ISSN 0001-4346, Mathematical Notes, 2019, Vol. 106, No. 5, pp. 727–739. © Pleiades Publishing, Ltd., 2019.

On Singular Operators in Vanishing Generalized Variable-Exponent Morrey Spaces and Applications to Bergman-Type Spaces*

A. N. Karapetyants[1,2], H. Rafeiro[3***], and S. G. Samko[4****]**

[1]Southern Federal University, Rostov-on-Don, 334006 Russia

[2]State University of New York at Albany, 12222 USA

[3]Department of Mathematical Sciences, College of Sciences, United Arab Emirates University, Al Ain, 15551 UAE

[4]University of Algarve, Faro, 8005-139 Portugal

Received September 13, 2018; in final form, February 16, 2019; accepted March 20, 2019

Abstract—We give a proof of the boundedness of the Bergman projection in generalized variable-exponent vanishing Morrey spaces over the unit disc and the upper half-plane. To this end, we prove the boundedness of the Calderón–Zygmund operators on generalized variable-exponent vanishing Morrey spaces. We give the proof of the latter in the general context of real functions on $\mathbb{R}^n$, since it is new in such a setting and is of independent interest. We also study the approximation by mollified dilations and estimate the growth of functions near the boundary.

DOI: 10.1134/S0001434619110075

Keywords: *singular operator, Morrey space, Bergman-type space, Calderón–Zygmund operator.*

1. INTRODUCTION

Let $dA(z)$ denote the normalized area measure on $\mathbb{D}$, so that the area of $\mathbb{D}$ is 1. As usual, $\mathcal{A}^p(\mathbb{D})$ stands for the Bergman space of functions holomorphic in $\mathbb{D}$ that belong to $L^p(\mathbb{D}) = L^p(\mathbb{D}; dA(z))$. The Bergman projection $B_{\mathbb{D}}$,

$$B_{\mathbb{D}} f(z) = \int_{\mathbb{D}} K(z,w) f(w) dA(w) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2}\, dA(w), \qquad z \in \mathbb{D}, \tag{1.1}$$

is well defined on $L^1(\mathbb{D})$ and bounded as an operator from $L^p(\mathbb{D})$ to $\mathcal{A}^p(\mathbb{D})$ for $1 < p < \infty$. Let $\mathbb{R}^2_+$ stand for the upper half-plane, and let $dA(z) = dx\,dy$ be the Lebesgue area measure. The symbol $\mathcal{A}^p(\mathbb{R}^2_+)$ stands for the Bergman space of functions f holomorphic in $\mathbb{R}^2_+$ and belonging to $L^p(\mathbb{R}^2_+) = L^p(\mathbb{R}^2_+; dA(z))$. The corresponding Bergman projection $B_{\mathbb{R}^2_+}$, which is defined on $f \in L^1(\mathbb{R}^2_+)$ as

$$B_{\mathbb{R}^2_+} f(z) = \int_{\mathbb{R}^2_+} K(z,w) f(w) dA(w) = -\frac{1}{\pi} \int_{\mathbb{R}^2_+} \frac{f(w)}{(z - \overline{w})^2}\, dA(w), \qquad z \in \mathbb{R}^2_+,$$

is bounded as an operator from $L^p(\mathbb{R}^2_+)$ to $\mathcal{A}^p(\mathbb{R}^2_+)$ for $1 < p < \infty$.

A boundedness result for more general weighted projections on weighted Bergman spaces is also well known. For references, see [1]–[4]. The existence of a bounded projection is very useful in the theory of Bergman-type spaces. There exist many extensions of the boundedness result to mixed-norm spaces,

*The article was submitted by the authors for the English version of the journal.

**E-mail: karapetyants@gmail.com

***E-mail: rafeiro@uaeu.ac.ae

****E-mail: ssamko@ualg.pt

general weights satisfying Bekolle–Bonami conditions, etc. Without any claim of completeness, we refer to the papers [5]–[7]; see also references therein. In general, most of the results on boundedness were proved by essentially using the holomorphic nature of the Bergman kernel function.

In the paper [14], a different approach was used, based on reducing the Bergman projection operator to a particular case of the Calderón–Zygmund operator, namely, the Riesz transforms, and then applying facts known for such operators.

This approach was further developed in the papers [10], [11] for the cases of variable-exponent Lebesgue space, Orlicz space, and generalized variable-exponent Morrey space on the unit disc and the half-plane (and also for the corresponding harmonic Bergman-type spaces). Note that, in the case of $L^{p(\cdot)}(\mathbb{D})$, a boundedness result was obtained in [17]. Our proof via singular operators is simpler in a sense.

The main idea of our paper is to show that the approach of [14] can be used to prove the boundedness of the Bergman projection in the case of vanishing Morrey space. As is known, variable-exponent spaces have become very popular in analysis during the last two decades. Following that approach, one can prove similar results for many other function spaces.

We also study the rate of growth of functions near the boundary in generalized Morrey-Bergman and generalized vanishing Morrey-Bergman spaces with variable exponent and their approximation by the so-called mollified dilations. In general, functions from the variable-exponent generalized Morrey-Bergman space cannot be approximated by continuous functions because of the nature of the function space equipped with the supremum norm. In the case of the unit disc, such an approximation is possible in the vanishing Morrey-Bergman space. In the case of domains of infinite measure, for example, a half-plane, a narrower subspace should be used (see Sec. 5).

2. BOUNDEDNESS OF THE CALDERÓN–ZYGMUND SINGULAR OPERATORS ON A GENERALIZED VARIABLE-EXPONENT VANISHING MORREY SPACE

In this section, both the Calderón–Zygmund operators and the generalized variable-exponent vanishing Morrey spaces are considered in a setting more general than in the rest of the paper.

