ISSN 0001-4346, Mathematical Notes, 2020, Vol. 108, No. 1, pp. 64–76. © Pleiades Publishing, Ltd., 2020.
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Variational Inequalities and Analogs of the Hopf Theorems

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Received October 13, 2018; in final form, May 29, 2019; accepted February 19, 2020

Abstract—Properties of the approximate rotation of vector fields generated by multivalued maps of monotone type are studied. Analogs of the Hopf theorems on the extension of multivalued maps without singular points and homotopy classification of the corresponding vector fields are proved. Applications to variational inequalities and operator inclusions are outlines.

DOI: 10.1134/S0001434620070068

Keywords: *variational inequality, operator inclusion, relative degree, multivalued map, convex set.*

1. INTRODUCTION

In the theory of degree a fundamental role is played by the Hopf theorems about the homotopy classification of continuous vector fields and about the extension of a vector field without singular points. The statements and proofs of these theorems, as well as some of their generalizations, can be found, e.g., in [1] and [2]. Most important for our purposes are the versions of the Hopf theorems for the relative degree of a completely continuous map leaving invariant a closed convex set, which were proved in [2].

The notion of relative topological degree [1]–[5] is used in studying variational inequalities. The variational inequality problem is usually stated as follows. Given a Banach space X and a closed convex subset Q of X, let F be a (generally nonlinear) operator mapping X to the dual space X^*. To solve the variational inequality means to find an element u of Q such that $\langle v-u, F(u)\rangle \geq 0$ for all v in Q.

Some authors studied variational inequalities by topological methods based on the notion of relative topological degree of a multivalued vector field; we mention only the papers [6]–[8], which are closest in the methods used. The topological methods have made it possible to more thoroughly study the question of the number of solutions of variational inequalities and the convergence of the Galerkin method and analyze the penalty method, which assigns a set of simpler operator inclusions to a given variational inequality. At the same time, the potential of the theory of relative topological degree is far from being completely exploited. The main reasons for this are both the insufficient knowledge of the qualitative properties of the corresponding characteristics and the lack of efficient algorithms for calculating them.

In the first section, we introduce two classes of maps of monotone type. The author believes that the properties of maps established in this section have not been recorded and are of independent interest. The second section is devoted to the definition of approximate rotation arising in the study of variational inequalities. Properties of approximate rotation are similar to the properties of the relative topological degree of multivalued maps that were described in [5]. The main results of the paper are gathered in the third section, in which analogs of the Hopf theorems about the extension of continuous maps without singular points and the homotopy classification of continuous vector fields are proved. In the concluding section, we study variational inequalities with a parameter and equivalent operator inclusions. We essentially employ the homotopy invariance of approximate rotation and the infinite-dimensional version of the Poincaré–Hopf theorem.

We use the following notation:

- $\overline{M}$ ($\mathring{M}$, ∂M) is the closure (interior, boundary) of a subset M of a metric space $(\mathfrak{R}, \rho)$;

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- $d_{\mathfrak{R}}(x, M) = \inf\{\rho(x, y),\ y \in M\}$ is the distance from a point x to a set M in the metric ρ, and
$$\operatorname{dist}(M_1, M_2) = \inf\{\rho(x_1, x_2)\},$$
where $x_1 \in M_1$ and $x_2 \in M_2$, is the distance between subsets M_1 and M_2 of the metric space $\mathfrak{R}$;
- the metric ρ on a Banach space X with norm $\|x\|$ is defined by $\rho(x, y) = \|x - y\|$;
- $\mathrm{Cv}(X)$ ($\mathrm{Kv}(X)$) is the set of nonempty closed (compact) convex subsets of a Banach space X;
- X^* is the space dual to X, $\langle x, x^*\rangle$ is the canonical bilinear form on $X \times X^*$, and
$$s(x^*, \mathscr{D}) = \sup\{\langle x, x^*\rangle, x \in \mathscr{D}\}$$
is the support function of a set $\mathscr{D} \subset X$;
- $\Gamma(X)$ is the set of finite-dimensional subspaces of X.

Given $C \in \mathrm{Cv}(X)$ and $x \in C$, the sets
$$T_C(x) = \overline{\bigcup_{h>0} h(C - x)} \qquad \text{and} \qquad N_C(x) = \{x^* \in X^*,\ \langle v - x, x^*\rangle \le 0\ \forall\, v \in C\}$$
are called, respectively, the *tangent* and the *normal cone to the set C at the point x*. The convex hull of a set $M \subset X$ is denoted by $\operatorname{co} M$. All Banach spaces under consideration are over the field $\mathbb{R}$ of real numbers.

A *multivalued map* $\mathscr{F}$ from a set $\mathscr{D}_1$ to a set $\mathscr{D}_2$ is an operator assigning a nonempty set $\mathscr{F}(x) \subset \mathscr{D}_2$ to each element x of $\mathscr{D}_1$; the set $\mathscr{F}(\mathscr{D}) = \cup \mathscr{F}(x)$, $x \in \mathscr{D}$, is called the *image* of $\mathscr{F}$ on $\mathscr{D} \subset \mathscr{D}_1$; the set
$$\mathscr{F}^{-1}(\mathscr{D}_0) = \{x \in \mathscr{D}_1,\ \mathscr{F}(x) \cap \mathscr{D}_0 \neq \varnothing\}$$
is the *inverse image* of $\mathscr{D}_0 \subset \mathscr{D}_2$; and the set
$$\mathrm{Gr}(\mathscr{F}) = \{(x, y) \in \mathscr{D}_1 \times \mathscr{D}_2,\ x \in \mathscr{D}_1,\ y \in \mathscr{F}(x)\}$$
is the *graph* of the map $\mathscr{F}$. A multivalued map $\mathscr{F}$ from a topological space X_1 to a topological space X_2 is *upper* (*lower*) *semicontinuous* if the inverse image $\mathscr{F}^{-1}(V)$ of any closed (respectively, open) set $V \subset X_2$ is closed (respectively, open) in X; a multivalued map $\mathscr{F}\colon X_1 \to X_2$ is *closed* if its graph is a closed subset of the Cartesian product $X_1 \times X_2$ of the spaces X_1 and X_2. A multivalued map $\mathscr{F}\colon \mathscr{D} \to Y$ from a subset $\mathscr{D}$ of a Banach space X to a Banach space Y is said to be *bounded* if the image $\mathscr{F}(M)$ of any bounded set $M \subset \mathscr{D}$ is a bounded subset of Y. The formula $\mathscr{F}\colon \mathscr{D} \to \mathrm{Cv}(Y)$ means that $\mathscr{F}(x) \in \mathrm{Cv}(Y)$ for any x in $\mathscr{D}$; the meaning of $\mathscr{F}\colon \mathscr{D} \to \mathrm{Kv}(Y)$ is similar.

