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On the Irrationality Measure of $\ln 7$

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Abstract—Using an integral construction based on symmetrized polynomials, we obtain a new estimate for the irrationality measure of the number ln 7. This estimate improves a result due to Wu, which was proved in 2002.

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1. INTRODUCTION

In 2002, Wu [1] obtained the following estimate:

$$|p + q_1 \ln 2 + q_2 \ln 3 + q_3 \ln 5 + q_4 \ln 7| > H^{-256.865\ldots}, \tag{1.1}$$

where $p, q_1, q_2, q_3, q_4 \in \mathbb{Z}$,

$$H = \max_{1 \le i \le 4} |q_i|, \qquad H \ge H_0.$$

Let us recall that by the *irrationality measure* $\mu(\gamma)$ of a real number γ we mean the lower bound of a set of numbers λ for which, beginning with some positive $q \ge q_0(\lambda)$, the following inequality holds:

$$\left|\gamma - \frac{p}{q}\right| > q^{-\lambda}, \qquad p \in \mathbb{Z}, \quad q \in \mathbb{N}.$$

For $q_1, q_2, q_3 = 0$, inequality (1.1) yields the estimate

$$\mu(\ln 7) < 257.865\ldots\,. \tag{1.2}$$

The purpose of the present paper is to prove the following statement.

Theorem 1. *For $p, q_1, q_2, q_3 \in \mathbb{Z}$, $H = \max_{1\le i\le 3} |q_i|$, and $H \ge H_0$, the following inequality holds*:

$$|p + q_1 \ln 2 + q_2 \ln 3 + q_3 \ln 7| > H^{-35.00999\ldots}.$$

Corollary 1. *The following estimate holds*:

$$\mu(\ln 7) \le 36.00999\ldots\,. \tag{1.3}$$

Estimate (1.3) follows from Theorem 1 for $q_1 = q_2 = 0$.

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2. PROOF OF THEOREM 1

We begin the proof of Theorem 1 with the following lemma proved by Wu in [1].

Lemma 1. *Let $m \in \mathbb{N}$, and let $\gamma_1, \dots, \gamma_m$ be real numbers. Let, for all $n \in \mathbb{N}$, there exist integers $r_n, p_n^{(1)}, \dots, p_n^{(m)}$ such that $\varepsilon_n^{(i)} = r_n\gamma_i - p_n^{(i)} \neq 0$, $i = 1, \dots, m$, and*

$$\lim_{n\to\infty} \frac{1}{n} \ln |r_n| \le \sigma, \qquad \lim_{n\to\infty} \frac{1}{n} \ln |\varepsilon_n^{(i)}| = -\tau^{(i)}, \tag{2.1}$$

where $\sigma, \tau^{(1)}, \dots, \tau^{(m)}$ are positive numbers.

Let $\tau = \min_{1\le i\le m}(\tau^{(i)})$, where $\tau^{(i)} \neq \tau^{(j)}$ for all $i \neq j$. Then the numbers $1, \gamma_1, \dots, \gamma_m$ are linearly independent over $\mathbb{Q}$ and, for any $\varepsilon > 0$, there exists an $H_0(\varepsilon) \in \mathbb{N}$ such that

$$|p + q_1\gamma_1 + \dots + q_m\gamma_m| \ge H^{-\sigma/\tau-\varepsilon} \tag{2.2}$$

for all integers $p, q_1, \dots, q_m$ and $H = \max_{1\le i\le m} |q_i| \ge H_0(\varepsilon)$.

Lemma 1 generalizes Lemma 2.1 from [2], in which the case $m = 2$ was studied.