Let $D \subseteq \mathbb{R}^n$ be an open set, and let $d = \operatorname{diam}\ D$ and x, y, z stand for points in $\mathbb{R}^n$. Let $p = p(x)$ be a measurable function on D with values in $[1, \infty]$. For details on the variable-exponent spaces $L^{p(\cdot)}(D)$, we refer the reader to [27], [28], [20], [21]. Recall that $\|f\|_{L^{p(\cdot)}(D)}$ is defined by

$$\|f\|_{L^{p(\cdot)}(D)} = \inf\{\lambda > 0 : \int_{D\setminus D_\infty} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\} + \sup_{x\in D_\infty} |f(x)|, \qquad 1 \le p(x) \le +\infty.$$

where $D_\infty = \{x : p(x) = \infty\}$.

We suppose that $1 \le p^- \le p(x) \le p^+ < \infty$, where the standard definitions $p^+ = \operatorname{esssup}_{x\in D} p(x)$ and $p^- = \operatorname{essinf}_{x\in D} p(x)$ are used. We say that the exponent p satisfies the log-condition on D if

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \qquad |x-y| \le \frac{1}{2}, \quad x, y \in D, \tag{2.1}$$

where $C > 0$ depends on p but not on x and y. The function p is said to satisfy the decay condition if there exists a number p_∞, denoted by the same symbol $p(\infty)$, such that

$$|p(x) - p_\infty| \le \frac{C}{\ln(e + |x|)}, \qquad |x| \to \infty, \quad x \in D. \tag{2.2}$$

To introduce the generalized variable-exponent Morrey space $L^{p(\cdot),\omega(\cdot)}(D)$ and its subspace, the generalized variable-exponent vanishing Morrey space $VL^{p(\cdot),\omega(\cdot)}(D)$, we define the modular

$$\mathfrak{M}_{p(\cdot),\omega(\cdot)}(f; x, r) := \frac{1}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(D(x,r)\cap D)},$$

where $\omega(x,r)$ is a nonnegative measurable function on $D\times[0,d]$. The generalized Morrey space $L^{p(\cdot),\omega(\cdot)}(D)$ over the open set D is defined as the set of functions f measurable on D and such that

$$\|f\|_{L^{p(\cdot),\omega(\cdot)}(D)}=\sup_{x\in D,r\in(0,d)}\mathfrak{M}_{p(\cdot),\omega(\cdot)}(f;x,r)<\infty.$$

For more details, we refer the reader to [21]. It is known that the space $L^{p(\cdot),\omega(\cdot)}(D)$ is nontrivial if

$$\inf_{r>\delta}\inf_{x\in D}\omega(x,r)>0 \qquad \text{for any} \quad \delta>0.$$

The vanishing Morrey space $VL^{p(\cdot),\omega(\cdot)}(D)$ is introduced as

$$VL^{p(\cdot),\omega(\cdot)}(D):=\{f\in L^{p(\cdot),\omega(\cdot)}(D):\lim_{r\to 0}\sup_{x\in D}\mathfrak{M}_{p(\cdot),\omega(\cdot)}(f;x,r)=0\}.$$

We set

$$p_*(x,r)=\begin{cases}2/p(x), & r\le 1,\\ 2/p(\infty), & r>1\end{cases} \quad\text{and}\quad p(x,r)=\begin{cases}p(x), & r\le 1,\\ p(\infty), & r>1.\end{cases}$$

By K we denote the Calderón–Zygmund singular operator

$$Kf(x)=\int_D K(x,y)f(y)\,dA(y)$$

with so-called "standard" kernel $K(x,y)$ (see, e.g., [39, p. 99]) i.e., a continuous function defined on $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, $x\ne y$, and satisfying the estimates

(a) $|K(z,w)|\le C|z-w|^{-n}$ for all $z\ne w$;

(b) $|K(x,y)-K(x,z)|\le C\dfrac{|y-z|^\sigma}{|x-y|^{n+\sigma}}$, $\sigma>0$, if $|x-y|>2|y-z|$;

(c) $|K(x,y)-K(z,y)|\le C\dfrac{|x-z|^\sigma}{|x-y|^{n+\sigma}}$, $\sigma>0$, if $|x-y|>2|x-z|$.

Everywhere in what follows we shall assume that the operator K is bounded on $L^2(\mathbb{R}^n)$; then it is bounded on $L^p(\mathbb{R}^n)$, $1<p<\infty$ ([39]).

The classical Calderón–Zygmund kernels $\Omega((x-y)/|x-y|)/|x-y|^n$, where $\Omega(x')$ satisfies Hölder condition of order $\sigma>0$ on the unit sphere $\mathbb{S}^{n-1}$, satisfy the above conditions (a)–(c).

In the theorem below, we use the notation

$$\psi(x,r)=\frac{w(x,r)}{r^{n/p(x,r)}}.$$

Theorem 1. *Suppose that $p^->1$, $p^+<\infty$, and p satisfies the log-condition* (2.1) *and the decay condition* (2.2) (*when D is unbounded*). *Suppose also that*

$$\int_r^\infty \frac{\operatorname{essinf}_{t<s<\infty}\omega(x,s)}{t^{1+n/p(x,t)}}\,dt\le C\frac{\omega(x,r)}{r^{n/p(x,r)}}, \tag{2.3}$$

where C is independent of x and r,

$$\int_r^{\delta_0}\frac{\psi(x,t)}{t}\,dt\le C\psi(x,r) \tag{2.4}$$

for some $\delta_0>0$, where C is independent of x,r,

$$\int_\delta^\infty\frac{\psi(x,t)}{t}\,dt\le C_\delta \tag{2.5}$$

for every $\delta > 0$, where C_δ is independent of x, and

$$\lim_{r\to 0}\sup_{x\in D}\frac{1}{\psi(x,r)} = 0. \tag{2.6}$$

Then the Calderón–Zygmund singular operator K is bounded on $VL^{p(\cdot),\omega(\cdot)}(D)$.