2. MULTIVALUED MAPS OF MONOTONE TYPE

Throughout the paper, X is a separable reflexive Banach space, $\|x\|$ and $\|y\|_*$ are the norms on X and on its dual X^*, and the symbols $\to$ and $\rightharpoonup$ denote, respectively, strong and weak convergence. Given a closed subset M of X, by $S(M)$ we denote the set of multivalued maps $F\colon M \to \mathrm{Cv}(X^*)$ satisfying the following condition:

(α) if sequences $x_n \in M$ and $y_n \in F(x_n)$ have the properties
$$x_n \rightharpoonup x, \qquad y_n \rightharpoonup y, \qquad \varlimsup_{n\to\infty} \langle x_n, y_n\rangle \le \langle x, y\rangle, \tag{1}$$
then
$$x_n \to x, \qquad y \in F(x). \tag{2}$$

Condition (α) means that the map F is closed in a certain strong sense. Such operators have been studied by many authors (see, e.g., [9]–[11] and the references therein).

In the sequel, by $S_0(M)$ we denote the set of multivalued maps $F\colon M \to \mathrm{Cv}(X^*)$ satisfying the following condition:

(α_0) if sequences $x_n \in M$ and $y_n \in F(x_n)$ have properties (1), then

$$x \in M, \qquad y \in F(x), \qquad \lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle. \tag{3}$$

Following [12], we refer to maps in the class $S_0(M)$ as *pseudomonotone*.

Condition (α) implies (α_0); therefore, $S(M) \subset S_0(M)$. For a finite-dimensional space X, the classes $S(M)$ and $S_0(M)$ coincide and consist of upper semicontinuous maps $F\colon M \to \mathrm{Kv}(X^*)$. For an infinite-dimensional space X, these classes are already different.

We mention some properties of multivalued maps in the classes defined above. Any pseudomonotone multivalued map F is *upper quasi-continuous* in the sense that the function

$$(v, x) \to s(v, F(x)) = \max\{\langle v, y\rangle,\ y \in F(x)\}$$

is jointly upper semicontinuous in the variables $v \in X$ and $x \in M$. The sum $F + F_0$ of a multivalued map F in the class $S(M)$ and a multivalued map F_0 in the class $S_0(M)$ belongs to $S(M)$. We say that a multivalued map $F_1 : M_1 \to X^*$ is a *closed part* of a multivalued map $F\colon M \to X^*$ if the graph $\mathrm{Gr}(F_1)$ is a subset of $\mathrm{Gr}(F)$ closed in the topology on $X \times X^*$ generated by the strong topology on X and the weak topology on X^*. Any closed part of a multivalued map in the class $S(M)$ belongs to the class $S(M_1)$.

Let $M \times T$ be the Cartesian product of a closed set $M \subset X$ and a compact metric space T, and let $\mathscr{F}$ be a multivalued map from $M \times T$ to $\mathrm{Cv}(X^*)$. We set $RB_X = \{x \in X,\ \|x\| \le R\}$. We say that the multivalued map $\mathscr{F}$ belongs to a class $S(M \times T)$ if

(I) $\mathscr{F}$ is bounded on each set of the form $(M \cap RB_X) \times T$ for any $R < \infty$;

(II) the following parametric version of condition (α) holds: if

$$x_n \in M, \qquad t_n \in T, \qquad y_n \in \mathscr{F}(x_n, t_n), \qquad t_n \to t \quad \text{in the metric of } T,$$

and sequences x_n and y_n have properties (1), then $x_n \to x$ and $y \in \mathscr{F}(x, t)$.

The *convexification* of a multivalued map $\mathscr{F} : M \times T \to \mathrm{Cv}(X^*)$ is the multivalued map

$$F\colon M \to \mathrm{Cv}(X^*) \quad \text{defined by} \quad F(x) = \overline{\mathrm{co}\biggl(\bigcup_{t\in T} \mathscr{F}(x, t)\biggr)}. \tag{4}$$

The support function $s(v, F(x)) = \max\{\langle v, y\rangle,\ y \in F(x)\}$ of each set $F(x)$ is related to the support functions of the sets $\mathscr{F}(x, t)$ by

$$s(v, F(x)) = \sup\{s(v, \mathscr{F}(x, t)),\ t \in T\}. \tag{5}$$

Relations (4) and (5) are equivalent.

Lemma 1. *The convexification of a multivalued map in the class $S(M \times T)$ is a multivalued map in the class $S(M)$.*

Proof. Let $\mathscr{F} \in S(M, T)$, and let F be the multivalued map defined by (4). It is easy to verify that $F(x) \in \mathrm{Cv}(X^*)$ for all $x \in M$ and F is bounded on each bounded subset of M. Let us check that F satisfies condition (α). Fix sequences $x_n \in M$ and $y_n \in F(x_n)$ and elements x and y with properties (1). Relations (1) and (4) imply the inequality

$$\langle x_n - x, y_n\rangle \ge \inf\{\langle x_n - x, u\rangle,\ u \in \mathscr{F}(x_n, t),\ t \in T\};$$

hence there exist $t_n \in T$ and $u_n \in \mathscr{F}(x_n, t_n)$ such that

$$\langle x_n - x, y_n\rangle \ge \langle x_n - x, u_n\rangle - \frac{1}{n}. \tag{6}$$

We can assume without loss of generality that $t_n \to t$ in the metric space T and $u_n \rightharpoonup u$. By virtue of (1) and (6), we have

$$\varlimsup_{n\to\infty} \langle x_n, u_n\rangle \le \langle x, u\rangle.$$

This inequality implies the convergence $x_n \to x$, because $\mathscr{F} \in S(M \times T)$.

To complete the proof, it remains to show that $y \in F(x)$. Fix an element v in X and, for each positive integer n, choose a parameter t_n in T so that $s(v, F(x_n)) \le s(v, \mathscr{F}(x_n, t_n)) + 1/n$. There exists a z_n in $\mathscr{F}(x_n, t_n)$ for which $s(v, \mathscr{F}(x_n, t_n)) = \langle v, z_n\rangle$. We can assume without loss of generality that $t_n \to t$ in the metric of T and $z_n \rightharpoonup z$. Since $x_n \to x$, it follows that

$$\lim_{n\to\infty} \langle x_n, z_n\rangle = \langle x, z\rangle.$$

These properties of the sequences x_n, z_n, and t_n imply $z \in \mathscr{F}(x, t)$, because $\mathscr{F}$ belongs to the class $S(M \times T)$. We have

$$\begin{aligned}\langle v, y\rangle &= \lim_{n\to\infty} \langle v, y_n\rangle \le \varlimsup_{n\to\infty} s(v, F(x_n)) \le \varlimsup_{n\to\infty} s(v, \mathscr{F}(x_n, t_n)) \\ &= \varlimsup_{n\to\infty} \langle v, z_n\rangle = \langle v, z\rangle \le s(v, \mathscr{F}(x, t)) \le s(v, F(x)).\end{aligned}$$

Since $F(x) \in \mathrm{Cv}(X^*)$, it follows from the validity of the inequality $\langle v, y\rangle \le s(v, F(x))$ for all $v \in X$ that $y \in F(x)$. This completes the proof of the lemma. $\square$

Corollary 1. *If $F_i \in S(M)$, $i = 1, \dots, k$, then the multivalued map*

$$F(x) = \overline{\mathrm{co}\biggl(\bigcup_{i=1}^{k} F_i(x)\biggr)}$$

belongs to the class $S(M)$.

