ISSN 0001-4346, Mathematical Notes, 2019, Vol. 106, No. 6, pp. 885–893. © Pleiades Publishing, Ltd., 2019.
Russian Text © The Author(s), 2019, published in Matematicheskie Zametki, 2019, Vol. 106, No. 6, pp. 837–847.

On the Parametrization of an Algebraic Curve

A. D. Bryuno[1*]

[1] *Keldysh Institute of Applied Mathematics of Russian Academy of Sciences, Moscow, 125047 Russia*

Received March 29, 2018; in final form, January 23, 2019; accepted March 20, 2019

Abstract—At present, a plane algebraic curve can be parametrized in the following two cases: if its genus is equal to 0 or 1 and if it has a large group of birational automorphisms. Here we propose a new polyhedron method (involving a polyhedron called a *Hadamard polyhedron* by the author), which allows us to divide the space $\mathbb{R}^2$ or $\mathbb{C}^2$ into pieces in each of which the polynomial specifying the curve is sufficiently well approximated by its truncated polynomial, which often defines the parametrized curve. This approximate parametrization in a piece can be refined by means of the Newton method. Thus, an arbitrarily exact piecewise parametrization of the original curve can be obtained.

DOI: 10.1134/S0001434619110233

Keywords: *algebraic curve, genus of a curve, piecewise parametrization, Hadamard polyhedron, Newton method.*

1. INTRODUCTION

Despite the extensive development and complication of mathematics, some of its main problems remain unsolved. One of such problems is the solution of an algebraic equation with two unknowns. We shall consider a polynomial $f(X)$, where $X = (x_1, x_2)$, with real or complex coefficients. It is required to find the solution of the equation

$$f(X) = 0 \tag{1.1}$$

for $X \in \mathbb{R}^2$ or $\mathbb{C}^2$ in the form of functions $X = \Phi(\xi)$ of a parameter ξ. At the same time, the solution set of Eq. (1.1) constitutes an algebraic curve $\mathcal{F}$ and may consist of several components (branches) of different dimensions in the real case. Here we describe algorithms for solving this problem and their computer realization.

2. POLYHEDRON AND NORMAL CONES

Let several points $\{Q_1, \dots, Q_k\} = \mathbf{S}$ in $\mathbb{R}^n$ be given. Their convex hull

$$\Gamma(\mathbf{S}) = \left\{ Q = \sum_{i=1}^{k} \mu_i Q_i,\ \mu_i \geq 0,\ \sum \mu_i = 1 \right\}$$

is a polyhedron. Its boundary $\partial\Gamma$ consists of vertices $\Gamma_j^{(0)}$, edges $\Gamma_j^{(1)}$, and faces $\Gamma_j^{(d)}$ of different dimensions d, $1 < d \leq n-1$. If a real n-vector $P = (p_1, \dots, p_n)$ is given, then the maximum and the minimum of the inner product

$$\langle P, Q \rangle = p_1 q_1 + \dots + p_n q_n$$

[*] E-mail: abruno@keldysh.ru

on $\mathbf{S}$ are attained at points Q_i lying on the boundary $\partial\Gamma$. For each face $\Gamma_j^{(d)}$ (including the vertices $\Gamma_j^{(0)}$ and the edges $\Gamma_j^{(1)}$), we distinguish a set of vectors P for which the maximum of $\langle P, Q\rangle$ is attained at the points $Q_i \in \Gamma_j^{(d)}$. This will be the *normal cone* of the face $\Gamma_j^{(d)}$:

$$\mathbf{U}_j^{(d)} = \big\{P : \langle P, Q'\rangle = \langle P, Q''\rangle > \langle P, Q'''\rangle \text{ for } Q', Q'' \in \Gamma_j^{(d)},\ Q''' \in \Gamma \setminus \Gamma_j^{(d)}\big\}.$$

It is noted that the vector P lies in the space $\mathbb{R}_*^n$ dual to space $\mathbb{R}^n$. In general, here we deal with affine geometry.