Proof. Note that (2.3) already guarantees boundedness in the $L^{p(\cdot),\omega(\cdot)}$-space:

$$\|Kf\|_{L^{p(\cdot),\omega(\cdot)}} \le c\|f\|_{L^{p(\cdot),\omega(\cdot)}}. \tag{2.7}$$

Clearly, (2.7) holds on the subspace $VL^{p(\cdot),\omega(\cdot)} \subset L^{p(\cdot),\omega(\cdot)}$, but this does not yet mean the boundedness of K on $VL^{p(\cdot),(\omega)}$, i.e., the preservation of the property

$$\lim_{r\to 0}\sup_{x\in D}\mathfrak{M}_{p(\cdot),\omega(\cdot)}(f;x,r) = 0$$

by the operator K. This is a known problem for spaces of vanishing type; see [9] for singular operators on vanishing Morrey spaces and [29] for fractional maximal operators on vanishing Orlicz–Morrey spaces.

We follow the scheme of argument in [9] (see Theorem 5.1). Assume that $f \in VL^{p(\cdot),\omega(\cdot)}(D)$. We use formula (3.13) from [8]:

$$\mathfrak{M}_{p(\cdot),\omega(\cdot)}(Kf;x,r) \le C\frac{1}{\psi(x,r)}\int_r^\infty \frac{\psi(x,t)}{t}\mathfrak{M}_{p(\cdot),\omega(\cdot)}(f;x,t)dt. \tag{2.8}$$

We split the integral on the right-hand side of (2.8) in two integrals over the intervals (r,δ) and (δ,∞) with some $\delta > 0$, which may be taken arbitrarily small, since we are interested in the behaviour of the left-hand side of (2.8) as $r \to 0$.

To estimate the first integral, we observe that, for $f \in VL^{p(\cdot),\omega(\cdot)}(D)$, the expression $\mathfrak{M}_{p(\cdot),\omega(\cdot)}(f;x,t)$ can be made arbitrarily small uniformly in $x \in D$ and $t \in (r,\delta)$ by an appropriate choice of δ; then we use (2.4).

Now fix δ. To estimate the second integral, we replace the expression $\mathfrak{M}_{p(\cdot),\omega(\cdot)}(f;x,t)$ with a constant and then use (2.5) and (2.6) to see that the second term goes to zero as $r \to 0$. □

In what follows, we shall consider generalized variable-exponent Morrey and vanishing Morrey spaces with ω given by

$$\omega(x,r) = \frac{r^{n/p(x)}}{\varphi(x,r)}, \qquad 0 < r < d, \quad x \in D.$$

Here and everywhere in what follows, the function $\varphi(x,r)$ is a nonnegative increasing function on $(0,d)$ such that

$$\inf_{r>\nu}\varphi(x,r) > 0 \qquad \text{for all} \quad \nu \in (0,d)$$

and

$$\lim_{r\to 0}\sup_{x\in D}\varphi(x,r) = 0. \tag{2.9}$$

We find it convenient to change the notation for such variable-exponent Morrey spaces in the following way:

$$L^{p(\cdot)}_\varphi(D) := L^{p(\cdot),w(\cdot)}(D)\Big|_{w(x,r)=\frac{r^{n/p(x)}}{\varphi(x,r)}}, \tag{2.10}$$

with $VL^{p(\cdot)}_\varphi(D)$ defined similarly.

In the case where the function w is $w(x,r) = r^{n/p(x)}/\varphi(x,r)$, the condition (2.6) follows from the condition (2.9) imposed on φ.

Corollary 1. *Let D be a bounded open set in $\mathbb{R}^n$. Suppose that $p^- > 1$, $p^+ < \infty$, and p satisfies the log-condition* (2.1)*. Suppose that*

$$\int_r^d \frac{dt}{t\varphi(x,t)} \le \frac{C}{\varphi(x,r)}, \tag{2.11}$$

where C is independent of r and x. Then the Calderón–Zygmund singular operator K is bounded on $VL^{p(\cdot)}_{\varphi}(D)$.

Corollary 2. *Let $D = \mathbb{R}^n_+$. Suppose that $p^- > 1$, $p^+ < \infty$, and p satisfies the log-condition* (2.1) *and the decay condition* (2.2)*. Suppose also that*

$$\int_r^\infty \frac{\operatorname{essinf}_{t<s<\infty} \frac{s^{n/p(x)}}{\varphi(x,s)}}{t^{1+n/p(x,t)}}\,dt \le C\frac{r^{n/p(x)-n/p(x,r)}}{\varphi(x,r)}, \tag{2.12}$$

where C is independent of x and r,

$$\int_r^{\delta_0} \frac{t^{n/p(x)-n/p(x,t)}}{\varphi(x,t)}\,\frac{dt}{t} \le C\frac{r^{n/p(x)-n/p(x,r)}}{\varphi(x,r)} \tag{2.13}$$

for some $\delta_0 > 0$, where C is independent of x and r, and

$$\int_\delta^\infty \frac{t^{n/p(x)-n/p(x,t)}}{\varphi(x,t)}\frac{dt}{t} \le C_\delta \tag{2.14}$$

for every $\delta > 0$, where C_δ is independent of x. Then the Calderón–Zygmund singular operator K is bounded on $VL^{p(\cdot)}_{\varphi}(\mathbb{R}^n_+)$.