Throughout the paper,

$$d = 17 \cdot 15 \cdot 13 = 3315, \qquad u = \left(x - \frac{d}{2}\right)^2,$$

$A \in \mathbb{N}$, $B \in \mathbb{Z}^+$, $(A, B) = 1$ in the case $B \neq 0$,

$$P = Au - \frac{B}{4} = A_2x^2 + A_1x + A_0, \qquad A_2 = A, \quad A_1 = -Ad, \quad A_0 = \frac{Ad^2 - B}{4}. \tag{2.3}$$

For an irreducible fraction a/b, where $a \in \mathbb{Z}$, $a \neq 0$, and $b \in \mathbb{N}$, we define the exponent of a prime p by setting $\nu_p = \nu_p(a/b) \in \mathbb{Z}$ so that

$$\frac{a}{b} = p^{\nu_p}\frac{a_1}{b_1}, \qquad \text{where} \quad a_1 \in \mathbb{Z}, \quad b_1 \in \mathbb{N}, \quad (a_1, p) = (b_1, p) = 1.$$

Finally, for a function $f(x)$ analytic at the point $x = 0$, we set

$$D_0(f(x)) = f(0), \qquad D_N(f(x)) = \frac{f^{(N)}(0)}{N!}, \quad N \in \mathbb{N}.$$

For a polynomial P from (2.3), we define

$$\begin{gathered} \nu_p(\mathrm{P}) = \min(2, \nu_p(A_0)), \qquad \text{where} \quad p \in \{17; 13; 5\}, \\ \nu_7(\mathrm{P}) = \min(1, \nu_7(A_0)), \qquad \nu_3(\mathrm{P}) = \min(4, \nu_3(A_0)), \qquad \nu_2(\mathrm{P}) = \min(3, \nu_2(A_0)). \end{gathered} \tag{2.4}$$

Lemma 2. *Let $m \in \mathbb{N}$, and let $N \in \mathbb{Z}^+$, $N \le 2m$. Then the following estimates hold:*

$$\nu_p(D_N(\mathrm{P}^m)) \ge m\nu_p(\mathrm{P}) - N, \qquad p \in \{17; 13; 5; 7\}, \tag{2.5}$$

$$\nu_p(D_N(\mathrm{P}^m)) \ge m\nu_p(\mathrm{P}) - 3N, \qquad p \in \{3; 2\}. \tag{2.6}$$

Proof. In what follows, $\overline{m} = (m_0, m_1, m_2) \in (\mathbb{Z}^+)^3$, $|\overline{m}| = m_0 + m_1 + m_2$, and

$$\gamma(\overline{m}) = \frac{|\overline{m}|!}{m_0!\,m_1!\,m_2!} \in \mathbb{N}.$$

From (2.3) we obtain

$$\mathrm{P}^m = \sum_{|\overline{m}|=m} \gamma(\overline{m}) A_2^{m_2} A_1^{m_1} A_0^{m_0} x^{m_1+2m_2},$$

$$D_N(\mathrm{P}^m) = \sum_{|\overline{m}|=m,\ m_1+2m_2=N} \gamma(\overline{m}) A_0^{m_0} d^{m_1} A_3, \qquad A_3 = (-1)^{m_1} A^{m_1+m_2} \in \mathbb{Z}.$$

For $p \in \{17; 13; 5\}$, we then have

$$\nu_p(D_N(\mathrm{P}^m)) \geq m_0\nu_p(A_0) + m_1 \geq (m_0 + m_1 + m_2)\nu_p(\mathrm{P}) - (m_1 + 2m_2) = m\nu_p(\mathrm{P}) - N,$$

because, by (2.4), we have $\nu_p(A_0) \geq \nu_p(P)$ and $\nu_p(P) \leq 2$.

Similarly,

$$\nu_7(D_N(\mathrm{P}^m)) \geq m_0\nu_7(A_0) \geq m\nu_7(\mathrm{P}) - N,$$

because it follows from (2.4) that

$$\nu_7(A_0) \geq \nu_7(P), \qquad \nu_7(P) \leq 1,$$

$$\begin{aligned}\nu_3(D_N(\mathrm{P}^m)) &\geq m_0\nu_3(A_0) + m_1 \\ &\geq (m_0 + m_1 + m_2)\nu_3(\mathrm{P}) - 3(m_1 + 2m_2) = m\nu_3(\mathrm{P}) - 3N,\end{aligned}$$