Corollary 2. *Let F_0 and F_1 be multivalued maps in the class $S(M)$, and let $\lambda\colon M \to \mathbb{R}$ be a continuous scalar function such that $0 \le \lambda(x) \le 1$ for all $x \in M$. Then the multivalued map F defined by*

$$F(x) = (1 - \lambda(x))F_0(x) + \lambda(x)F_1(x), \qquad x \in M,$$

belongs to the class $S(M)$.

Below we give several lemmas on superposition, which will be useful in what follows. In Lemmas 2 and 3, M and M_1 are closed subsets of X.

Lemma 2. *Let $A\colon M \to X$ be a completely continuous map. Suppose that*

$$\Phi(x) = x - A(x), \quad x \in M, \qquad \Phi(M) \subset M_1,$$
$$F_1 \in S(M_1), \qquad F(x) = F_1(\Phi(x)) = (F_1 \circ \Phi)(x), \quad x \in M.$$

Then $F \in S(M)$.

Proof. Obviously, $F(x) \in \mathrm{Cv}(X^*)$ and the multivalued map $F\colon M \to \mathrm{Cv}(X^*)$ is bounded. Let us verify condition (α). Take any sequences $x_n \in M$ and $y_n \in F(x_n)$ and elements x and y with properties (1). We set $v_n = A(x_n)$, $n = 1, 2, \dots$. We can assume without loss of generality that $v_n \to v$. We have

$$x_n - v_n \rightharpoonup x - v, \qquad y_n \in F_1(x_n - v_n), \qquad y_n \rightharpoonup y, \qquad \varlimsup_{n\to\infty} \langle x_n - v_n, y_n\rangle \le \langle x - v, y\rangle.$$

The assumption $F_1 \in S(M_1)$ implies $x_n - v_n \to x - v$ and $y \in F_1(x - v)$. Since the map A is continuous and $x_n \to x$, it follows that $v = A(x)$ and $y \in F_1(x - v) = F(x)$. This proves the lemma. $\square$

The following lemma is a parametric version of Lemma 2.

Lemma 3. *Let T be a compact metric space, and let $A\colon (M \times T) \to X$ be a completely continuous map. Suppose that $\Phi(x,t) = x - A(x,t)$, $x \in M$, $t \in T$,*

$$\Phi(M,T) \subset M_1, \qquad F_1 \in S(M_1 \times T),$$
$$F(x,t) = F_1(\Phi(x,t),t) = (F_1 \circ \Phi)(x,t) \qquad x \in M, \quad t \in T.$$

Then $F \in S(M \times T)$.

The proof of this lemma is similar to that of Lemma 2 and is therefore omitted.

Below we give corollaries of Lemmas 2 and 3 for the case in which the external superposition operator coincides with the dual map. First, we recall some facts and definitions. On a separable reflexive space X, a norm equivalent to the initial one can be defined with respect to which X and X^* are locally uniformly convex spaces [13]. Therefore, we can assume without loss of generality that the initial norm on X has this property. This ensures the differentiability of the functional $f_0(x) = (1/2)\|x\|^2$ on the entire space X [14]. The operator $J\colon X \to X^*$ defined by $J(x) = f_0'(x)$ is called the *dual map*. It is characterized by the relations

$$\|J(x)\|_* = \|x\|, \quad \langle x, J(x)\rangle = \|x\|^2.$$

Properties of the dual map have been studied by many authors. As is known (see, e.g., [15]), $J \in S(X)$; more general results were obtained in [16]. If X is a Hilbert space, then the dual map J coincides with the identity transformation I.

Corollary (of Lemma 2)**.** *If $A\colon M \to X$ is a completely continuous operator, then the map $F = J(I - A)$ belongs to the class $S(M)$.*

Corollary (of Lemma 3)**.** *Let T be a compact metric space, and let $A\colon (M \times T) \to X$ be a completely continuous operator. Then the map F defined by*

$$F(x,t) = J(x - A(x,t)), \qquad x \in M, \quad t \in T,$$

belongs to the class $S(M \times T)$.

3. VARIATIONAL INEQUALITIES AND APPROXIMATE ROTATION

Let $Q \in \mathrm{Cv}(X)$, and let $X_0 = \mathrm{Lin}(Q - Q)$ be the linear span of the set $Q - Q$. In what follows, it is assumed that $\overline{X_0} = X$. We use $\Gamma(Q)$ to denote the part of $\Gamma(X)$ consisting of those finite-dimensional spaces E for which the interior $\mathrm{ri}_E(Q \cap E)$ of $Q \cap E$ relative to the space E is nonempty. Some of the properties of the class $\Gamma(Q)$ are as follows:

(1) if $E \in \Gamma(Q)$, $z \in Q$, and $\mathscr{H} = \mathrm{Lin}\{E, z\}$ is the linear span of E and z, then $\mathscr{H} \in \Gamma(Q)$;

(2) if $E_0 \in \Gamma(X)$ and $Q \cap E_0 \neq \varnothing$, then there exists a space E in $\Gamma(Q)$, $E \subset E_0$, for which $Q \cap E = Q \cap E_0$;

(3) there exists a sequence E_n, $n = 1, 2, \dots$, with the properties

$$E_n \subset E_{n+1}, \qquad E_n \in \Gamma(Q), \qquad \overline{\bigcup_{n=1}^{\infty} (Q \cap E_n)} = Q;$$

in what follows, we refer to a sequence E_n with these properties as a sequence *exhausting* the set Q.

The existence of exhausting sequences is easy to derive from the separability of the space X.

Let M be a closed subset of Q, and let $S_0(M)$ and $S(M)$ be the corresponding classes of multivalued maps on M. An element $x \in M$ is called a *Q-singular* point of a multivalued map F in the class $S_0(M)$ if

$$0 \in F(x) + N_Q(x). \tag{7}$$

The relation (7) is equivalent to the existence of an element y in $F(x)$ such that

$$\langle v - x, y\rangle \geq 0 \qquad \text{for any } v \in Q. \tag{8}$$

Relation (8) means that the Q-singular points of a multivalued map F are the solutions of the corresponding variational inequality [15]; the converse is also true. Given a bounded set Q and a multivalued map F in the class $S_0(Q)$, the set of solutions of (7) is nonempty. We sometimes denote the solution set of the variational inequality (8) by $\mathrm{Sol}(F, Q)$.

Let us study variational inequalities for multivalued maps satisfying condition (α) in more detail. Let T be a compact metric space, and let M be a closed subset of Q. We say that a map $\mathscr{F}$ in the class $S(M \times T)$ is *nondegenerate* on a set $\mathfrak{M} \subset M$ if, for any t in T, the set $\mathfrak{M}$ contains no Q-singular points of the multivalued map $F_t = \mathscr{F}(\,\cdot\,, t)\colon M \to \mathrm{Cv}(X^*)$.