Observe that, in Corollary 1, we replaced $\operatorname{essinf}_{s>t}\psi(x,t)$ by

$$\psi(x,t) = \frac{t^{n/p(x)-n/p(x,r)}}{\varphi(x,t)} = \frac{1}{\varphi(x,r)}, \qquad r<1,$$

to obtain a simpler condition, while in Corollary 2, we prefer to keep $\operatorname{essinf}_{s>t}$ because of integration over an infinite interval; note that, under a similar replacement, condition (2.12) would turn into (2.11) with $d = \infty$ only if $p^+ = p(\infty)$.

The classical variable-exponent Morrey space $L^{p(\cdot),\lambda}(\mathbb{R}^n)$ is usually defined by the norm

$$\|f\|_{L^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n}\sup_{r>0}\frac{1}{r^{\lambda/p(x)}}\|\chi_{B(x,r)}f\|_{L^{p(\cdot)}(B(x,r))}. \tag{2.15}$$

It is also reasonable to consider the modification of such a space denoted by $\widetilde{L}^{p(\cdot),\lambda}(\mathbb{R}^n)$ and defined by

$$\|f\|_{\widetilde{L}^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n}\sup_{r>0}\frac{1}{r^{\lambda/p(x,r)}}\|\chi_{B(x,r)}f\|_{L^{p(\cdot)}(B(x,r))}. \tag{2.16}$$

The spaces $L^{p(\cdot),\lambda}(\mathbb{R}^n)$ and $\widetilde{L}^{p(\cdot),\lambda}(\mathbb{R}^n)$ correspond to the cases

$$\varphi(x,r) = r^{(n-\lambda)/p(x)} \qquad \text{and} \qquad \varphi(x,r) = r^{(n-\lambda)/p(x,r)},$$

respectively.

Corollary 3. *Suppose that p satisfies the conditions $p^- > 1$, $p^+ < \infty$, and p satisfies the log-condition* (2.1) *and the decay condition* (2.2)*. Then the Calderón–Zygmund singular operator K is bounded on the spaces*

$$L^{p(\cdot),\lambda}(\mathbb{R}^n) \quad \textit{and} \quad VL^{p(\cdot),\lambda}(\mathbb{R}^n) \qquad \textit{if} \quad 0<\lambda<\frac{np^-}{p_\infty}$$

and on

$$\widetilde{L}^{p(\cdot),\lambda}(\mathbb{R}^n) \quad \textit{and} \quad V\widetilde{L}^{p(\cdot),\lambda}(\mathbb{R}^n) \qquad \textit{if} \quad 0<\lambda<n.$$

Proof. The corollary follows from the fact that

$$\operatorname{essinf}_{s>t}\frac{s^{n/p(x)}}{\varphi(x,s)}=\inf_{s>t}s^{\lambda/p(x)}=t^{\lambda/p(x)};$$

thus, condition (2.12) turns into

$$\int_r^\infty\frac{t^{n/p(x)-\lambda/p(x,r)}}{t}\,dt\le Cr^{n/p(x)-\lambda/p(x,r)},$$

and it remains to consider the cases where $r<1$ and $r\ge 1$. □

3. THE GENERALIZED VARIABLE-EXPONENT VANISHING MORREY SPACE $V\mathcal{A}_\varphi^{p(\cdot)}(\mathbb{D})$ OF HOLOMORPHIC FUNCTIONS

By $V\mathcal{A}_\varphi^{p(\cdot)}(\mathbb{D})$ we denote the space of functions holomorphic in $\mathbb{D}$ that belong to $VL_\varphi^{p(\cdot)}(\mathbb{D})$.

Here and in what follows, we identify $\mathbb{R}^2=\mathbb{C}$, so that z stands for $(x_1,x_2)=z=x_1+\mathrm{i}x_2$ and $w=(y_1,y_2)=y_1+\mathrm{i}y_2$.

We denote $D_0=\{z\in\mathbb{C}:|z|<1/2\}$, $D_1=\{z\in\mathbb{C}:1/2<|z|<1\}$, $D_2=\{z\in\mathbb{C}:1<|z|<2\}$. Let Q denote the inversion operator with respect to the unit circle, i.e., $Qf(z)=f(1/\overline{z})$. Given a function ϕ defined on $(0,1)$, we say that it satisfies the doubling condition on $(0,1)$ if its extension by $\phi(1)$ to $t\ge 1$ satisfies the doubling condition for $t>0$, i.e., if there exists a $C_{(2)}>0$ such that $\phi(2t)\le C_{(2)}\phi(t)$, $t>0$.

The authors of [14] used a certain representation of the Bergman projection. We find it important to formulate it as the following theorem, where $K(z,w)$ stands for the Bergman kernel given in (1.1).

Theorem 2. *The Bergman projection $B_\mathbb{D}$ has the following representation in terms of the Calderón–Zygmund singular operator: $B_\mathbb{D}f(z)=K_1f(z)+K_2f(z)$, where K_1 is the integral operator with bounded kernel $K_1(z,w)=(\chi_{D_0}(z)+\chi_{D_1}(z)\chi_{D_0}(w))K(z,w)$ and*

$$K_2f(z)=\chi_{D_1}(z)\int_{D_2}\frac{\Omega\left(\frac{\zeta-z}{|\zeta-z|}\right)}{|\zeta-z|^2}g(\zeta)dA(\zeta)=:Tg(z),\tag{3.1}$$

where $\Omega(z)=\overline{z}^2$, $g(\zeta)=Qf(\zeta)\zeta^2/|\zeta|^4$, and $\zeta\in D_2$.