because

$$\nu_3(A_0) \geq \nu_3(\mathrm{P}), \qquad \nu_3(\mathrm{P}) \leq 4.$$

Finally,

$$\nu_2(D_N(\mathrm{P}^m)) \geq m_0\nu_2(A_0) \geq (m_0 + m_1 + m_2)\nu_2(\mathrm{P}) - 3(m_1 + 2m_2) = m\nu_2(\mathrm{P}) - 3N,$$

because from (2.4) we see that $\nu_2(A_0) \geq \nu_2(\mathrm{P})$, $\nu_2(\mathrm{P}) \leq 3$, and the lemma is proved. $\square$

We will also need some quadratic polynomials similar to (2.3).

Let $A, B, C \in \mathbb{N}$, and let $(A, B, C) = 1$; we set

$$\mathrm{P} = Au^2 - \frac{B}{4}u + \frac{C}{16} = A\left(x - \frac{d}{2}\right)^4 - \frac{B}{4}\left(x - \frac{d}{2}\right)^2 + \frac{C}{16} = \sum_{i=0}^{4} A_i x^i, \tag{2.7}$$

where

$$\begin{aligned}&A_4 = A, \qquad A_3 = -2Ad, \qquad A_2 = \frac{6Ad^2 - B}{4}, \\ &A_1 = \frac{-2Ad^3 + Bd}{4}, \qquad A_0 = \frac{Ad^4 - Bd^2 + C}{16}.\end{aligned}$$

For the polynomial (2.7), we define the exponents

$$\begin{aligned}&\nu_p(\mathrm{P}) = \min(4, \nu_p(A_0), \nu_p(A_1) + 1, \nu_p(A_2) + 2), \qquad p \in \{17; 13; 5\}, \\ &\nu_7(\mathrm{P}) = \min(2, \nu_7(A_0), \nu_7(A_1) + 1), \\ &\nu_3(\mathrm{P}) = \min(8, \nu_3(A_0), \nu_3(A_1) + 3, \nu_3(A_2) + 6), \\ &\nu_2(\mathrm{P}) = \min(6, \nu_2(A_0), \nu_2(A_1) + 3, \nu_2(A_2) + 6).\end{aligned} \tag{2.8}$$

Lemma 3. *For the polynomial* (2.7)*, estimates* (2.5) *and* (2.6) *in which $N \leq 4m$ and the exponents $\nu_p(\mathrm{P})$, $p \in \{17; 13; 5; 7; 3; 2\}$, are defined by* (2.8)*, hold.*

Proof. In what follows,

$$\overline{m} = (m_0, m_1, \dots, m_4) \in (\mathbb{Z}^+)^5, \qquad |\overline{m}| = m_0 + m_1 + \dots + m_4,$$
$$\gamma(\overline{m}) = \frac{|\overline{m}|!}{m_0!\, m_1! \dots m_4!}.$$

We have

$$P^m = \sum_{|\overline{m}|=m} \gamma(\overline{m}) \prod_{i=0}^{4} (A_i x^i)^{m_i}, \qquad D_N(P^m) = \sum_{|\overline{m}|=m,\, \sum_{i=1}^4 i m_i = N} \gamma(\overline{m}) \prod_{i=0}^{4} A_i^{m_i}.$$

We shall consider several cases.

Case 1: $p \in \{17; 13; 5\}$. We have

$$\nu_p(D_N(P^m)) \geq \sum_{i=0}^{4} m_i \nu_p(A_i) \geq m\nu_p(\mathrm{P}) - \sum_{i=0}^{4} i m_i = m\nu_p(\mathrm{P}) - N,$$

because

$$\nu_p(A_i) + i \geq \nu_p(\mathrm{P}), \qquad i = 0, 1, \dots, 4.$$

For $i \in \{0; 1; 2\}$, this follows from the definition of $\nu_p(\mathrm{P})$ in (2.8), for $i = 3$,