Proposition 1. *Let a multivalued map $\mathscr{F}$ in the class $S(M \times T)$ be nondegenerate on a closed bounded set $\mathfrak{M} \subset M$. Then there exists an E in $\Gamma(Q)$, a bounded continuous map $w\colon M \times T \to X$, and a constant $\eta > 0$ such that $w(x,t) \in \mathrm{ri}_E(Q \cap E)$ for all $(x,t) \in M \times T$ and*

$$s(w(x,t) - x, \mathscr{F}(x,t)) < -\eta \qquad \textit{for all } (x,t) \in \mathfrak{M} \times T. \tag{9}$$

Proposition 1 follows from results of [6] and [18]. This proposition indicates that the single-valued map $\Phi(x,t) = x - w(x,t)$ makes an acute angle with the multivalued map $\mathscr{F}(x,t)$ on $\mathfrak{M} \times T$; in this respect, Proposition 1 is close to known results about acute-angled approximations (see, e.g., [5]). The main novelty is the finite dimensionality of the map w; we refer to the map $\Phi(x,t) = x - w(x,t)$ as the *finite-dimensional acute-angled approximation* of the multivalued map $\mathscr{F}$. For E we can take the space E_m, where $\{E_n\}$ is a sequence of finite-dimensional subspaces exhausting the set Q and $m > n_0$, n_0 being a sufficiently large number.

We endow Q with the standard metric $\rho(x,y) = \|x - y\|$ and the relative topology. In what follows, $\omega(Q)$ is the set of bounded open (in the relative topology) subsets of Q and $\partial_Q \mathscr{K}$ is the relative boundary of a set $\mathscr{K} \subset Q$. By $\omega(X)$ we denote the set of bounded open subsets of X.

Let $\Omega \in \omega(Q)$, and let $\partial_Q\Omega$ be the relative boundary of Ω, $\partial_Q\Omega \neq \varnothing$; we set $\overline{\Omega} = \Omega \cup \partial_Q\Omega$. If a mapping F belongs to the class $S(\overline{\Omega})$ and Q is nondegenerate on $\partial_Q\Omega$, then, according to Proposition 1, there exists a space E in the class $\Gamma(Q)$ and a continuous vector field $w\colon \overline{\Omega} \to X$ such that

$$w(x) \in \mathrm{ri}_E(Q \cap E) \quad \text{for any } x \in \overline{\Omega}, \qquad \langle x - w(x), x^*\rangle > 0 \quad \text{for any } x \in \partial_Q\Omega,\ x^* \in F(x). \tag{10}$$

In particular, the vector field $\Phi(x) = x - w(x)$ is nondegenerate on $\partial_Q\Omega$; therefore, the relative topological degree $\deg_Q(\Phi, \partial_Q\Omega)$ is defined [1]–[5]. This degree does not depend on the choice of the space E and the continuous vector field w satisfying requirements (10). The number $\deg_Q(\Phi, \partial_Q\Omega)$ is called the *approximate rotation* of the field F on $\partial_Q\Omega$ and denoted by $\gamma_Q(F, \Omega)$. If the set Q is bounded, then the case $\Omega = Q$ must be considered separately. In the situation under consideration, the relative boundary $\partial_Q\Omega$ is the empty set; we put $\gamma_Q(F, Q) = 1$.

Somewhat different but yielding the same results definitions of $\gamma_Q(F, \Omega)$ are contained in [16]. For technical reasons, the terminology of the present paper slightly differs from that of [16].

Approximate rotation has many properties of relative topological degree. We specify and discuss some of them. In the formulations given below, Ω, Ω_1, and Ω_2 are domains in the class $\omega(Q)$, $\partial_Q\Omega$ is the relative boundary of the domain Ω, $\overline{\Omega} = \Omega \cup \partial_Q\Omega$, and F, F_0, and F_1 are multivalued maps in the class $S(\overline{\Omega})$. If a map $\mathscr{F}$ belongs to the class $S(\overline{\Omega} \times [0,1])$, is nondegenerate on the set

$$M_1 \subset \mathrm{cl}_Q\,\Omega, \qquad \mathrm{Sol}(\mathscr{F}(\,\cdot\,, t), Q) \cap M_1 = \varnothing,$$

and satisfies the conditions

$$\mathscr{F}(x, 0) = F_0(x), \quad \mathscr{F}(x, 1) = F_1(x) \qquad \text{for any } x \in M_1,$$

then we say that the maps F_0 and F_1 are *homotopic* on the set M_1. If $\mathscr{F}$ belongs to the class $S(M \times [0,1])$, then it is called a *deformation* of the map F_0 into F_1.

The homotopy relation between maps is an equivalence, i.e., it is reflexive, symmetric and transitive. We write $F \cong G$ if maps F and G are homotopic. The set $S(M)$ can be represented as the union of pairwise disjoint classes of maps homotopic on M_1.

(I) **The zero rotation principle**. *If a map F is nondegenerate on $\overline{\Omega}$ (i.e., $\mathrm{Sol}(F,Q) \cap \overline{\Omega} = \varnothing$), then $\gamma_Q(F,\Omega) = 0$.*

(II) **The homotopy invariance of rotation**. *If maps F_0 and F_1 are homotopic on $\partial_Q \Omega$, then $\gamma_Q(F_0,\Omega) = \gamma_Q(F_1,\Omega)$.*

(III) **The additivity of rotation**. *If $\Omega_1 \cap \Omega_2 = \varnothing$, $\Omega_1 \cup \Omega_2 \subset \Omega$, and a map F is nondegenerate on $\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$, then $\gamma_Q(F,\Omega) = \gamma_Q(F,\Omega_1) + \gamma_Q(F,\Omega_2)$.*

(IV) **The normalization property**. *If $\langle x - x_0, y\rangle > 0$ for all $x \in \partial_Q\Omega$ and $y \in F(x)$, then*

$$\gamma_Q(F,\Omega) = \begin{cases} 1 & \textit{if } x_0 \in \Omega, \\ 0 & \textit{if } x_0 \in Q \setminus \overline{\Omega}. \end{cases}$$

(V) **The oddness of rotation**. *If sets Q and Ω are symmetric with respect to zero, a map F is nondegenerate on $\partial_Q\Omega$, and, for any x in $\partial_Q\Omega$,*

$$\Psi(x) \cap \Psi(-x) = \varnothing, \qquad \textit{where} \quad \Psi(x) = \{v : v = y/\|y\|_*,\ y \in F(x)\},$$

then $\gamma_Q(F,\Omega)$ is an odd number.

Property (I) implies the *nonzero rotation principle*: if $\gamma_Q(F,\Omega) \neq 0$, then the map F has a Q-singular point. In particular, if the set Q is bounded, then $\gamma_Q(F,\Omega) = 1$. Therefore, the variational inequality (8) has a solution. It seems to be important that the solvability of inequality (8) is a consequence of the equality $\gamma_Q(F,\Omega) = 1$.