Note that the boundedness of the kernel $K_1(z,w)$ is obvious; the operator K_2 in the form (3.1) is obtained by the change of variables $w\mapsto 1/\overline{\zeta}$, which takes D_1 to D_2.

The above representation allows one to prove the boundedness of the Bergman projection in a variety of Banach spaces with the lattice property for which the boundedness of the operators T and Q (in the corresponding setting) is known.

In the following theorem, we consider the spaces $VL_\varphi^{p(\cdot)}(\mathbb{D})$ and $V\mathcal{A}_\varphi^{p(\cdot)}(\mathbb{D})$ with the function φ depending only on $r:\varphi=\varphi(r)$. This restriction is required to justify the implication (3.3) in the proof of the theorem, which involves a change of variables in the integrals with respect to the corresponding norms and the passage from the domain D_1 to the domain D_2.

Theorem 3. *Suppose that p satisfies the log-condition* (2.1) *on $\mathbb{D}$, $p^->1$, and $p^+<\infty$. Let $\varphi=\varphi(r)$ satisfy the condition* (2.11) *with $d=1$, the doubling condition on the interval $(0,1)$, and the condition $\lim_{r\to 0}\varphi(r)=0$. Then the Bergman projection $B_\mathbb{D}$ is bounded as an operator from $VL_\varphi^{p(\cdot)}(\mathbb{D})$ to $V\mathcal{A}_\varphi^{p(\cdot)}(\mathbb{D})$.*

Proof. The proof is based on the representation of the Bergman projection provided by Theorem 2 and on the boundedness result provided by Corollary 1.

Evidently, the operator K_1 is bounded on $VL_\varphi^{p(\cdot)}(\mathbb{D})$, that is, it is bounded on $L_\varphi^{p(\cdot)}(\mathbb{D})$ and preserves the vanishing property, which follows directly from the definition of $VL_\varphi^{p(\cdot)}(\mathbb{D})$.

To handle the operator K_2, we must first make some arrangements. Since p is bounded and uniformly continuous on $\mathbb{D}$, it extends to a continuous function up to the boundary. We extend the exponent p to the set D_2 by the rule $p(\zeta) = p(1/\overline{\zeta})$, $\zeta \in D_2$. We shall use the same symbol p for the function thus extended. It can be directly verified that p satisfies the log-condition (2.1) on $\overline{\mathbb{D}} \cup D_2$.

In our estimates, we deal with the norm on $L_\varphi^{p(\cdot)}(D)$, where D is one of the domains D_1, D_2, and $\mathbb{D}$. In this case, for p satisfying the log-condition (2.1), the norm on the space $L_\varphi^{p(\cdot)}(D)$ defined by (2.10) is equivalent to the norm

$$\|f\|^*_{L_\varphi^{p(\cdot)}(D)} = \sup_{\mathcal{D}_r \in \mathfrak{D}_{\varepsilon_0}} \frac{\varphi(r)}{r^{2/p^-(\mathcal{D}_r \cap D)}} \|f\|_{L^{p(\cdot)}(\mathcal{D}_r \cap D)}, \tag{3.2}$$

where $\mathfrak{D}_{\varepsilon_0}$ is the set of discs $\mathcal{D}_r$ of radius $r \le \varepsilon_0$ and $p^-(\mathcal{D}_r \cap D) = \min_{z \in \mathcal{D}_r \cap D} p(z)$. We also have

$$f \in VL_\varphi^{p(\cdot)}(D_1) \ \text{ implies } \ g \in VL_\varphi^{p(\cdot)}(D_2) \tag{3.3}$$

and

$$\|g\|^*_{L_\varphi^{p(\cdot)}(D_2)} \le C_1 \|f\|^*_{L_\varphi^{p(\cdot)}(D_1)} \le C_2 \|f\|_{L_\varphi^{p(\cdot)}(\mathbb{D})}.$$

To obtain the first inequality, we used the fact that φ satisfies the doubling condition on $(0,1)$ and also the fact that $p(\zeta) = p(z)$ for $\zeta = 1/\overline{z}$, $z \in D_1$, $\zeta \in D_2$.

By Corollary 1, we obtain

$$K_2 f \in VL_\varphi^{p(\cdot)}(\mathbb{D}) \qquad \text{and} \qquad \|K_2 f\|_{L_\varphi^{p(\cdot)}(\mathbb{D})} \le C \|g\|_{L_\varphi^{p(\cdot)}(D_2)}, \tag{3.4}$$

which, in view of the equivalence of the norms $\|\cdot\|_{L_\varphi^{p(\cdot)}(D)}$ and $\|\cdot\|^*_{L_\varphi^{p(\cdot)}(D)}$, completes the proof. □

Corollary 4. *Under the conditions of Theorem 3, the space $V\mathcal{A}_\varphi^{p(\cdot)}(\mathbb{D})$ is a closed subspace in $VL_\varphi^{p(\cdot)}(\mathbb{D})$.*

Theorem 4. *Suppose that p satisfies (2.1) and $1 < p^- \le p(z) \le p^+ < \infty$. If $f \in \mathcal{A}_\varphi^{p(\cdot)}(\mathbb{D})$, then, for $z \in \mathbb{D}$,*

$$|f(z)| \le \frac{\eta(z)}{\varphi(z, 1-|z|)}, \qquad \text{where} \quad \eta(z) = C\|f\|_{L_\varphi^{p(\cdot)}(D(z,1-|z|))}, \tag{3.5}$$

so that $\eta(z) \le C\|f\|_{L_\varphi^{p(\cdot)}(\mathbb{D})}$ and $\eta(z) \to 0$ as $|z| \to 1$ if $f \in V\mathcal{A}_\varphi^{p(\cdot)}(\mathbb{D})$.