$$\nu_p(A_3) + 3 \geq 1 + 3 \geq \nu_p(\mathrm{P}),$$

and, for $i = 4$,

$$\nu_p(A_4) + 4 \geq 4 \geq \nu_p(\mathrm{P}).$$

Case 2: $p = 7$. Here, as above,

$$\nu_7(A_i) + i \geq \nu_7(\mathrm{P}), \qquad i = 0, 1, \dots, 4.$$

For $i \in \{0; 1\}$ this inequality follows from (2.8) and for $i \in \{2; 3; 4\}$,

$$\nu_7(A_i) + i \geq 2 \geq \nu_7(\mathrm{P}).$$

Case 3: $p \in \{3; 2\}$. In this case,

$$\nu_p(D_N(P^m)) \geq \sum_{i=0}^{4} m_i \nu_p(A_i) \geq m\nu_p(\mathrm{P}) - 3\sum_{i=1}^{4} i m_i = m\nu_p(\mathrm{P}) - 3N,$$

because

$$\nu_p(A_i) + 3i \geq \nu_p(\mathrm{P}), \qquad i = 0, 1, \dots, 4.$$

The lemma is proved. □

In the proof of Theorem 1, we shall apply the following nine polynomials, seven of which are of the form (2.3) and two are of the form (2.7).

We denote

$$P_0 = 4u = 4x^2 - 4dx + d^2,$$

$$P_1 = u - \frac{15^2 13^2}{4} = (x - 9 \cdot 15 \cdot 13)(x - 8 \cdot 15 \cdot 13),$$

$$P_2 = u - \frac{17^2 13^2}{4} = (x - 8 \cdot 17 \cdot 13)(x - 7 \cdot 17 \cdot 13),$$

$$P_3 = u - \frac{17^2 15^2}{4} = (x - 7 \cdot 17 \cdot 15)(x - 6 \cdot 17 \cdot 15),$$

$$P_4 = 5u^2 - \frac{78714}{4} u + \frac{45d^2}{16}$$
$$= 5x^4 - 10dx^3 + 3^2 \cdot 9155501x^2 - 7 \cdot 3^3 \cdot 2^3 \cdot 18157dx + 7^2 \cdot 3^7 \cdot 2^5 \cdot d^2,$$

$$P_5 = 8u - 17 \cdot 15^2 \cdot 13 = 8x^2 - 8dx + 17 \cdot 13 \cdot 5^2 \cdot 7^2 \cdot 3^4,$$

$$P_6 = 101u - \frac{17^2 \cdot 15 \cdot 13^2 \cdot 3}{4} = 101x^2 - 101dx + 17^2 \cdot 13^2 \cdot 5 \cdot 7 \cdot 3^4 \cdot 2,$$

$$P_7 = 19u - \frac{17 \cdot 13^2 \cdot 3^2 \cdot 11}{4} = 19x^2 - 19dx + 17 \cdot 13^2 \cdot 7 \cdot 3^4 \cdot 2^5,$$

$$P_8 = 16u^2 - \frac{16 \cdot 22887}{4} u + \frac{9 \cdot 16d^2}{16}$$

$$= 16x^4 - 32dx^3 + 4 \cdot 3 \cdot 21970821x^2 - 7 \cdot 3^3 \cdot 2^2 \cdot 116167dx + 7^2 \cdot 3^6 \cdot 307 \cdot d^2.$$

We set

$$(p_1; \dots ; p_6) = (17; 13; 5; 7; 3; 2), \qquad \Pi_k = \prod_{j=1}^{6} p_j^{\nu_{p_j}(P_k)}, \quad k = 0; 1; \dots ; 8.$$