A set $\mathfrak{N} \subset \Omega$ is said to be *singular* for a map F in the class $S(\overline{\Omega})$ if any element of $\mathfrak{N}$ is a Q-singular point of F. A singular set is *isolated* if it has a neighborhood containing no other Q-singular points. The set of all singular points in the domain Ω is compact [16]. For all neighborhoods $\mathscr{U}$ of an isolated singular set $\mathfrak{N}$ such that the map F is nondegenerate on $\overline{\mathscr{U}} \setminus \mathfrak{N}$, the rotation $\gamma_Q(F,\mathscr{U})$ is the same; it is called the *index* of the singular set $\mathfrak{N}$ and denoted by $\mathrm{ind}_Q(\mathfrak{N},F)$.

Proposition 2 (on the algebraic number of singular sets). *Let Ω be a domain in the class $\omega(Q)$. Suppose that a map $F \in S(\overline{\Omega})$ is nondegenerate on $\partial_Q\Omega$ and $\mathfrak{N}_1,\dots,\mathfrak{N}_m$ are isolated singular sets of F. Then*

$$\gamma_Q(F,\Omega) = \mathrm{ind}_Q(\mathfrak{N}_1,F) + \dots + \mathrm{ind}_Q(\mathfrak{N}_m,F).$$

Proposition 2 can be used to obtain upper and lower bounds for the number of singular points of a map F. The main difficulty arising here is involved in calculating the rotation $\gamma_Q(F,\Omega)$ and the indices of singular sets. If, e.g., a set Q is bounded and $\mathrm{ind}_Q(\mathfrak{N},F) \neq 1$ for some singular set $\mathfrak{N}$, then there exists a singular point of F not belonging to $\mathfrak{N}$. A number of results concerning the calculation of the rotation and indices of single-valued maps of monotone type were obtained in [1] and [9]–[11]. Similar questions for multivalued maps have been much less studied; we mention statements concerning potential operators [6], [16].

The properties applied most frequently are (I)–(IV). On these properties some axiomatic definitions of the rotation of a vector field are based; the relevant references can be found in [1]. The homotopy invariance of rotation makes it possible to pass from the initial map F to simpler maps.

Instead of finite-dimensional acute-angled approximations, their completely continuous analogs can be used. The theory of relative topological degree is well developed for completely continuous vector fields too [1]–[5].

Lemma 4. *Suppose that $\Omega \in \omega(Q)$, $\varnothing \neq \Omega \neq Q$, $\partial_Q\Omega$ is the relative boundary of Ω, and $\overline{\Omega} = \Omega \cup \partial_Q\Omega$. Let $F \in S(\overline{\Omega})$, and let $A\colon \operatorname{cl}_Q \Omega \to X$ be a completely continuous operator satisfying the following conditions similar to* (10):

$$A(x) \in Q \quad \text{for any } x \in \overline{\Omega}, \qquad \langle x - A(x), x^* \rangle > 0 \quad \text{for any } x \in \partial_Q\Omega,\, x^* \in F(x). \tag{11}$$

Then the map F is Q-nondegenerate on $\partial_Q\Omega$, the completely continuous vector field given by $\Phi(x) = x - A(x)$ is nondegenerate on $\partial_Q\Omega$, and

$$\gamma_Q(F, \Omega) = \deg_Q(\Phi, \Omega). \tag{12}$$

Proof. By virtue of the corollary of Lemma 2, the map $F_0 = J(I - A)$ belongs to the class $S(\overline{\Omega})$. On the set $\partial_Q\Omega$, the maps F and F_0 make an acute angle with the field Φ, which implies their homotopy equivalence: $F \cong F_0$. Choose a finite-dimensional continuous operator $A_1 : \overline{\Omega} \to X$ with the properties

$$A_1(x) \in Q \quad \text{for any } x \in \overline{\Omega}, \qquad \|A(x) - A_1(x)\| < \frac{1}{2} \inf\{\|x - A(x)\|,\, x \in \partial_Q\Omega\};$$

such an operator A_1 can be for constructed by using Schauder's well-known construction. It follows from properties of the operators A and A_1 that the vector fields $\Phi = I - A$ and $\Phi_1 = I - A_1$ are linearly homotopic on $\partial_Q\Omega$:

$$\Phi \cong \Phi_1.$$

Homotopy equivalence implies the equality $\deg_Q(\Phi_1, \Omega) = \deg_Q(\Phi, \Omega)$ of the corresponding relative degrees and the Q-homotopy of the maps $F_0 = J\Phi$ and $F_1 = J\Phi_1$ on $\partial_Q\Omega$:

$$F_0 \cong F_1.$$

Equality (12) follows from the relations

$$\gamma_Q(F, \Omega) = \gamma_Q(F_0, \Omega) = \gamma_Q(F_1, \Omega) = \deg_Q(\Phi_1, \Omega) = \deg_Q(\Phi, \Omega).$$

This completes the proof of the lemma. □

4. ANALOGS OF THE HOPF THEOREMS

In this section, we prove analogs of the Hopf theorems for multivalued maps of monotone type. First, we consider the problem of extension without Q-singular points.

Theorem 1. *Suppose that $Q \in \mathrm{Cv}(X)$ and $\Omega \subset Q$ is a bounded connected set open in the relative topology of Q. Let F be a map in the class $S(\overline{\Omega})$ nondegenerate on the boundary $\partial_Q\Omega$ of Ω, and let $\gamma_Q(F, \Omega) = 0$. Then there exists a map G in the class $S(\overline{\Omega})$ coinciding with F on $\partial_Q\Omega$ and nondegenerate on $\overline{\Omega}$.*

Proof. Let $\mathscr{K} = \mathrm{Sol}(F, Q) \cap \Omega$ be the set of solutions of $0 \in F(x) + N_Q(x)$ belonging to Ω. As mentioned above, $\mathscr{K}$ is compact. Since $\mathscr{K}$ does not intersect $\partial_Q\Omega$, it follows that the number

$$d = \operatorname{dist}(\mathscr{K}, \partial_Q\Omega) = \inf\{\|v_1 - v_2\|,\, v_1 \in \mathscr{K},\, v_2 \in \partial_Q\Omega\} > 0$$

is positive. The compactness of $\mathscr{K}$ implies the existence of a finite ε-net for any $\varepsilon > 0$. In particular, we can find elements $x_1, \dots, x_N$ of $\mathscr{K}$ such that $\mathscr{K}$ is contained in the union of the relatively open convex sets

$$U_i = \left\{v \in Q,\, \|v - x_i\| < \frac{d}{2}\right\}, \qquad i = 1, \dots, N.$$

Since the set Ω is connected, there exists a continuous path $z = z(t)$, $t \in [0, 1]$, passing through all points x_i, $i = 1, \dots, N$, and contained in Ω. We have $z(t) \in \Omega$ for all $t \in [0, 1]$; therefore, the number

$$d_1 = \inf\{\|z(t) - v\|,\, t \in [0, 1],\, v \in \partial_Q\Omega\}$$

is positive. For any t in $[0,1]$, the set $V(t)=\{v\in Q,\ \|z(t)-v\|<d_1/2\}$ is relatively open. The union

$$U_{N+1}=\bigcup_{t\in[0,1]}V(t)$$

of these sets is relatively open as well.