Proof. Let $f \in \mathcal{A}_\varphi^{p(\cdot)}(\mathbb{D})$. For all $0 \le \rho < 1 - |z|$, we have

$$|f(z)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z + \rho e^{i\alpha})| \, d\alpha, \qquad z \in \mathbb{D}.$$

Integration in the variable ρ over $(0, \delta)$ for some $\delta \le 1 - |z|$ with respect to the measure $2\rho d\rho$ gives

$$|f(z)| \le \frac{1}{\delta^2} \int_{D(z,\delta)} |f(w)| dA(w) \le \frac{2}{\delta^2} \|f\|_{L^{p(\cdot)}(D(z,\delta))} \|\chi_{D(z,\delta)}\|_{L^{q(\cdot)}(D(z,\delta))}, \qquad z \in \mathbb{D}. \tag{3.6}$$

Here we applied the Hölder inequality for variable-exponent Lebesgue spaces to the functions f and $\chi_{D(z,\delta)}$. Since p satisfies the log-condition (2.1), it follows that (cf. [20, Lemma 1.4]) we obtain

$$\|\chi_{D(z,\delta)}\|_{L^{q(\cdot)}(D(z,\delta))} \le C_1 |D(z,\delta))|^{2/q(z)} \le C_2 \delta^{2/q(z)}, \qquad z \in \mathbb{D}.$$

From this inequality and (3.6) it follows that

$$|f(z)| \le \frac{C}{\delta^{2/p(z)}}\|f\|_{L^{p(\cdot)}(D(z,\delta))} \le \frac{C_1}{\varphi(z,\delta)}\|f\|_{L^{p(\cdot),\varphi}(D(z,\delta))}, \qquad z \in \mathbb{D},$$

which proves (3.5). It remains to note that $\eta(z) \to 0$ as $|z| \to 1$ for $f \in VL^{p(\cdot)}_{\varphi}(\mathbb{D})$ by the definition of the vanishing space. □

Remark 1. In the case $p(z) = p = const$, the value $p = 1$ is allowed in Theorem 4. We assume that $p^- > 1$ for simplicity.

4. THE GENERALIZED VARIABLE-EXPONENT VANISHING MORREY SPACE $V\mathcal{A}^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$ OF HOLOMORPHIC FUNCTIONS

By $V\mathcal{A}^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$ we denote the space of functions holomorphic in $\mathbb{R}^2_+$ that belong to $VL^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$.

Theorem 5. *Suppose that p satisfies the log-condition* (2.1) *and the decay condition* (2.2)*, $p^- > 1$, and $p^+ < \infty$, and let conditions* (2.12)–(2.14) *be satisfied for $x \in \mathbb{R}^2_+$. Then the Bergman projection $B_{\mathbb{R}^2_+}$ is bounded as an operator from $VL^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$ to $V\mathcal{A}^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$.*

Proof. Our proof is based on the following representation for the Bergman projection $B_{\mathbb{R}^2_+}$ obtained in [10]. Let Q denote the operator of reflection with respect to the real line $Qf(z) = f(\overline{z})$. Let $\mathbb{R}^2_-$ denote the lower half-plane, and let $E_{\mathbb{R}^2_-}$ denote the operator of extension by zero from $\mathbb{R}^2_-$ to the whole space $\mathbb{R}^2$. The Bergman projection $B_{\mathbb{R}^2_+}$ has the following representation in terms of the Calderón–Zygmund singular operator:

$$B_{\mathbb{R}^2_+}f(z) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\Omega\left(\frac{z-w}{|z-w|}\right)}{|z-w|^2}\, g(w)dA(w), \qquad z \in \mathbb{R}^2_+, \quad g = E_{\mathbb{R}^2_-}Qf, \quad \Omega(z) = \overline{z}^2. \tag{4.1}$$

By representation (4.1) and Corollary 2, the proof is completed by an argument similar to that in Theorem 3. □

Corollary 5. *Under the conditions of Theorem 5, the space $V\mathcal{A}^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$ is a closed subspace in $VL^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$.*

Theorem 6. *Suppose that p satisfies the log-condition* (2.1) *and the decay condition* (2.2) *and $1 < p^- \le p(z) \le p^+ < \infty$. If $f \in \mathcal{A}^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$, then*

$$|f(z)| \le \frac{C}{\varphi(z,y)}\|f\|_{L^{p(\cdot)}_{\varphi}(D(z,y))}, \qquad z = x+iy \in \mathbb{R}^2_+, \quad z = x+iy \in \mathbb{R}^2_+, \quad y \ge 1.$$

If, moreover, $f \in V\mathcal{A}^{p(\cdot)}_{\varphi}(\mathbb{R}^2_+)$, then

$$f(z) \;=\; o\left(\frac{1}{\varphi(z,y)}\right), \qquad \textit{where} \quad y \to 0, \quad z = x+iy \in \mathbb{R}^2_+.$$

Proof. The line of reasoning is the same as in the proof of Theorem 4. The only thing to explain is the following. Direct estimation gives

$$|f(z)| \le \frac{C}{\varphi(z,y)}\|f\|_{L^{p(\cdot)}_{\varphi}(D(z,y))}$$

for $y < 1$ and

$$|f(z)| \le C\frac{y^{2/p(z)-2/p(\infty)}}{\varphi(z,y)}\|f\|_{L^{p(\cdot)}_{\varphi}(D(z,y))}$$

for $y \ge 1$. The factor $y^{2/p(z)-2/p(\infty)}$ can be omitted, which is obvious when $2/p(z) - 2/p(\infty) \le 0$. Otherwise, $y^{2/p(z)-2/p(\infty)} \le |z|^{2/p(z)-2/p(\infty)}$, and it suffices to use the decay condition. □

Remark 2. In the case $p(z) = p = const$, the value $p = 1$ is allowed in Theorem 6. We assume that $p^- > 1$ for simplicity.