For the polynomials $P_0, \dots, P_8$ defined above, using the definitions of the exponents (2.4) and (2.8), we obtain

$$\begin{aligned} &\Pi_0 = 17^2 \cdot 13^2 \cdot 5^2 \cdot 3^2, && \Pi_1 = 13^2 \cdot 5^2 \cdot 3^4 \cdot 2^3, \\ &\Pi_2 = 17^2 \cdot 13^2 \cdot 7^1 \cdot 2^3, && \Pi_3 = 17^2 \cdot 5^2 \cdot 7^1 \cdot 3^3 \cdot 2^1, \\ &\Pi_4 = 17^2 \cdot 13^2 \cdot 5^2 \cdot 7^2 \cdot 3^7 \cdot 2^5, && \Pi_5 = 17^1 \cdot 13^1 \cdot 5^2 \cdot 7^1 \cdot 3^4, \\ &\Pi_6 = 17^2 \cdot 13^2 \cdot 5^1 \cdot 7^1 \cdot 3^4 \cdot 2^1, && \Pi_7 = 17^1 \cdot 13^2 \cdot 7^1 \cdot 3^4 \cdot 2^3, \\ &\Pi_8 = 17^2 \cdot 13^2 \cdot 5^2 \cdot 7^2 \cdot 3^7. \end{aligned} \tag{2.9}$$

Let

$$\begin{aligned} &\alpha_0 = 0.180555, && \alpha_1 = 0.306, && \alpha_2 = 0.27464, \\ &\alpha_3 = 0.31048, && \alpha_4 = 0.17339, && \alpha_5 = 0.00001, \\ &\alpha_6 = 0.05377, && \alpha_7 = 0.00896, && \alpha_8 = 0.00268. \end{aligned} \tag{2.10}$$

The parameters $\alpha_0, \dots, \alpha_8$ satisfy the following relations:

$$\begin{aligned} \alpha_0 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8 &= 2, \\ 2\alpha_0 + 2\alpha_1 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_8 &= 2, \\ 2\alpha_0 + 2\alpha_1 + 2\alpha_3 + 2\alpha_4 + 1\alpha_5 + \alpha_6 + 2\alpha_8 &= 2, \\ \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_8 &= 1, \\ 2\alpha_0 + 4\alpha_1 + 3\alpha_3 + 7\alpha_4 + 4\alpha_5 + 4\alpha_6 + 4\alpha_7 + 7\alpha_8 &= 4, \\ 3\alpha_1 + 3\alpha_2 + \alpha_3 + 5\alpha_4 + \alpha_6 + 3\alpha_7 &= 3. \end{aligned} \tag{2.11}$$

We now consider the rational function

$$R_n(x) = \frac{\prod_{k=0}^{8} P_k^{\alpha_k n}}{x^{n+1}(d-x)^{n+1}}, \tag{2.12}$$

where $n \in \mathbb{N}$ and n is a multiple of 10^6, so that $\alpha_k n \in \mathbb{N}$ for all $k = 0, 1, \dots, 8$.

We define the following integrals:

$$\omega_1 = \frac{1}{2}\int_{8\cdot 15\cdot 13}^{9\cdot 15\cdot 13} R_n(x)\,dx, \qquad \omega_2 = \int_{9\cdot 15\cdot 13}^{8\cdot 17\cdot 13} R_n(x)\,dx, \qquad \omega_3 = \int_{8\cdot 17\cdot 13}^{7\cdot 17\cdot 15} R_n(x)\,dx. \tag{2.13}$$

The function $R_n(x)$ is symmetric about the point $x = d/2$. Therefore, its partial-fraction expansion is of the form

$$R_n(x) = Q_n(x) + \sum_{i=1}^{n+1}\left(\frac{a_i}{x^i} + \frac{a_i}{(d-x)^i}\right), \tag{2.14}$$

where $Q_n(x) \in \mathbb{Z}[x]$ and all $a_i \in \mathbb{Q}$.

The following lemma is similar to Lemma 1 from [3], in which symmetric polynomials were first used.