All the vectors of the normal cone $\mathbf{U}_j^{(d)}$ are orthogonal to the faces $\Gamma_j^{(d)}$. In view of the homogeneity of the normal cones, it suffices to consider their intersections with two hyperplanes (e.g., $p_n = \pm 1$) on which we mark the intersections

$$\widetilde{\mathbf{U}}_{j\pm}^{(d)} \overset{\text{def}}{=} \mathbf{U}_j^{(d)} \cap \{p_n = \pm 1\},$$

called *reduced normal cones*. There are standard programs for both the calculation of the convex hulls and the calculation of the normal cones of their faces [1], [2]. In particular, they are included in the `Maple` system.

3. GLOBAL ANALYSIS OF THE CURVE

We assume that the polynomial $f(X) = f(x_1, x_2)$ is irreducible. Then its root set $\mathcal{F}$ is a plane algebraic curve. For complex values of x_1 and x_2, the curve $\mathcal{F}$ is a two-dimensional surface in the four-dimensional space with coordinates $\operatorname{Re} x_1, \operatorname{Im} x_1, \operatorname{Re} x_2, \operatorname{Im} x_2$. If to this space we add the points at infinity, then the curve $\mathcal{F}$ becomes topologically equivalent to the sphere with $\mathfrak{g}$ handles, $\mathfrak{g} \geq 0$ (Riemann, 1840); further,

- if $\mathfrak{g} = 0$, then this is the usual sphere;
- if $\mathfrak{g} = 1$, then this is the surface of a torus (thick doughnut);
- if $\mathfrak{g} = 2$, then this is the surface of a pretzel, etc.

Thus, the algebraic curve $\mathcal{F}$ has an integer topological invariant, namely, the genus $\mathfrak{g} \geq 0$, $\mathfrak{g} \in \mathbb{Z}$. We shall consider situations for various values of $\mathfrak{g}$.

3.1. The Case $\mathfrak{g} = 0$

In this case, the curve $\mathcal{F}$ is birationally equivalent to a straight line, i.e., there exists a parametrization

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \qquad t = \eta(x_1, x_2), \tag{3.1}$$

where φ_1, φ_2, and η are rational functions.

Example 1. For the folium of Descartes

$$f(x_1, x_2) \overset{\text{def}}{=} x_1^3 + x_2^3 - 3x_1x_2 = 0, \tag{3.2}$$

we have

$$x_1 = \frac{3t}{1+t^3}, \qquad x_2 = \frac{3t^2}{1+t^3}, \qquad t = \frac{x_2}{x_1}. \tag{3.3}$$

3.2. The Case $\mathfrak{g} = 1$ *[3]*

In this case, the curve $f(X) = 0$ is called *elliptic*. After the birational change $x_1, x_2 \leftrightarrow y_1, y_2$, it becomes the Weierstrass normal form

$$y_2^2 = 4y_1^3 - g_2 y_1 - g_3, \tag{3.4}$$

where g_2 and g_3 are constants (the moduli).

The uniformization of the normal form of the curve is provided by the Weierstrass function $\wp(t)$, which is the solution of the differential equation

$$[\wp'(t)]^2 = 4\wp^3(t) - g_2\wp(t) - g_3, \tag{3.5}$$

where $\wp' = d\wp(t)/dt$, $g_2, g_3 = \text{const}$, and, near $t = 0$,

$$\wp(t) = \frac{1}{t^2} + \sum_{k=1}^{\infty} b_k t^{2k}, \qquad b_k = \text{const}. \tag{3.6}$$

Thus, we obtain the uniformization

$$y_1 = \wp(t), \qquad y_2 = \wp'(t). \tag{3.7}$$

3.3. The Hyperelliptic Case with $\mathfrak{g} > 1$ *[3, Chap. 13]*

In this case, the birational transformation $x_1, x_2 \leftrightarrow y_1, y_2$ reduces the equation of the curve to the normal form

$$y_2^2 = R(y_1), \tag{3.8}$$

where $R(y_1)$ is a polynomial of degree $2\mathfrak{g} + 1$ or $2\mathfrak{g} + 2$ without multiple roots. The solutions of this equation can be written as $y_1 = \varphi(t)$, $y_2 = \psi(t)$, where the automorphic functions $\varphi(t)$ and $\psi(t)$ can be expressed in a certain way in terms of the theta function. If $\mathfrak{g} = 2$, then the curve is always hyperelliptic (see examples in [4]–[11]), but, for $\mathfrak{g} > 2$, there exist nonhyperelliptic curves [12, Chap. 7].