Remark 3. We use this opportunity to correct a misprint in the paper [10]. In Theorems 4.1 and 6.1, the condition $1 \le p(z) \le p^+ < \infty$ should be replaced by $1 < p^- \le p(z) \le p^+ < \infty$. In the case $p(z) = p = const$, the value $p = 1$ is allowed in these theorems. Further, in Remarks 4.1 and 6.1, the third formula is true for f in the corresponding vanishing Morrey space.

By $\mathcal{A}^{p(\cdot),\lambda}(\mathbb{R}^n)$ and $\widetilde{\mathcal{A}}^{p(\cdot),\lambda}(\mathbb{R}^n)$ we denote the corresponding subspaces in $L^{p(\cdot),\lambda}(\mathbb{R}^n)$ and $\widetilde{L}^{p(\cdot),\lambda}(\mathbb{R}^n)$ of holomorphic functions.

Corollary 6. *Under the assumptions of Theorem* 6 *on the exponent function* p,

$$|f(z)| \le \frac{C}{y^{(2-\lambda)/p(z)}}\|f\|_{L^{p(\cdot),\lambda}(D(z,y))} \qquad \text{for} \quad f \in \mathcal{A}^{p(\cdot),\lambda}(\mathbb{R}^2_+)$$

and

$$|f(z)| \le \frac{C}{y^{(2-\lambda)/p(z,y)}}\|f\|_{\widetilde{L}^{p(\cdot),\lambda}(D(z,y))} \qquad \text{for} \quad f \in \widetilde{\mathcal{A}}^{p(\cdot),\lambda}(\mathbb{R}^2_+).$$

In the next theorem, we show how the estimate should be changed for $p^- = 1$ for the case of $\mathcal{A}^{p(\cdot),\lambda}(\mathbb{R}^2_+)$.

Theorem 7. *Suppose that* $1 \le p(z) \le p^+ < \infty$ *for* $z \in \mathbb{R}^2_+$ *and* $p(z)$ *satisfies the log-condition* (2.1) *and the decay condition* (2.2) *with* $0 \le \lambda < 2$. *Then*

$$|f(z)| \le C\|f\|_{\mathcal{A}^{p(\cdot),\lambda}(D(z,y))} \cdot \begin{cases} y^{(\lambda-2)/p(\infty)} & \text{if} \quad y \ge 1, \\ y^{(\lambda-2)/p(z)} & \text{if} \quad |B(z,y) \cap E_1| = 0 \text{ and } y < 1, \\ y^{\lambda/p(z)-2} & \text{if} \quad |B(z,y) \cap E_1| > 0 \text{ and } y < 1, \end{cases}$$

where $E_1 = \{z : p(z) = 1\}$.

Proof. The line of reasoning is the same as in the proof of Theorem 4. The main difference is that now we have to use an estimate for $\|\chi_{B(x,r)}\|_{L^{q(\cdot)}(\mathbb{R}^2_+)}$ when $q(x)$ is not bounded. In this case,

$$\|\chi_{B(x,r)}\|_{L^{q(\cdot)}(\mathbb{R}^2_+)} = \inf\{\eta > 0 : \int_{B(x,r)} \frac{dy}{\eta^{q(y)}} \le 1\} + \sup_{y \in E_1} \chi_{B(x,r)},$$

where the first term is estimated, in the familiar way, by $r^{2/p(x)}$, while the second one is equal to 1 when $|B(x,r) \cap E_1| > 0$. For the case $y \ge 1$, we use the decay condition (2.2), from which it follows that $|1/p(z) - 1/p(\infty)| \ln y \le C$ and, consequently, $y^{1/p(z)} \sim y^{1/p(\infty)}$. □

5. ON MOLLIFIERS IN THE SPACES $\mathcal{A}^{p,\psi}(\mathbb{D})$ AND $\mathcal{A}^{p,\psi}(\mathbb{R}^2_+)$, $1 < p < \infty$

In this section, we shall define the space $L^{p,\psi}(\Omega)$ as the space of measurable functions on Ω equipped with the norm

$$\|f\|_{L^{p,\psi}(\Omega)} := \sup_{\substack{x\in\Omega \\ r>0}} \mathfrak{M}_{p,\psi}(f; x, r)^{1/p}, \qquad \text{where} \quad \mathfrak{M}_{p,\psi}(f; x, r) = \frac{1}{\psi(r)} \int_{B(x,r)} |f(y)|^p dy.$$

The corresponding subspace of holomorphic functions will be denoted by $\mathcal{A}^{p,\psi}(\Omega)$.

5.1. Mollifiers in the Case of the Disk

The standard approximation of the functions $f(z)$ by the dilated function $f(\frac{z}{1+\varepsilon})$, which is well known for the Bergman spaces, works also for the Morrey spaces in the case where p is constant, as can be seen from the proof of the next theorem. We wish to show that mollifiers which use both translation and dilation can also be applied for this purpose. Let $\eta(z)$ be any $C_0^\infty(\mathbb{R}^2)$ function supported on $B(0,1/2)$ and such that $\int_{B(0,1/2)} \eta(z)dA(z) = 1$. Consider a mollifier of the form

$$H_\varepsilon f(z) = \frac{1}{\varepsilon^2}\int_{\mathbb{D}} f\left(\frac{w}{1+\varepsilon}\right)\eta\left(\frac{z-w}{\varepsilon}\right)dA(w) = \int_{B(0,1/2)} \eta(w) f\left(\frac{z-\varepsilon w}{1+\varepsilon}\right)dA(w),$$

where $f \in \mathcal{A}^{p,\psi}(\mathbb{D})$.