Lemma 4. *For all* $i = 1, \dots, n+1$*, the following representation holds*:

$$a_i = 17^{i-2}13^{i-2}5^{i-2}7^{i-1}3^{3i-4}2^{3i-3}M_i, \quad \textit{where} \quad M_i \in \mathbb{Z}. \tag{2.15}$$

Proof. Let

$$\overline{m} = (m_0, m_1, \dots, m_9) \in (\mathbb{Z}^+)^{10}, \qquad |\overline{m}| = m_0 + m_1 + \dots + m_9.$$

Then, using (2.12) and (2.14), we obtain

$$a_i = D_{n+1-i}(R_n(x)x^{n+1}) = \sum_{|\overline{m}|=n+1-i} \prod_{k=0}^{8} D_{m_k}((P_k(x))^{\alpha_k n}))D_{m_9}((d-x)^{-n-1}).$$

We have

$$D_{m_9}((d-x)^{-n-1}) = \binom{n+m_9}{m_9} d^{-n-m_9-1}.$$

Just as in Lemma 1 from [3], we need to find lower bounds for the exponents $\nu_{p_j}(a_i)$, $j = 1, \dots, 6$. We shall apply Lemmas 2 and 3, equalities (2.9) and (2.10), and relations (2.11). For the exponent ν_{17}, we have

$$\begin{aligned}\nu_{17}(a_i) &\geq 2n\alpha_0 - m_0 + 2n\alpha_2 - m_2 + 2n\alpha_3 - m_3 + 2n\alpha_4 - m_4 + n\alpha_5 - m_5 \\ &\quad + 2n\alpha_6 - m_6 + n\alpha_7 - m_7 + 2n\alpha_8 - m_8 - n - m_9 - 1 \\ &\geq n(2\alpha_0 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8) - n - 1 - (n+1-i) \\ &= i - 2.\end{aligned}$$

The other equalities are verified in a similar way.

The lemma is proved. □

We denote $\delta_n = \mathrm{LCM}(1, 2, \dots, n)$, $C_n = 2d\delta_n$, and $\varepsilon_n^{(i)} = C_n\omega_i$, $i = 1, 2, 3$, where the integrals ω_1, ω_2, and ω_3 are defined by the equalities in (2.13).

Lemma 5. *The following expressions are valid*:

$$\varepsilon_n^{(1)} = r_n \ln\frac{9}{8} - p_n^{(1)}, \qquad \varepsilon_n^{(2)} = r_n\left(\ln\frac{8}{7} - \ln\frac{9}{8}\right) - p_n^{(2)},$$

$$\varepsilon_n^{(3)} = r_n\left(\ln\frac{7}{6} - \ln\frac{8}{7}\right) - p_n^{(3)},$$

where $r_n = C_n a_1 \in \mathbb{Z}$, *all* $p_n^{(1)}, p_n^{(2)}, p_n^{(3)} \in \mathbb{Z}$.

Proof. It is necessary to integrate identity (2.14) and use Lemma 4. We have

$$\begin{aligned}\varepsilon_n^{(1)} &= d\delta_n\left(a_1 \ln\frac{x}{d-x}\Big|_{8\cdot15\cdot13}^{9\cdot15\cdot13} - 2\sum_{i=2}^{n+1}\frac{a_i}{i-1}\left(\frac{1}{(9\cdot15\cdot13)^{i-1}} - \frac{1}{(8\cdot15\cdot13)^{i-1}}\right)\right. \\ &\qquad\left. + \int_{8\cdot18\cdot13}^{9\cdot15\cdot13} Q_n(x)\,dx\right) \\ &= C_n a_1 \ln\frac{9}{8} - p_n^{(1)}, \qquad p_n^{(1)} \in \mathbb{Q}.\end{aligned}$$

For $i = 1$, using (2.14), we obtain $da_1 \in \mathbb{Z}$, i.e., $r_n = C_n a_1 \in \mathbb{Z}$.