Consider the set

$$\Omega_1=\bigcup_{i=1}^{N+1}U_i.$$

It is bounded, connected, and open in the relative topology of Q; moreover, Ω_1 contains $\mathscr{K}$, and its distance

$$r=\operatorname{dist}(\Omega_1,\partial_Q\Omega)=\inf\{\|u-v\|,\ u\in\Omega_1,\ v\in\partial_Q\Omega\}$$

from the boundary $\partial_Q\Omega$ is positive.

Since $\mathscr{K}\subset\Omega_1$, the map F is nondegenerate on the closed bounded set $\mathfrak{M}=\overline{\Omega}\setminus\Omega_1$. Properties (I) and (III) of rotation imply $\gamma_Q(F,\Omega_1)=0$. Since the map F is nondegenerate on $\mathfrak{M}$, it follows by Proposition 1 that there exists a set E in $\Gamma(Q)$ and a bounded continuous map $w\colon\overline{\Omega}\to X$ such that

$$w(x)\in Q\cap E\quad\text{for any } x\in\overline{\Omega},\qquad\langle w(x)-x,y\rangle>0\quad\text{for any } x\in\mathfrak{M},\ y\in F(x).\tag{13}$$

By virtue of (13), the vector field $\Phi(x)=x-w(x)$, $x\in\overline{\Omega}$, makes an acute angle with the map $F(x)$ on the set $\mathfrak{M}$. We have

$$\deg_Q(\Phi,\partial_Q\Omega_1)=\gamma_Q(F,\Omega_1)=0.$$

Theorem 3 of [2] implies the existence of a completely continuous operator $w_1\colon\overline{\Omega}_1\to Q$ coinciding with w on $\partial_Q\Omega_1$ and having no fixed points in $\overline{\Omega}_1$, i.e., such that $w_1(x)\neq x$ for each $x\in\overline{\Omega}_1$.

The operator defined by

$$A(x)=\begin{cases}w(x) & \text{if } x\in\mathfrak{M},\\ w_1(x) & \text{if } x\in\Omega_1\end{cases}$$

is single-valued and completely continuous, and it has no fixed points in $\overline{\Omega}$. On the set $\mathfrak{M}$, the vector field $\Psi(x)=x-A(x)$ coincides with Φ and therefore makes an acute angle with the map F. The map $F_0=J\Psi$ belongs to the class $S(\overline{\Omega})$.

Consider the scalar function

$$\lambda(x)=\frac{\operatorname{dist}(x,\overline{\Omega}_1)}{\operatorname{dist}(x,\partial_Q\Omega)+\operatorname{dist}(x,\overline{\Omega}_1)},\qquad x\in\overline{\Omega}.$$

The function $\lambda\colon\overline{\Omega}\to\mathbb{R}$ is continuous, $0\le\lambda(x)\le1$, and

$$\partial_Q\Omega=\{x\in\overline{\Omega},\ \lambda(x)=1\},\qquad\overline{\Omega}_1=\{x\in\overline{\Omega},\ \lambda(x)=0\}.$$

The sought-for map G can be defined by

$$G(x)=\lambda(x)F(x)+(1-\lambda(x))F_0(x),\qquad x\in\overline{\Omega}.$$

Let us check that it has the required properties. The relation $G\in S(\overline{\Omega})$ follows from Corollary 2 of Lemma 1. Since $\lambda(x)=1$ on $\partial_Q\Omega$, we have $G(x)=F(x)$ for all $x\in\partial_Q\Omega$.

The field $\Psi(x)=x-A(x)$ makes an acute angle with the map $G(x)$ everywhere in $\overline{\Omega}$:

$$\langle\Psi(x),y\rangle>0,\qquad x\in\overline{\Omega},\quad y\in G(x).\tag{14}$$

Indeed, for $x\in\mathfrak{M}$, inequality (14) follows from (13), and for $x\in\Omega_1$, we have

$$G(x)=J\Psi(x),\qquad\langle\Psi(x),G(x)\rangle=\|\Psi(x)\|^2>0.$$

Since $A(x)\in Q$ for all x, the map G is nondegenerate on $\overline{\Omega}$. This completes the proof of the theorem. □

By virtue of the zero rotation principle, the condition $\gamma_Q(F,\Omega)=0$ is also necessary for the existence of a map G in $S(\overline{\Omega})$ that coincides with F on $\partial_Q\Omega$ and is nondegenerate on $\overline{\Omega}$. The condition that the domain Ω is simply connected becomes redundant.

We have introduced the notion of homotopy equivalence between maps in the class $S(M)$ with respect to a set M_1. We are interested in the case where $M=\overline{\Omega}$, $M_1=\partial_Q\Omega$, and $\Omega\in\omega(Q)$.

Theorem 2. *Let $Q\in\mathrm{Cv}(X)$, and let $\Omega\subset Q$ be a bounded connected set open relative to Q and such that the set $Q\setminus\overline{\Omega}$ is connected. Suppose that maps F_0 and F_1 belong to the class $S(\overline{\Omega})$ and are nondegenerate on $\partial_Q\Omega$. Then F_0 and F_1 are homotopic if and only if $\gamma_Q(F_0,\Omega)=\gamma_Q(F_1,\Omega)$.*

Proof. Thanks to the homotopy invariance of rotation (property (II)), it suffices to show that the equality $\gamma_Q(F_0,\Omega)=\gamma_Q(F_1,\Omega)$ of rotations implies the homotopy equivalence of the maps F_0 and F_1 on ∂Q. If $\Omega=Q$, then all maps are homotopic on $\partial_Q Q=\varnothing$; therefore, we shall assume that $\partial_Q\Omega\neq\varnothing$. Since the maps F_i, $i=0,1$, are nondegenerate on $\partial_Q\Omega$, there exist spaces E_i in $\Gamma(Q)$, continuous maps $w_i\colon\overline{\Omega}\to E_i$, and a constant $\delta>0$ such that

$$w_i(x)\in Q\cap E_i\quad\text{for all } x\in\overline{\Omega},\qquad s(w_i(x)-x,F_i(x))<-\delta\quad\text{for all } x\in\partial_Q\Omega.$$

