ii}\diamond P_j\diamond I)+\Phi(B_{ii}\diamond P_j\diamond I)\\ &\quad=\Phi(A_{ii})\diamond P_j\diamond I+A_{ii}\diamond\Phi(P_j)\diamond I+A_{ii}\diamond P_j\diamond\Phi(I)+\Phi(B_{ii})\diamond P_j\diamond I\\ &\qquad+B_{ii}\diamond\Phi(P_j)\diamond I+B_{ii}\diamond P_j\diamond\Phi(I)\\ &\quad=(\Phi(A_{ii})+\Phi(B_{ii}))\diamond P_j\diamond I+(A_{ii}+B_{ii})\diamond\Phi(P_j)\diamond I+(A_{ii}+B_{ii})\diamond P_j\diamond\Phi(I).\end{aligned}$$

Therefore,

$$T\diamond P_j\diamond I=0.$$

Thus, $T_{ij}=T_{ji}=T_{jj}=0$.

On the other hand, for every $C_{ij}\in\mathcal{A}_{ij}$, we have

$$\begin{aligned}&\Phi(A_{ii}+B_{ii})\diamond C_{ij}\diamond I+(A_{ii}+B_{ii})\diamond\Phi(C_{ij})\diamond I+(A_{ii}+B_{ii})\diamond C_{ij}\diamond\Phi(I)\\ &\quad=\Phi((A_{ii}+B_{ii})\diamond C_{ij}\diamond I)\\ &\quad=\Phi(A_{ii}\diamond C_{ij}\diamond I)+\Phi(B_{ii}\diamond C_{ij}\diamond I)\\ &\quad=(\Phi(A_{ii})+\Phi(B_{ii}))\diamond C_{ij}\diamond I+(A_{ii}+B_{ii})\diamond\Phi(C_{ij})\diamond I\\ &\qquad+(A_{ii}+B_{ii})\diamond C_{ij}\diamond\Phi(I).\end{aligned}$$

Thus, $T \diamond C_{ij} \diamond I = 0$; then $T_{ii} \diamond C_{ij} \diamond I = 0$. We have $T_{ii}^* C_{ij} + C_{ij}^* T_{ii} = 0$. We know that if $\mathcal{A}$ is prime, then $T_{ii} = 0$. Hence the additivity of $\Phi$ follows from the above claims. □

In the rest of this paper, we show that $\Phi$ is a $*$-derivation.

**Claim 9.** $\Phi$ *preserves stars.*

Since $\Phi(I/2) = 0$, we have

$$\Phi\left(\frac{I}{2} \diamond \frac{I}{2} \diamond A\right) = \frac{I}{2} \diamond \frac{I}{2} \diamond \Phi(A).$$

Therefore,

$$\Phi(A + A^*) = \Phi(A) + \Phi(A)^*.$$

Thus, we have shown that $\Phi$ preserves stars.

**Claim 10.** $\Phi$ *is a derivation.*

For every $A, B \in \mathcal{A}$, we have

$$\begin{aligned}\Phi(AB + A^*B + B^*A + B^*A^*) &= \Phi(I \diamond A \diamond B)\\ &= I \diamond \Phi(A) \diamond B + I \diamond A \diamond \Phi(B)\\ &= (\Phi(A) + \Phi(A)^*) \diamond B + (A + A^*) \diamond \Phi(B)\\ &= \Phi(A)B + \Phi(A)^*B + B^*\Phi(A) + B^*\Phi(A)^*\\ &\qquad + A\Phi(B) + A^*\Phi(B) + \Phi(B)^*A + \Phi(B)^*A^*.\end{aligned}$$

Therefore,

$$\begin{aligned}\Phi(AB + A^*B + B^*A + B^*A^*) = \Phi(A)B &+ \Phi(A)^*B + B^*\Phi(A) + B^*\Phi(A)^* + A\Phi(B)\\ &+ A^*\Phi(B) + \Phi(B)^*A + \Phi(B)^*A^*. \end{aligned} \tag{2.15}$$

Also

$$\begin{aligned}\Phi(AB - A^*B - B^*A + B^*A^*) &= \Phi(I \diamond (-iA) \diamond iB)\\ &= I \diamond \Phi(-iA) \diamond iB + I \diamond (-iA) \diamond \Phi(iB)\\ &= \Phi(A)B - \Phi(A)^*B - B^*\Phi(A) + B^*\Phi(A)^*\\ &\qquad + A\Phi(B) - A^*\Phi(B) - \Phi(B)^*A + \Phi(B)^*A^*.\end{aligned}$$

So we have

$$\begin{aligned}\Phi(AB - A^*B - B^*A + B^*A^*) = \Phi(A)B &- \Phi(A)^*B - B^*\Phi(A)\\ &+ B^*\Phi(A)^* + A\Phi(B) - A^*\Phi(B)\\ &- \Phi(B)^*A + \Phi(B)^*A^*. \end{aligned} \tag{2.16}$$

By adding equations (2.15) and (2.16), we obtain

$$\Phi(AB + B^*A^*) = \Phi(A)B + A\Phi(B) + \Phi(A)^*B^* + A^*\Phi(B)^*. \tag{2.17}$$

From (2.17), Claims 3 and 9, it follows that

$$\begin{aligned}\Phi(AB - B^*A^*) &= i\Phi(A(-iB) + (-iB)^*A^*)\\ &= i(\Phi(A)(-iB) + A\Phi(-iB) + \Phi(A)^*(-iB)^* + A^*\Phi(-iB)^*)\\ &= \Phi(A)B + A\Phi(B) - \Phi(A)^*B^* - A^*\Phi(B)^*.\end{aligned}$$

Therefore,

$$\Phi(AB - B^*A^*) = \Phi(A)B + A\Phi(B) - \Phi(A)^*B^* - A^*\Phi(B)^*. \tag{2.18}$$

From (2.17) and (2.18), we obtain

$$\Phi(AB) = \Phi(A)B + A\Phi(B).$$

This completes the proof. □

## FUNDING

The research of the first author was supported by the Talented Young Scientist Program of the Ministry of Science and Technology of China (Iran-19-001).

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