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On a Property of the Franklin System in $C[0,1]$ and $L^1[0,1]$

V. G. Mikayelyan[1*]

[1] *Yerevan State University, Yerevan, 375025 Armenia*

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Abstract—A problem posed by J. R. Holub is solved. In particular, it is proved that if $\{\widetilde{f}_n\}$ is the normalized Franklin system in $L^1[0,1]$, $\{a_n\}$ is a monotone sequence converging to zero, and $\sup_{n\in\mathbb{N}}\|\sum_{k=0}^{n} a_k\widetilde{f}_k\|_1 < +\infty$, then the series $\sum_{n=0}^{\infty} a_n\widetilde{f}_n$ converges in $L^1[0,1]$. A similar result is also obtained for $C[0,1]$.

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A basis $\{e_n\}_{n=0}^{\infty}$ in a Banach space X is said to be *boundedly complete* if, for any number sequence $\{a_n\}_{n=0}^{\infty}$ satisfying the condition

$$\sup_{n\in\mathbb{N}}\left\|\sum_{k=0}^{n} a_k e_k\right\| < +\infty, \tag{1}$$

the series $\sum_{n=0}^{\infty} a_n e_n$ converges. If a space contains a boundedly complete basis, then it is isomorphic to its dual space (see [1, p. 70]). In particular, the spaces $C[0,1]$ and $L^1[0,1]$ have no boundedly complete bases. Holub [2] introduced the notion of a monotonically bounded basis, which is weaker than the notion of a bounded basis. Let us recall that a basis $\{e_n\}_{n=0}^{\infty}$ in a Banach space X is said to be *seminormalized* if there exists a constant $C > 0$ such that $C^{-1} \le \|e_n\| \le C$, $n \in \mathbb{N}$.

Definition 1. A seminormalized basis $\{e_n\}_{n=0}^{\infty}$ in a Banach space X is said to be is *monotonically boundedly complete* if, for any monotone number sequence $\{a_n\}_{n=0}^{\infty}$ converging to zero and satisfying condition (1), the series $\sum_{n=0}^{\infty} a_n e_n$ converges.

Holub [2] proved that the Schauder basis in $C[0,1]$ is monotonically boundedly complete. He posed the following question: Are the Haar and Franklin systems monotonically boundedly complete bases in $L^1[0,1]$? In [3], Kadets proved that the Haar system is a monotonically boundedly complete basis in $L^1[0,1]$. In the present paper, we prove that the Franklin system is monotonically boundedly complete in $L^1[0,1]$ and $C[0,1]$; moreover, in the case of $L^1[0,1]$, we prove an even stronger property than the fact that the Franklin system is monotonically boundedly complete.

Let us recall the definition of the Franklin system.

Let $n = 2^\mu + \nu$, where $\mu = 0,1,2,\dots$ and $1 \le \nu \le 2^\mu$. We denote

$$s_{n,i} = \begin{cases} \dfrac{i}{2^{\mu+1}}, & 0 \le i \le 2\nu, \\ \dfrac{i-\nu}{2^\mu}, & 2\nu < i \le n. \end{cases}$$

We also set $s_{n,-1} = s_{n,0} = 0$ and $s_{n,n+1} = s_{n,n} = 1$.

By S_n we denote the space of continuous piecewise linear functions on $[0,1]$ with nodes $\{s_{n,i}\}_{i=0}^{n}$; in other words, $f \in S_n$ if $f \in C[0,1]$ and f is linear on each of the intervals $[s_{n,i-1}, s_{n,i}]$, $i = 1,2,\dots,n$.

*E-mail: mik.vazgen@gmail.com

Obviously, $\dim S_n = n+1$ and the set $\{s_{n,i}\}_{i=0}^{n}$ is obtained by adding $s_{n,2\nu-1}$ to the set $\{s_{n-1,i}\}_{i=0}^{n-1}$. Therefore, there exists a unique (up to sign) function $f_n \in S_n$ orthogonal to S_{n-1} with $\|f_n\|_2 = 1$. Setting $f_0(x) = 1$ and $f_1(x) = \sqrt{3}(2x-1)$, $x \in [0,1]$, we obtain the orthonormal system $\{f_n(x)\}_{n=0}^{\infty}$, which was defined in an equivalent way by Franklin in [4].

It is well known that the Franklin system is a basis in $C[0,1]$, $L^1[0,1]$ (see [4]) and an unconditional basis in $L^p[0,1]$, $1 < p < \infty$ (see [5]).