3.4. The Superelliptic Case with $\mathfrak{g} > 2$

In this case, a birational change of coordinates reduces the equation to the normal form

$$y_2^m = R(y_1), \tag{3.9}$$

where $m \geq 3$ is an integer and $R(y_1)$ is a polynomial. Uniformization in terms of the theta function exists in this case, too. But it is not clear how to find it; see [13], [14].

Apparently, there exist curves that are not superelliptic. But there is no normal form for them so far.

Example 2. Consider the Fermat curve

$$x^n + y^n = 1. \tag{3.10}$$

Its genus is $\mathfrak{g} = n(n-3)/2 + 1$. The values of $\mathfrak{g}$ for some n are given in the table.

Table. The genus of the Fermat curve (3.10)

n	2	3	4	5	6	7	8	9	10
$\mathfrak{g}$	0	1	3	6	10	15	21	28	36

This curve is rational only for $n = 2$. In addition to the case $n = 3$, its uniformization is also known for $n = 4$ and 8 [6].

3.5. The General Case

For an arbitrary curve $f(x_1, x_2) = 0$, only the theorem on the existence of its global uniformization $x_1 = \varphi(t)$, $x_2 = \psi(t)$ is known, but there is no analytic algorithm for its calculation. At present, for $\mathfrak{g} > 1$, explicit uniformizations are known only for curves having a sufficiently large symmetry group, i.e., that of birational automorphisms [4]–[6]. Moreover, even for hyperelliptic curves, such a uniformization can be found mainly in cases of additional symmetries [4]–[9].

3.6. Implementation

All the calculations presented in Secs. 3.1–3.3 exist in the computer algebra system `Maple`. Using the package `algcurves`, one can calculate the genus of the curve $\mathfrak{g}$. If $\mathfrak{g} = 0$ or 1, then one can obtain the corresponding birational change of coordinates. If $\mathfrak{g} > 2$, then one can find whether or not the curve is hyperelliptic. For a hyperelliptic curve, one can find the birational transform to the normal form, but there is no search algorithm for uniformization. Questions related to superelliptic curves are not considered in `Maple`.

Example 3. The paper [10] presents the uniformization of the curve

$$w^3 - 3A(z)w - 2B(z) = 0,$$

where

$$A(z) = \frac{1}{z^2 - 1}, \quad B(z) = \frac{z}{(z^2 - a^2)(z^2 - 1)}, \qquad 0 < a < 1.$$

The calculations in the `Maple` system showed that the genus of this curve is 2 and its normal form is

$$y^2 = x^6 - 3a^2x^4 + 3a^4x^2 - 2a^4 + a^2.$$

3.7. The Hadamard Polyhedron Method

Since a parametrization for the curve $\mathcal{F}\colon f(x_1, x_2) = 0$ has not been found, we can obtain several simpler approximate curves $\breve{\mathcal{F}}_l\colon \breve{f}_l(x_1, x_2) = 0$, $l = 1, \dots, m$, that approximate the original curve in different sets $\mathcal{W}_l$ of the space $\mathbb{R}^2$ or $\mathbb{C}^2$.

It may happen that the curve $\breve{\mathcal{F}}_l$ can be parametrized: $x_1 = \breve{\varphi}_l(t)$, $x_2 = \breve{\psi}_l(t)$. For the initial curve $\mathcal{F}$, this parametrization can be refined.

Let

$$g(t, \varepsilon) \overset{\text{def}}{=} f(\breve{\varphi}_l(t) + \beta_1\varepsilon(t), \breve{\psi}_l(t) + \beta_2\varepsilon(t)),$$

where $\beta_i = \text{const}$ and $|\beta_1| + |\beta_2| \neq 0$. We shall find $\varepsilon(t)$ by using the Newton method in the form of an expansion

$$\varepsilon(t) = \sum_{k=1}^{\infty} \varepsilon_k(t).$$

Here $\varepsilon_1(t)$ is determined from the equation

$$f(\breve{\varphi}_l(t), \breve{\psi}_l(t)) + \varepsilon_1(t)\frac{\partial g}{\partial \varepsilon}(\breve{\varphi}_l(t), \breve{\psi}_l(t)) = 0.$$