Consider the maps

$$G_i(x)=J(x-w_i(x)),\qquad x\in\overline{\Omega},\quad i=0,1.$$

The maps G_i and F_i make an acute angle with $\Phi_i(x)=x-w_i(x)$; therefore, they are homotopic to each other:

$$F_i\cong G_i.$$

By the very definition of approximate rotation, we have $\gamma_Q(F_i,\Omega)=\deg_Q(\Phi_i,\partial_Q\Omega)$, $i=0,1$. Since $\gamma_Q(F_0,\Omega)=\gamma_Q(F_1,\Omega)$, it follows that

$$\deg_Q(\Phi_0,\partial_Q\Omega)=\deg_Q(\Phi_1,\partial_Q\Omega). \tag{15}$$

Theorem 4 of [2] and equality (15) imply the relative homotopy equivalence of the continuous vector fields $\Phi_0=I-w_0$ and $\Phi_1=I-w_1$. Hence there exists a completely continuous deformation of the form $\Phi_t=I-w_t$, $0\le t\le 1$, where $w_t(x)\in Q$, $x\in\overline{\Omega}$, $t\in[0,1]$. The equality $G_t=J\Phi_t$ defines a deformation joining G_0 and G_1. Thus,

$$F_0\cong G_0\cong G_1\cong F_1.$$

This completes the proof of the theorem. □

Theorem 2 is an analog of the Hopf classification theorem for multivalued maps of monotone type. If $\dim(X)>1$, then, given any integer k, there exists a map $\mathfrak{F}_k$ in $S(\overline{\Omega})$ for which $\gamma_Q(\mathfrak{F}_k,\Omega)=k$. For example, in the cases $k=0,1$, we can set

$$\mathfrak{F}_k(x)=J(x-x_k),\qquad\text{where}\quad x_1\in\Omega,\quad x_0\in Q\setminus\overline{\Omega}.$$

A more general example is

$$\mathfrak{F}_k(x)=J(x-w(x)),$$

where $w\colon\overline{\Omega}\to Q\cap E$ is a continuous map and E is a two-dimensional space in the class $\Gamma(Q)$. Appropriately choosing a space E and a map w, we can achieve that $\gamma_Q(\mathfrak{F}_k,\Omega)=k$ for an integer k given in advance.

Let us introduce the notion of trace of a map. Given a closed subspace E of X, let $i_E\colon E\to X$ be the inclusion operator, and let $i_E^*\colon X^*\to E^*$ be its adjoint. If $Q\cap E\neq\varnothing$, then each operator $F\colon Q\to X^*$ can be assigned its trace on $Q\cap E$, which is denoted by F_E and defined by $F_E=i_E^*Fi_E$. It is easy to see that the trace of an operator F in $S(Q)$ on a space E belongs to the class $S(Q\cap E)$.

The notion of trace of an operator naturally arises in studying potential operators. Let us recall some notions of nonsmooth analysis [19].

Let $f\colon Q \to \mathbb{R}$ be a functional satisfying the local Lipschitz condition, and let us take $x \in Q$ and $v \in X_0 = \mathrm{Lin}(Q - Q)$. The set of sequences $\{y, t\}$ such that $y \to x$ as $t \to +0$ and $[y, y + tv] \subset Q$ is nonempty [20]. We set

$$f^\circ(x, v) = \overline{\lim_{y\to x,\, t\to+0,\, [y,y+tv]\subset Q}} \frac{f(y + tv) - f(y)}{t}.$$

The number $f^\circ(x, v)$ is called the *Clarke derivative* of the functional f at the point x in the direction v. The functional $f^\circ(x, \cdot\,)\colon X_0 \to \mathbb{R}$ is convex and positively homogeneous, and it satisfies the Lipschitz condition and admits a one-to-one extension to $X = \overline{X_0}$ with the same properties. For the extended functional we use the same notation. The functional $f^\circ(x, \cdot\,)\colon X \to \mathbb{R}$ is the support function of a set in $\mathrm{Cv}(X^*)$, which is denoted by $\partial f(x)$ and called the *Clarke gradient* of the functional f at the point x [19], [20].

By $\Lambda_1(Q)$ we denote the set of functionals $f\colon Q \to \mathbb{R}$ satisfying the local Lipschitz condition for which the gradient map $F(x) = \partial f(x)$, $x \in Q$, belongs to the class $S(Q)$. Such a functional f is called the *potential* of F, and the map F itself is said to be a *potential map*.

If E is a closed subspace of X and $Q \subset E \neq \varnothing$, then the restriction to $Q \cap E$ of a functional f in the class $\Lambda_1(Q)$ is a functional in the class $\Lambda_1(Q \cap E)$, which we denote by f_E. We have $\partial f_E \subset (\partial f)_E$; if the functional f is regular [19, p. 44 (Russian transl.)] at a point x, then $\partial f_E(x) = (\partial f)_E(x)$. In the case of a finite-dimensional space E, the functional f_E is called a *finite-dimensional approximation of* f.

Let $SP(Q)$ denote the part of $S(Q)$ that consists of potential maps. The class $SP(Q)$ is substantially narrower than $S(Q)$. When deformations are constructed by using only maps in the class $SP(Q)$, the corresponding homotopy classification becomes finer; in addition to approximate rotation, there arise other homotopy invariants.

5. VARIATIONAL INEQUALITIES WITH A PARAMETER

The homotopy invariance of rotation suggests a version of the *index switch principle* [1].

Proposition 3. *Let $\mathscr{F}$ be a multivalued map in the class $S(\overline{\Omega} \times [a, b])$, and let $F_t = \mathscr{F}(\,\cdot\,, t)$ for $t \in [a, b]$. If*

$$\gamma_Q(F_a, \Omega) \neq \gamma_Q(F_b, \Omega), \tag{16}$$

then, for some t in (a, b), the inclusion

$$0 \in F_t(x) + N_Q(x) \tag{17}$$

has a solution belonging to $\partial_Q\Omega$.

Condition (16) holds, e.g., if $\gamma_Q(F_b, \Omega) \neq k$, where k is 0 or 1, and the map F_a satisfies the acute angle condition in the form

$$\langle x - u, y\rangle > 0 \qquad \text{for any } x \in \partial_Q\Omega,\ y \in F_a(x),$$

where u is a fixed element such that $u \in \Omega$ if $k = 1$ and $u \in Q \setminus \overline{\Omega}$ if $k = 0$.

In what follows, we need some geometric notions and facts. Let $\mathbb{H}$ be a finite-dimensional Euclidean space with norm $|\cdot|$, and let $\mathbb{B} = \{x \in \mathbb{H},\ |x| < 1\}$ be the open ball of radius 1 centered at zero. According to [19, p. 59 (Russian transl.)], an element v in $\mathbb{H} \setminus \{0\}$ is said to be *hypertangent* to the set $\mathbb{D} \subset \mathbb{H}$ at a point $x \in \mathbb{D}$ if there exists an $\varepsilon > 0$ such that

$$y + tw \in \mathbb{D} \qquad \text{for any} \quad y \in (x + \varepsilon\mathbb{B}) \cap \mathbb{D}, \quad w \in v + \varepsilon\mathbb{B}, \quad 0 < t < \varepsilon.$$

If there exists at least one hypertangent to the set $\mathbb{D}$ at a point x, then the set $\mathscr{K}_{\mathbb{D}}(x)$ of all such hypertangents is a nonempty open convex cone in $\mathbb{H}$ [19, p. 59 (Russian transl.)]. Obviously, $\mathscr{K}_{\mathbb{D}}(x) = \mathbb{H}$ for x in $\mathring{\mathbb{D}}$; therefore, of interest is the case where x is a boundary point of the set $\mathbb{D}$. If $\mathbb{D} \in \mathrm{Cv}(\mathbb{H})$ and $\mathring{\mathbb{D}} \neq \varnothing$, then $\mathscr{K}_{\mathbb{D}}(x) = \mathring{T_{\mathbb{D}}}(x)$. In [19, Chap. 2], methods for constructing the cone $\mathscr{K}_{\mathbb{D}}(x)$ were discussed for the case where $\mathbb{D}$ coincides with the lower Lebesgue set of a function $g\colon \mathbb{H} \to \mathbb{R}$, that is, where $\mathbb{D} = \{x \in \mathbb{H},\ g(x) \le c\}$.