By $C_1, C_2, \dots, C_p$ we denote positive constants depending only on their subscripts.

In studying the properties of the Franklin system, the following Ciesielski exponential estimates play an important role:

Lemma 1 (see [6]). *Let $n = 2^\mu + i$, $\mu \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \dots, 2^\mu$. Then there exists a $q \in (0,1)$ such that, for all $k \in \{0, 1, \dots, n\}$,*

$$C_1 2^{\mu/2} q^{|k-(2i-1)|} \le (-1)^{k+1} f_n(s_{n,k}) \le C_2 2^{\mu/2} q^{|k-(2i-1)|}.$$

Corollary 1. *For all $n \ge 1$, the following estimates hold:*

$$\frac{C_3}{\sqrt{n}} \le \|f_n\|_1 \le \frac{C_4}{\sqrt{n}}.$$

The following theorem holds.

Theorem 1. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that*

$$\frac{|a_n|}{n^\alpha} \le C_5 \frac{|a_k|}{k^\alpha}, \qquad n \ge k,$$

for some $\alpha \ge 0$. If $\sup_{n\in\mathbb{N}} \|\sum_{k=0}^{n} a_k f_k\|_1 < +\infty$, then the series $\sum_{n=0}^{\infty} a_n f_n$ converges in $L^p[0,1]$ for all $1 \le p < \infty$.

It is easy to see that Theorem 1 implies the following theorem.

Theorem 2. *The Franklin system in $L^1[0,1]$ is a monotonically boundedly complete basis.*

Indeed, let $\{\widetilde{f}_n\}_{n=0}^{\infty}$ be the normalized Franklin system in $L^1[0,1]$, and let a_n decrease, tending to zero. Then, by Corollary 1, the assumptions of Theorem 1 hold for $\alpha = 1/2$. Therefore, the series $\sum_{n=0}^{\infty} a_n f_n$ converges in $L^1[0,1]$.

Proof of Theorem 1. In [7], it was proved that if the estimate $\sup_{n\in\mathbb{N}} \|\sum_{k=0}^{n} a_k f_k\|_1 < +\infty$ holds, then the series $\sum_{n=0}^{\infty} a_n f_n$ converges almost everywhere on $[0,1]$. And it was shown in [8] that a series $\sum_{n=0}^{\infty} a_n f_n$ in the Franklin system converges almost everywhere on $E \subset [0,1]$ if and only if $\sum_{n=0}^{\infty} a_n^2 f_n^2(x) < +\infty$ for almost all $x \in E$. Therefore, $\sum_{n=0}^{\infty} a_n^2 f_n^2(x) < \infty$ for almost all $x \in [0,1]$. We choose a point $x_0 \in [0,1]$ so that

$$\sum_{n=0}^{\infty} a_n^2 f_n^2(x_0) < \infty. \tag{2}$$

We note that if $2^{2\mu} \le n \le 2^{2\mu+2}$, then

$$|a_n| \ge \frac{1}{C_5}\left(\frac{n}{2^{2\mu+2}}\right)^{\alpha} |a_{2^{2\mu+2}}| \ge C_6 |a_{2^{2\mu+2}}|,$$

where $C_6 = 1/(C_5 4^\alpha)$. Hence, (2) and the estimate

$$\sum_{n=2^\mu+1}^{2^{\mu+2}} f_n^2(x) > 2^{\mu-3}, \qquad \mu \in \mathbb{N} \cup \{0\}, \quad x \in [0,1],$$

which was derived in [8], imply

$$\sum_{\mu=1}^{\infty} a_{2^{2\mu}}^2 2^{2\mu} \le 32 \sum_{\mu=0}^{\infty} a_{2^{2\mu+2}}^2 \sum_{n=2^{2\mu}+1}^{2^{2\mu+2}} f_n^2(x_0) \le \frac{32}{C_6^2} \sum_{n=0}^{\infty} a_n^2 f_n^2(x_0) < \infty.$$

Thus, $\sum_{\mu=1}^{\infty} a_{2^{2\mu}}^2 2^{2\mu} < \infty$. We now note that, for all $\mu \in \mathbb{N} \cup \{0\}$,

$$|a_{2^{2\mu+1}}| \le C_5 \biggl(\frac{2^{2\mu+1}}{2^{2\mu}}\biggr)^{\alpha} |a_{2^{2\mu}}| = C_7 |a_{2^{2\mu}}|,$$

where $C_7 = C_5 2^{\alpha}$. Therefore,

$$\sum_{\mu=2}^{\infty} a_{2^{\mu}}^2 2^{\mu} = \sum_{\mu=1}^{\infty} a_{2^{2\mu}}^2 2^{2\mu} + \sum_{\mu=1}^{\infty} a_{2^{2\mu+1}}^2 2^{2\mu+1} \le (1 + 2C_7^2) \sum_{\mu=1}^{\infty} a_{2^{2\mu}}^2 2^{2\mu} < \infty. \tag{3}$$