Further, we obtain $\varepsilon_2(t)$ from the equation

$$f[\breve{\psi}_l(t) + \beta_1\varepsilon_1(t), \breve{\psi}_l(t) + \beta_2\varepsilon_1(t)] + \varepsilon_2(t)\frac{\partial g_1}{\partial \varepsilon_2}[\breve{\psi}_l(t) + \beta_1\varepsilon_1(t), \breve{\psi}_l(t) + \beta_2\varepsilon_1(t)] = 0,$$

where

$$g_1(t, \varepsilon_2) = f[\breve{\psi}_l(t) + \beta_1\varepsilon_1(t) + \beta_1\varepsilon_2(t), \breve{\psi}_l(t) + \beta_2\varepsilon_1(t) + \beta_2\varepsilon_2(t)],$$

etc. Here all the successive additions $\varepsilon_1(t), \varepsilon_2(t), \ldots$ are rational functions of $\breve{\varphi}_l(t)$ and $\breve{\psi}_l(t)$.

We can find these curves $\breve{\mathcal{F}}_l$ as follows. Let

$$f(X) = \sum a_Q X^Q, \tag{3.11}$$

where $X = (x_1, x_2)$, $Q = (q_1, q_2)$, and $X^Q = x_1^{q_1} x_2^{q_2}$. To each summand $a_Q X^Q$ we assign the point $\breve{Q} = (Q, \ln|a_Q|)$ in $\mathbb{R}^3$. The collection of these points constitutes a *supersupport* $\breve{\mathbf{S}} \subset \mathbb{R}^3$. Its convex hull will be denoted by $\mathbf{H}$. The union of the faces $\breve{\Gamma}_l^{(2)}$ of the upper part of its boundary $\partial\mathbf{H}$, i.e., with $p_3 > 0$ in the normal cone $\mathbf{U}_j^{(2)}$, will be called an *Hadamard polyhedron*. To each such face $\breve{\Gamma}_l^{(2)}$ there corresponds the truncated polynomial

$$\breve{f}_l^{(2)}(X) = \sum a_Q X^Q \qquad \text{by} \quad \breve{Q} \in \breve{\Gamma}_l^{(2)} \cap \breve{\mathbf{S}}. \tag{3.12}$$

Here the sum is taken over all Q such that

$$\breve{Q} = (Q, \ln|a_Q|) \in \breve{\Gamma}_l^{(2)} \cap \breve{\mathbf{S}}.$$

The faces whose normal cone contains $p_3 \le 0$ are eliminated from our study, because the corresponding truncated polynomials approximate the original polynomial only for x_1 or x_2, or both x_1 and x_2 tending to 0 or ∞,. The normal cones of the remaining faces intersect with the plane $p_3 = 1$, i.e., they possess reduced normal cones. It is these faces which are considered in what follows and constitute an Hadamard polyhedron.

For the normal form of the superelliptic curve $x_2^m = R(x_1)$, only the values of x_1 are bounded in the sets $\mathcal{W}_l$, while the values of x_2 are arbitrary there. The accuracy of the approximation $\breve{\mathcal{F}}_l$ to $\mathcal{F}$ can be estimated by the accuracy of approximation of that of the roots of the equation $R(x_1) = 0$ by roots of the equation $\breve{R}_l(x_1) = 0$ [15] (see examples below).

Example 4. Consider the curve $\mathcal{F}$:

$$f \stackrel{\text{def}}{=} -y^2 + 9x - 10x^3 + x^5 = 0. \tag{3.13}$$

Its genus is 2. By [6, Proposition 18], the curve (3.13) has the following uniformization in theta functions:

$$x = 3\frac{\theta_3^2(3t)}{\theta_3^2(t)}, \qquad y = 48\sqrt{3}\, i \frac{\theta_3^3(3t)}{\theta_3^3(t)} \frac{\theta_2^2(t)\theta_4^2(t)}{9\theta_3^4(3t) - \theta_3^4(t)}.$$

Its polyhedron $\mathbf{H}$ is spanned by the four vertices $(0,2,0)$, $(1,0,\ln 9)$, $(3,0,\ln 10)$, $(5,0,0)$. The projection of the Hadamard polyhedron on the plane q_1, q_2 is shown in Fig. 1, and the projection on the plane $q_1, q_3 = \ln|a_Q|$ is shown in Fig. 2.