A compact set $\mathbb{D} \subset \mathbb{H}$ is said to be *strongly Lipschitz* if

$$\mathscr{K}_{\mathbb{D}}(x) \neq \varnothing \qquad \text{for all } x \in \mathbb{D}.$$

The structure of a strongly Lipschitz set is comparatively simple. It satisfies the condition $\mathbb{D} = \overline{(\mathring{\mathbb{D}})}$, all Betti numbers $b_l(\mathbb{D})$ are finite, and $b_l(\mathbb{D}) = 0$ for $l > \dim(H)$. Therefore, the Euler characteristic $\chi(\mathbb{D})$ of $\mathbb{D}$ is defined [21].

Suppose given $C \in \mathrm{Cv}(\mathbb{H})$ with $\mathring{C} \neq \varnothing$ and a compact set $\mathbb{D} \subset C$. A map $G\colon \partial_C\mathbb{D} \to \mathrm{Cv}(H)$ *satisfies condition* (N) if the relations $x \in \partial_C\mathbb{D}$, $u \in \mathring{T}_C(x)$, and $s(u, G(x)) < 0$ imply $u \in \mathscr{K}_{\mathbb{D}}(x)$.

Let us introduce an infinite-dimensional version of condition (N). Let $\Omega \in \omega(Q)$ be such that $\varnothing \neq \Omega \neq Q$. A map F in the class $S(Q)$ is said to be *conormal* to the domain Ω if F is nondegenerate on $\partial_Q\Omega$ and, for any space E in $\Gamma(Q)$ (endowed with the Euclidean structure), the trace $F_E = i_E^* F i_E$ satisfies condition (N) for the sets $C = Q \cap E$ and $\mathbb{D} = \mathrm{cl}_Q\,\Omega \cap E$. Since

$$\partial_{Q\cap E}(\mathrm{cl}_Q\,\Omega \cap E) \subset (\partial_Q\Omega) \cap E \subset \partial_Q\Omega,$$

it follows that the conormality condition refers only to the points of $\partial_Q\Omega$.

As an example, consider the case where F is a potential operator in the class $S(Q)$ with potential $f\colon Q \to \mathbb{R}$ and the domain Ω is $\Omega = \{x \in Q,\ f(x) < c\}$; its relative boundary is

$$\partial_Q\Omega = \{x \in Q,\ f(x) = c\}.$$

If the map F is nondegenerate on $\partial_Q\Omega$, then it is conormal to the domain Ω [6].

Proposition 4. *Let $\Omega \in \omega(Q)$, and let F be a multivalued map belonging to the class $S(Q)$ and conormal to Ω. Then the Euler characteristic $\chi(\overline{\Omega})$ of the space $\overline{\Omega}$ is defined, and $\gamma_Q(F, \Omega) = \chi(\overline{\Omega})$.*

A more general result was obtained in [6]. Proposition 4 is an infinite-dimensional set-valued version of the Poincaré–Hopf theorem ([22, p. 172 (Russian transl.)]).

Below we state two corollaries of Proposition 4 as theorems. We assume that $\Omega \in \omega(Q)$ and $\partial_Q\Omega$ is the relative boundary of the domain Ω.

Theorem 3. *If multivalued maps F_0 and F_1 in the class $S(Q)$ are homotopic on $\partial_Q\Omega$ and one of them is conormal to the domain Ω, then*

$$\gamma_Q(F_0, \Omega) = \gamma_Q(F_1, \Omega) = \chi(\overline{\Omega}).$$

Proof. Theorem 3 follows from the homotopy invariance of approximate rotation and Proposition 4. □

Theorem 4. *Let $\mathscr{F}$ be a multivalued map in the class $S(Q \times [a, b])$, and let $\mathscr{F}_t(x) = \mathscr{F}(x, t)$. If $\mathscr{F}_a$ is conormal to the domain Ω, $\mathscr{F}_b$ is nondegenerate on $\partial_Q\Omega$, and $\gamma(\mathscr{F}_b, \Omega) \neq \chi(\overline{\Omega})$, then the conclusion of Proposition* 3 *holds.*

Proof. Theorem 4 follows from Propositions 3 and 4. □

Consider the situation in which the inclusion (17) has only the zero solution for any t in $[a, b]$. A number τ in $[a, b]$ is said to be *regular* for (17) if there exists an $\varepsilon > 0$ such that, for t in $(\tau - \varepsilon, \tau + \varepsilon) \cap [a, b]$, the inclusion (17) has a unique (zero) solution in the ball $\|v\| < \varepsilon$. Since 0 is an isolated singular point of the multivalued map F_t for $t \in (\tau - \varepsilon, \tau + \varepsilon) \cap [a, b]$, the topological index $\mathrm{ind}(0, F_t) = \mathrm{ind}_Q(\{0\}, F_t)$ is defined. The homotopy invariance of approximate rotation implies the following assertion.

Lemma 5. *If τ is a regular point for* (17)*, then there exists an $\varepsilon > 0$ such that the function $t \to \mathrm{ind}(0, F_t)$ is constant on the interval $(\tau - \varepsilon, \tau + \varepsilon) \cap [a, b]$.*

To a number λ_0 which is not regular for (17) we refer as a *point of bifurcation*. Thus, λ_0 is a point of bifurcation if there exist sequences $\lambda_n \to \lambda_0$ and $x_n \to 0$, $x_n \neq 0$, for which $0 \in F_{\lambda_n}(x_n) + N_Q(x_n)$.

Let 0 be an isolated solution of the inclusion (17) for $t = \lambda_n$ and $t = \mu_n$, $n = 1, 2, \dots$, where $\lambda_n \to \lambda_0$ and $\mu_n \to \lambda_0$. If $\operatorname{ind}(0, F_{\lambda_n}) \neq \operatorname{ind}(0, F_{\mu_n})$, $n = 1, 2, \dots$, then we say that λ_0 is a *point of index switch*.

Theorem 5. *Let λ_0 be a point of index switch for* (17). *Then λ_0 is a point of bifurcation for* (17).

Proof. It suffices to note that, under the assumptions of the theorem, the function $t \to \operatorname{ind}(0, F_t)$ is nonconstant in each neighborhood of λ_0. □

If $x \in \mathring{Q}$, then $N_Q(x) = \{0\}$ and the inclusion (17) acquires the substantially simpler form $0 \in F_t(x)$. In [23], for this inclusion, an analog of Theorem 5 was proved. In the same paper, the question about bifurcation points of solutions of the inclusion $0 \in F_t(x)$ for a potential operator F_t was considered. The corresponding assertions were stated in terms of the typical numbers of critical points. Approaches to generalizing these results to inclusions of the form (17) are rather obvious.

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