In the same way, we prove that $|a_n| \le C_7 |a_{2^{\mu}}|$ for all $n \in \{2^{\mu}, 2^{\mu}+1, \dots, 2^{\mu+1}\}$ and $\mu \in \mathbb{N} \cup \{0\}$. It follows that, for $x \in [0,1]$ and any natural number $k \ge 2$,

$$\sum_{n=k}^{\infty} a_n^2 f_n^2(x) \le C_7^2 \sum_{\mu=[\log_2(k-1)]}^{\infty} a_{2^{\mu}}^2 \sum_{n=2^{\mu}+1}^{2^{\mu+1}} f_n^2(x). \tag{4}$$

It follows from (4) and the inequality (see [8])

$$\sum_{n=2^{\mu}+1}^{2^{\mu+1}} f_n^2(x) \le C_8 2^{\mu}, \qquad \mu \in \mathbb{N} \cup \{0\}, \quad x \in [0,1],$$

that

$$\sum_{n=k}^{\infty} a_n^2 f_n^2(x) \le C_9 \sum_{\mu=[\log_2(k-1)]}^{\infty} a_{2^{\mu}}^2 2^{\mu}, \tag{5}$$

where $C_9 = C_7^2 C_8$.

It was proved in [9] that, for all $p > 1$ and any number sequence $\{b_n\}_{n=0}^{\infty}$, the following inequality holds:

$$\int_0^1 \biggl|\sum_{n=0}^{\infty} b_n f_n(x)\biggr|^p dx \le C_p \int_0^1 \biggl(\sum_{n=0}^{\infty} b_n^2 f_n^2(x)\biggr)^{p/2} dx.$$

Therefore, from (3) and (5) we obtain

$$\int_0^1 \biggl|\sum_{n=k}^{\infty} a_n f_n(x)\biggr|^p dx \le C_p \biggl(C_9 \sum_{\mu=[\log_2(k-1)]}^{\infty} a_{2^{\mu}}^2 2^{\mu}\biggr)^{p/2} \to 0$$

as $k \to \infty$. The theorem is proved.

Theorem 3. *The Franklin system in $C[0,1]$ is a monotonically boundedly complete basis.*

Proof. Let $\{\overline{f}_n\}_{n=0}^{\infty}$ be the normalized Franklin system in $C[0,1]$, let $\{a_n\}_{n=0}^{\infty}$ be a monotone sequence of numbers tending to zero, and let

$$\sup_{n\in\mathbb{N}} \biggl\|\sum_{k=0}^{n} a_k \overline{f_k}\biggr\|_C < +\infty. \tag{6}$$

It follows from Lemma 1 that $\sum_{n=2^k+1}^{2^{k+1}} |\overline{f_n}(x)| \le C_{10}$ for all $k \in \mathbb{N} \cup \{0\}$ and $f_n(0) \le 0$ for all $n \in \mathbb{N}$. It was proved in [10] that $|\overline{f}_{2^n+1}(0)| = 1$. Combining this with (6), we see that

$$\sum_{n=0}^{\infty} a_{2^n+1} \le \sum_{n=0}^{\infty} a_n |\overline{f}_n(0)| < \infty. \tag{7}$$

Now let $n > m$, $2^p + 1 \le m \le 2^{p+1}$, and let $2^q + 1 \le n \le 2^{q+1}$. We note that

$$\left| \sum_{k=m}^{n} a_k \overline{f}_k(x) \right| \le \sum_{k=2^p+1}^{2^{q+1}} a_k |\overline{f}_k(x)| = \sum_{l=p}^{q} \sum_{k=2^l+1}^{2^{l+1}} a_k |\overline{f}_k(x)| \le C_{10} \sum_{l=p}^{q} a_{2^l+1}. \tag{8}$$

It follows from (7) and (8) that

$$\left\| \sum_{k=m}^{n} a_k \overline{f}_k(x) \right\|_C \to 0 \qquad \text{as} \quad n, m \to \infty.$$

The theorem is proved.

It should be noted that, by the same method, it can be proved that Theorem 1 holds for a Haar basis in $L^1[0,1]$.

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