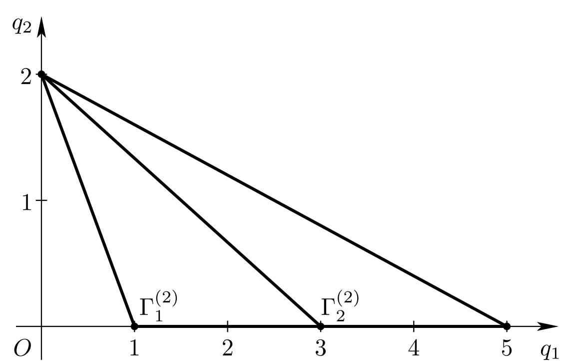

Fig. 1. The projection of the Hadamard polyhedron of the polynomial (3.13) on the plane q_1, q_2.

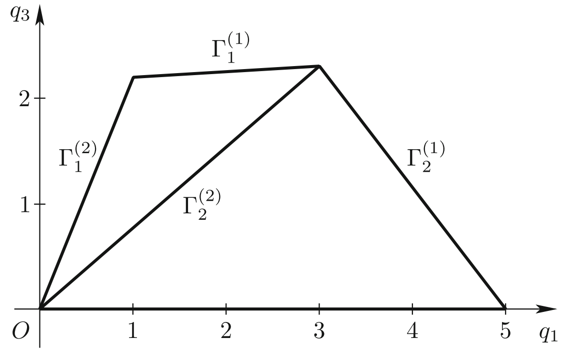

Fig. 2. The projection of the Hadamard polyhedron of the polynomial (3.13) on the plane $q_1, q_3 = \ln |a_Q|$.

It is seen from Fig. 1 that, for $q_2 \neq 0$, the polyhedron $\mathbf{H}$ has exactly two upper two-dimensional faces $\Gamma_1^{(2)}$ and $\Gamma_2^{(2)}$. They correspond to the following two truncated polynomials:

$$\breve{f}_1^{(2)} = -y^2 + 9x - 10x^3, \qquad \breve{f}_2^{(2)} = -y^2 - 10x^3 + x^5.$$

Their reduced normal cones are the points $\widetilde{\mathbf{U}}_{1+}^{(2)} = (\omega, \gamma)$ and $\widetilde{\mathbf{U}}_{2+}^{(2)} = (\beta, \delta)$, where

$$\omega = \ln \sqrt{0.9} \approx -0.05268, \qquad \gamma = \frac{3\ln 9 - \ln 10}{4} \approx 1.07227,$$

$$\beta = \ln \sqrt{10} \approx 1.15129, \qquad \delta = \frac{5 \ln 10}{4} \approx 2.87231.$$

They are shown in Fig. 3 together with the reduced normal cones of the vertices and edges.

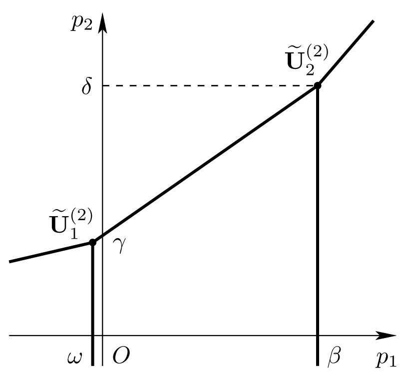

Fig. 3. The reduced normal cones of the vertices, edges, and faces of the Hadamard polyhedron of the polynomial (3.13).

The domains are assumed to be

$$\mathcal{W}_1 = \left\{(x, y) : \ln |x| < \frac{\omega + \beta}{2} \approx 0.5493\right\},$$

i.e., $\{|x| < 1.73204$, y is arbitrary$\}$, and

$$\mathcal{W}_2 = \left\{(x, y) : \ln |x| > \frac{\omega + \beta}{2} \approx 0.5493\right\},$$

i.e., $\{|x| > 1.73204$, y is arbitrary$\}$. The curves $\breve{\mathcal{F}}_1$ and $\breve{\mathcal{F}}_2$ have genus 1.