Let us now show that $p_n^{(1)} \in \mathbb{Z}$. Let us first calculate the degree of the polynomial $Q_n(x)$. From (2.11) and (2.13) we obtain

$$\begin{aligned}\deg Q_n(x) &= n(2\alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + 4\alpha_8) - 2n - 2 \\ &= 0.97311n - 2.\end{aligned}$$

Therefore,

$$\delta_n \int_{8\cdot 15\cdot 13}^{9\cdot 15\cdot 13} Q_n(x)\,dx \in \mathbb{Z}.$$

Further, $\delta_n(1/(i-1)) \in \mathbb{N}$, $i = 2, \dots, n+1$; using (2.15), we can write

$$\frac{da_i}{(9\cdot 15\cdot 13)^{i-1}} \in \mathbb{Z}, \qquad \frac{da_i}{(8\cdot 15\cdot 13)^{i-1}} \in \mathbb{Z}.$$

Therefore, $p_n^{(1)} \in \mathbb{Z}$. Similarly,

$$\begin{aligned}
\varepsilon_n^{(2)} &= 2d\delta_n\biggl(a_1 \ln\frac{x}{d-x}\bigg|_{9\cdot 15\cdot 13}^{8\cdot 17\cdot 13} - \sum_{i=2}^{n+1}\frac{a_i}{i-1}\biggl(\frac{1}{(8\cdot 17\cdot 13)^{i-1}} - \frac{1}{(7\cdot 17\cdot 13)^{i-1}} \\
&\qquad - \frac{1}{(9\cdot 15\cdot 13)^{i-1}} + \frac{1}{(8\cdot 15\cdot 13)^{i-1}}\biggr) \\
&\qquad + \int_{9\cdot 15\cdot 13}^{8\cdot 7\cdot 13} Q_n(x)\,dx\biggr) \\
&= r_n\biggl(\ln\frac{8}{7} - \ln\frac{9}{8}\biggr) - p_n^{(2)}, \qquad p_n^{(2)} \in \mathbb{Z}, \\
\varepsilon_n^{(3)} &= 2d\delta_n\biggl(a_1 \ln\frac{x}{d-x}\bigg|_{8\cdot 17\cdot 13}^{7\cdot 17\cdot 15} - \sum_{i=2}^{n+1}\frac{a_i}{i-1}\biggl(\frac{1}{(7\cdot 17\cdot 15)^{i-1}} - \frac{1}{(6\cdot 17\cdot 15)^{i-1}} \\
&\qquad - \frac{1}{(8\cdot 17\cdot 13)^{i-1}} + \frac{1}{(7\cdot 17\cdot 13)^{i-1}}\biggr) \\
&\qquad + \int_{8\cdot 17\cdot 13}^{7\cdot 17\cdot 15} Q_n(x)\,dx\biggr) \\
&= r_n\biggl(\ln\frac{7}{6} - \ln\frac{8}{7}\biggr) - p_n^{(3)}, \qquad p_n^{(3)} \in \mathbb{Z}
\end{aligned}$$

The lemma is proved. □

Now, using Lemma 1, we can complete the proof of Theorem 1. Using the Laplace theorem, it is easy to calculate the asymptotics of the linear forms $\varepsilon_n^{(1)}$, $\varepsilon_n^{(2)}$, and $\varepsilon_n^{(3)}$, and applying the saddle-point method, we can find the asymptotics of $|r_n|$, which has already been done previously (see, e.g., [1]–[4]). Since this procedure is standard, we restrict ourselves to just a few comments.

In the integrals (2.13), we make the change $u = (x - d/2)^2$. We have

$$\omega_1 = \int_0^{u_1} \varphi(u)(g(u))^n\,du,$$

where

$$g(u) = \frac{\prod_{k=0}^{8}(P_k(u))^{\alpha_k}}{d^2/4 - u}, \qquad \varphi(u) = \frac{1}{2\sqrt{u}(d^2/4 - u)}, \qquad u_1 = \frac{15^2 13^2}{4}.$$

Similarly,

$$\omega_2 = \int_{u_1}^{u_2} \varphi(u)(g(u))^n\,du, \qquad u_2 = \frac{17^2 13^2}{4},$$

$$\omega_3 = \int_{u_2}^{u_3} \varphi(u)(g(u))^n\,du, \qquad u_3 = \frac{17^2 15^2}{4}.$$

Let us calculate the constants σ and $\tau^{(i)}$ defined by equalities (2.1) from Lemma 1 for the linear forms $\varepsilon_n^{(i)}$, $i=1,2,3$.