The transformation $x_1 = -10x$, $y_1 = -20y$ reduces the equation $\breve{f}_1 = 0$ to its Weierstrass normal form

$$y_1^2 = 4x_1^3 - 360x_1,$$

for which $g_2 = 360$ and $g_3 = 0$. Therefore, the uniformization for the curve $\breve{\mathcal{F}}_1$ is as follows:

$$x = -\frac{1}{10}\,\wp(t|360,0) \overset{\text{def}}{=} \breve{\varphi}_1(t), \qquad y = -\frac{1}{20}\,\wp'(t|360,0) \overset{\text{def}}{=} \breve{\psi}_1(t). \tag{3.14}$$

The transformation $x = x_2$, $y = x_2y_2/2$ reduces the equation $\breve{f}_2 = 0$ to the Weierstrass normal form

$$y_2^2 = 4x_2^3 - 40x_2,$$

for which $g_2 = 40$ and $g_3 = 0$. Therefore, the uniformization for the curve $\breve{\mathcal{F}}_2$ is as follows:

$$x = \wp(t|40,0) \overset{\text{def}}{=} \breve{\varphi}_2(t), \quad y = \frac{1}{2}\,\wp(t|40,0)\wp'(t|40,0) \overset{\text{def}}{=} \psi_2(t). \tag{3.15}$$

The curves $\mathcal{F}$, $\breve{\mathcal{F}}_1$, and $\breve{\mathcal{F}}_2$ are shown in Fig. 4.

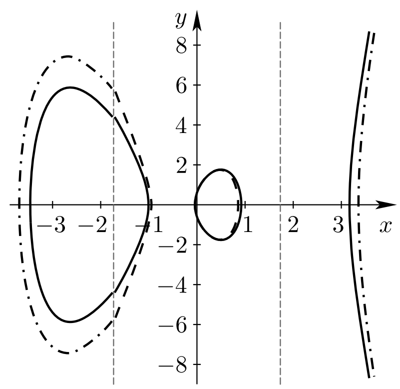

Fig. 4. The curves of Example 4: $\mathcal{F}$ (solid), $\breve{\mathcal{F}}_1$ (dotted line), $\breve{\mathcal{F}}_2$ (dot-and-dash line). The curves $\breve{\mathcal{F}}_1$ and $\breve{\mathcal{F}}_2$ are only shown in their domains $\mathcal{W}_1$ and $\mathcal{W}_2$.

For $y = 0$, the roots of Eq. (3.13) are $x = 0, \pm 1, \pm 3$, the roots of the equation $\breve{f}_1^{(2)} = 0$ are $x = 0, \pm\sqrt{0.9} \approx \pm 0.948683$, and the roots of the equation $\breve{f}_2^{(2)} = 0$ are $x = 0, \pm\sqrt{10} \approx \pm 3.162278$.

It is seen that the roots of the truncated equations are close to the roots of the complete equation. Therefore, we could assume that the curves $\breve{\mathcal{F}}_1$ and $\breve{\mathcal{F}}_2$ are close to the curve $\mathcal{F}$ in their domains W_1 and W_2, respectively. This can be seen from Fig. 4.

Let us refine the curve (3.14) as an approximation to the curve (3.13). To do this, we set $x = \breve{\varphi}_1(t) + \varepsilon$ and $y = \breve{\psi}_1(t)$. Then, using the Newton method for ε, we obtain the equation

$$-\breve{\psi}_1^2 + \varepsilon(9 - 30\breve{\varphi}_1^2 + 5\breve{\varphi}_1^4) + 9\breve{\varphi}_1 - 10\breve{\varphi}_1^3 + \breve{\varphi}_1^5 = 0.$$

Using the equation of the curve $\breve{\mathcal{F}}_1$, we find that

$$\varepsilon(9 - 30\breve{\varphi}_1^2 + 5\breve{\varphi}_1^4) + \breve{\varphi}_1^5 = 0, \qquad \text{i.e.,} \quad \varepsilon = \varepsilon_1 = -\frac{\breve{\varphi}_1^5}{9 - 30\breve{\varphi}_1^2 + 5\breve{\varphi}_1^4}.$$

We can also calculate subsequent corrections.

Similarly, we refine the curve (3.15) as an approximation to the curve (3.13). To do this, we set $x = \breve{\varphi}_2(t) + \varepsilon$ and $y = \breve{\psi}_2(t)$. Then, in the first approximation to ε, we obtain

$$-\breve{\psi}_2^2 + \varepsilon(9 - 30\breve{\varphi}_2^2 + 5\breve{\varphi}_2^4) + 9\breve{\varphi}_2 - 10\breve{\varphi}_2^3 + \breve{\varphi}_2^5 = 0.$$

But now $-\breve{\psi}_2^2 = -10\breve{\varphi}_2^3 + \breve{\varphi}_2^5$; therefore,

$$\varepsilon = \varepsilon_1 = -\frac{9\breve{\varphi}_2}{9 - 30\breve{\varphi}_2^2 + 5\breve{\varphi}_2^4}.$$

Example 5. Consider the curve $\mathcal{F}$:

$$f \stackrel{\text{def}}{=} -y^2 - x - \frac{5}{6}x^3 + x^5 = 0. \tag{3.16}$$

Its genus is equal to 2 and the parametrization is unknown. Its supersupport $\mathbf{S}$ consists of the four points $(0,2,0)$, $(1,0,0)$, $(3,0,\ln(5/6))$, and $(5,0,0)$. The Hadamard polyhedron $\mathbf{H}$ has only one upper face $\Gamma_1^{(2)}$ containing the points $(0,2,0)$, $(1,0,0)$, and $(5,0,0)$, with reduced normal cone $\widetilde{\mathbf{U}}_{1+}^{(2)} = (0,0)$. To this face there corresponds one truncated equation:

$$\breve{f}_1^{(2)} \stackrel{\text{def}}{=} -y^2 - x + x^5 = 0. \tag{3.17}$$

This is the Burnside curve, its explicit parametrization $x = \breve{\varphi}_1(t)$, $y = \breve{\psi}_1(t)$ in the functions $\wp$ and $\wp'$ is given in the complicated formula (3) of the paper [5].

For $y = 0$, the roots of Eq. (3.16) are $x = 0, \pm\sqrt{3/2} \approx \pm 1.2241$, and those of Eqs. (3.17) are $x = 0, \pm 1$. They are close to each other, just as the corresponding curves in Fig. 5.

Let us find the correction ε to the curve $\breve{\mathcal{F}}_1$. We set $g(t,\varepsilon) = f(\breve{\varphi}_1(t) + \varepsilon, \breve{\psi}_1(t))$. Then

$$\frac{\partial g}{\partial \varepsilon} = \left.\frac{\partial f}{\partial x}\right|_{(\breve{\varphi}_1, \breve{\psi}_1)},$$

and, in the first approximation, we obtain the following equation for ε:

$$f(\breve{\varphi}_1, \breve{\psi}_1) + \varepsilon \frac{\partial f}{\partial x}(\breve{\varphi}_1, \breve{\psi}_1) = 0, \qquad \text{i.e.,} \quad \varepsilon = \frac{(5/6)\breve{\varphi}_1^3(t)}{-1 - (5/2)\breve{\varphi}_1^2 + 5\breve{\varphi}_1^4}.$$

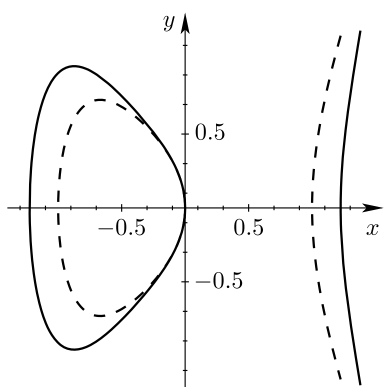

Fig. 5. The curves of Example 5: $\mathcal{F}$ and $\breve{\mathcal{F}}_1$.

This procedure can be extended to obtain subsequent additions $\varepsilon_2, \varepsilon_3, \ldots$ as rational functions of $\breve{\varphi}_1(t)$ and $\breve{\psi}_1(t)$. Here the domain $\mathcal{W}_1$ coincides with the whole space.

Using this approach, we obtain the parametrization problem for curves of the form

$$y^m = a_k x^k + a_l x^l, \qquad k < l, \quad a_k, a_l = \text{const} \neq 0.$$

They can have any genus (see Example 2), but they have many symmetries, i.e., birational automorphisms.