We have

$$\max_{[0;u_1]}|g(u)| = |g(4427.463)| = 0.276764\dots,$$
$$\max_{[u_1;u_2]}|g(u)| = |g(10954.689)| = 0.177446\dots,$$
$$\max_{[u_2;u_3]}|g(u)| = |g(14784.423)| = 0.266166\dots.$$

Further, $\lim_{n\to\infty}((1/n)\ln\delta_n) = 1$. Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln|\varepsilon_n^{(1)}|\right) = \lim_{(n\to\infty)}\frac{1}{n}\ln\delta_n + \ln 0.276764\dots = -0.284589\dots,$$
$$\lim_{n\to\infty}\left(\frac{1}{n}\ln|\varepsilon_n^{(2)}|\right) = \lim_{(n\to\infty)}\frac{1}{n}\ln\delta_n + \ln 0.177446\dots = -0.729\dots,$$
$$\lim_{n\to\infty}\left(\frac{1}{n}\ln|\varepsilon_n^{(3)}|\right) = \lim_{(n\to\infty)}\frac{1}{n}\ln\delta_n + \ln 0.266166\dots = -0.323\dots.$$

Thus, $\tau^{(1)}$, $\tau^{(2)}$, and $\tau^{(3)}$ are pairwise distinct, and

$$\tau = \min(\tau^{(1)},\tau^{(2)},\tau^{(3)}) = 0.284589\dots.$$

Further,

$$\sigma = \lim_{n\to\infty}\frac{1}{n}\ln|r_n| = \lim_{n\to\infty}\frac{1}{n}\ln\delta_n + \ln|g(8376691.4\dots)|$$
$$= 1 + \ln(7812.546\dots) = 9.963486\dots,$$

and

$$\frac{\sigma}{\tau} = 35.00999\dots.$$

The requirement $n\in\mathbb{N}$ is not essential, because it suffices to make the change $n = 10^6 n_1$, $n_1\in\mathbb{N}$. Finally, for arbitrary $q_1', q_2', q_3' \in \mathbb{Z}$, we have

$$q_1\ln\frac{9}{8} + q_2\left(\ln\frac{8}{7} - \ln\frac{9}{8}\right) + q_3\left(\ln\frac{7}{6} - \ln\frac{8}{7}\right)$$
$$= (-3q_1 + 6q_2 - 4q_3)\ln 2 + (2q_1 - 2q_2 - q_3)\ln 3 + (-q_2 + 2q_3)\ln 7,$$

$$q_1'\ln 2 + q_2'\ln 3 + q_3'\ln 7$$
$$= (5q_1' + 8q_2' + 14q_3')\ln\frac{9}{8} + (4q_1' + 6q_2' - 11q_3')\left(\ln\frac{8}{7} - \ln\frac{9}{8}\right)$$
$$+ (2q_1' + 3q_2' + 6q_3')\left(\ln\frac{7}{6} - \ln\frac{8}{7}\right),$$

and the assertion of Theorem 1 follows from inequality (2.2) in Lemma 1 for the numbers

$$\gamma_1 = \ln\frac{9}{8}, \qquad \gamma_2 = \ln\frac{8}{7} - \ln\frac{9}{8}, \qquad \gamma_3 = \ln\frac{7}{6} - \ln\frac{8}{7}.$$

3. REMARKS

Remark 1. Estimate (1.3) is considerably sharper than estimate (1.2); however, the assertion of Theorem 1 is somewhat less general than inequality (1.1).

Remark 2. We can further refine the values of the parameters (2.10), which will lead to a slight improvement in the result of Theorem 1.

Remark 3. General symmetrized polynomials of first degree of the form $P(t) = At - B$, where $t = (x - d)^2$, were first used in [4]. The polynomials of the form (2.3) used in the present paper yield a sharper result.

Remark 4. Quadratic polynomials of the form 2.7 are applied here for the first time. Of particular importance is the use of the polynomial P_4, which has complex roots.

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