Remark 1. A similar technique also applies to the case $n = 3$ for the global parametrization of a two-dimensional algebraic manifold given by one polynomial in three variables. If there is no such global parametrization, then it can be written in the form of several approximate parametrizations that are obtained by using the three-dimensional upper part of the boundary of a four-dimensional polyhedron.

Remark 2. Certainly, to ensure parametrization by means of an Hadamard polyhedron, it is desirable to reduce the polynomial $f(X)$ to its simplest form. To this end, in Examples 4 and 5, we took hyperelliptic polynomials, which are of normal form. Their proposed parametrization is reduced to the parametrizations of curves of the form $y^2 = x^k + 1$, which can, apparently, be parametrized in terms of theta functions. But even such a realization will be a remarkable achievement.

Remark 3. It is seen from Examples 4 and 5 that the curves obtained by the proposed method approximate both connected components of the original curves in the real sense. Apparently, this will always be so, because this method applies to the complex case as well.

Remark 4. For the polynomial (3.11), the set of points $\{Q\}$ is a support, and their convex hull is a Newton polygon. For that reason, the set of points $\breve{Q} = (Q, \ln|a_Q|)$ was called a supersupport. In general, an Hadamard polyhedron differs from a Newton polygon in that it takes into account the values of the coefficients of the polynomial.

REFERENCES

1. K. Fukuda, "Exact algorithms and software in optimization and polyhedral computation," in *XXI International Symposium on Symbolic and Algebraic Computations* (ACM, New York, 2008), pp. 333–334.
2. C. B. Barber, D. P. Dobkin, and H. Huhdanpaa, "The Quickhull algorithm for convex hulls," ACM Trans. Math. Software **22** (4), 469–483 (1996).
3. *Higher Transcendental Functions*, Vol. 3: *Elliptic and Modular Functions, Lamé and Mathieu Functions* (McGraw–Hill, New York–Toronto–London, 1955; Nauka, Moscow, 1967).
4. Yu. V. Brezhnev, "On uniformization of algebraic curves," Moscow Math. J. **8** (2), 233–271 (2008).
5. Yu. V. Brezhnev, "On uniformization of Burnside's curve $y^2 = x^5 - x$," J. Math. Phys. **50** (103519) (2009).
6. Yu. V. Brezhnev, "The sixth Painlevé transcendent and uniformization of algebraic curves," J. Differential Equations **260** (3), 2507–2556 (2016).
7. E. D. Belokolos and V. Z. Enolskii, "Reduction of abelian functions and algebraically integrable systems. I," J. Math. Sci. (New York) **106** (6), 3395–3486 (2001).
8. E. D. Belokolos and V. Z. Enolskii, "Reduction of abelian functions and algebraically integrable systems. II," J. Math. Sci. (New York) **108** (3), 295–374 (2002).
9. Y. Ônishi, "Complex multiplication formulae for hyperelliptic curves of genus three," Tokyo J. Math. **21** (2), 381–431 (1998).
10. A. I. Aptekarev, D. N. Tulyakov, and M. L. Yattselev, "On the parametrization of a certain algebraic curve of genus 2," Mat. Zametki **98** (5), 782–785 (2015) [Math. Notes **98** (5), 843–846 (2015)].
11. A. I. Aptekarev, D. N. Toulyakov, and W. Van Assche, "Hyperelliptic uniformization of algebraic curves of third order," J. Comput. Appl. Math. **284**, 38–49 (2015).
12. R. J. Walker, *Algebraic Curves* (Dover Publ., New York, 1962; ("Librokom," Moscow, 2009).
13. J. C. Eilbeck, V. Z. Enolski, S. Matsutani, Y. Ônishi, and E. Previato, "Abelian functions for trigonal curves of genus three," Int. Math. Res. Not. IMRN, No. 1 (2008), Art. ID rnm140.
14. J. C. Eilbeck, *Weierstrass Functions for Higher Genus Curves*, `http://www.ma.hw.ac.uk/Weierstrass (2011)`.
15. A. D. Bryuno, "Algorithms for solving an algebraic equation," Programming and Computer Software **44** (6), 533–545 (2018).