Clearly, if f is holomorphic in $\{z : |z| < 1\}$, then $H_\varepsilon f$ is holomorphic in $\{z : |z| < 1+\varepsilon\}$ and $H_\varepsilon f$ converges to f for all $z \in \mathbb{D}$.

We say that a measurable function $\varphi : (0,\infty) \to (0,\infty)$ belongs to the class Φ if

(a) φ is almost increasing;

(b) $\varphi(t)/t^2$ is almost decreasing;

(c) $\inf_{t>\delta} \varphi(t) > 0$ for every $\delta > 0$.

The conditions defining the class Φ are standard in the theory of generalized Morrey spaces; see [26], [23]. For $\psi \in \Phi$, we see that the space $L^{p,\psi}(\mathbb{R}^n)$ is nontrivial.

Theorem 8. *Let $1 < p < \infty$, and let $f \in V\mathcal{A}^{p,\psi}(\mathbb{D})$, where $\psi \in \Phi$. Then*

$$\lim_{\varepsilon\to 0} \|H_\varepsilon f - f\|_{L^{p,\psi}(\mathbb{D})} = 0. \tag{5.1}$$

Proof. By Minkowski's integral inequality for Morrey spaces, we have

$$\|H_\varepsilon f - f\|_{L^{p,\psi}(\mathbb{D})} \le \int_{|w|\le 1/2} |\eta(w)|\, \|f\left(\frac{z-\varepsilon w}{1+\varepsilon}\right) - f(z)\|_{L^{p,\psi}(\mathbb{D})}\, dA(w),$$

where the norm of the right-hand side is taken with respect to z. This norm is uniformly bounded in $w \in B(0,1/2)$ and $\varepsilon < 1$; hence, by the Lebesgue dominated convergence theorem, it suffices to check that this norm tends to zero as ε tends to zero for all $w \in B(0,1/2)$. We have

$$\|f\left(\frac{z-\varepsilon w}{1+\varepsilon}\right) - f(z)\|_{L^{p,\psi}(\mathbb{D})} \le \|f\left(\frac{z-\varepsilon w}{1+\varepsilon}\right) - f(z-\varepsilon w)\|_{L^{p,\psi}(\mathbb{D})} + \|f(z-\varepsilon w) - f(z)\|_{L^{p,\psi}(\mathbb{D})}.$$

It is known that the translation operator approximates functions in vanishing Morrey spaces in the Morrey norm (see [38], [12] for the case $\psi(r) = r^\lambda$ and [13] for the general case), which gives the required convergence of the second term. The same property of the vanishing space with respect to dilation is also true and can be proved in the same way as for translation, namely, by using the splitting

$$\|f\|_{L^{p,\psi}(\mathbb{D})} = \max\left\{ \sup_{\substack{0<r<\delta \\ x\in\mathbb{D}}} \frac{1}{\psi(r)}\|f\|_{L^p(B(x,r)\cap\mathbb{D})},\quad \sup_{\substack{r\ge\delta \\ x\in\mathbb{D}}} \frac{1}{\psi(r)}\|f\|_{L^p(B(x,r)\cap\mathbb{D})} \right\},$$

which gives the convergence of the first term after an evident change of variables and completes the proof of the theorem. □

Let $T_m(f)$ denote the Taylor polynomial of degree m of a function f.

Corollary 7. *Let $1 < p < \infty$, and let $\psi \in \Phi$. Then the set of holomorphic polynomials is dense in the space $V\mathcal{A}^{p,\psi}(\mathbb{D})$. More precisely, the two-parameter family $\{T_m(H_\varepsilon f)\}$, $f \in V\mathcal{A}^{p,\psi}(\mathbb{D})$ of polynomials is dense in $V\mathcal{A}^{p,\psi}(\mathbb{D})$.*

Proof. Indeed, we have

$$\|T_m(H_\varepsilon f) - f\|_{L^{p,\psi}(\mathbb{D})} \le \|T_m(H_\varepsilon f) - H_\varepsilon f\|_{L^{p,\psi}(\mathbb{D})} + \|H_\varepsilon f - f\|_{L^{p,\psi}(\mathbb{D})}.$$

It suffices to choose ε sufficiently small, so that the second term is less than $\delta/2$ by Theorem 8. Since $H_\varepsilon f$ is holomorphic in the bigger disc $D(0, 1+\varepsilon)$, we have

$$\sup_{z\in\mathbb{D}} |T_m(H_\varepsilon f)(z) - H_\varepsilon f(z)| < \frac{\delta}{2},$$

for sufficiently large m. This completes the proof of the theorem. □

5.2. Mollifiers in the Case of the Upper Half-Plane

We consider the standard mollifier

$$U_\varepsilon f(z) = \int_e \eta(w) f(z - \varepsilon w)\, dA(w), \tag{5.2}$$

where η is an infinitely differentiable function supported on e such that

$$e \cap \overline{\mathbb{R}^2_+} = \varnothing \qquad \text{and} \qquad \int_e \eta(w)\, dA(w) = 1.$$

We take $e = \{w : |w + 2\mathrm{i}| < 1\}$ for simplicity. The function $U_\varepsilon f$ is holomorphic in $\Im(z) > -\varepsilon$.

Let $V^{(*)}_{0,\infty} L^{p,\psi}(\mathbb{R}^2_+)$ denote the subset of $VL^{p,\psi}(\mathbb{R}^2_+)$ of functions which satisfy the vanishing type condition at infinity (5.3) and the vanishing type condition (5.4):

$$\lim_{r\to\infty} \frac{1}{\psi(r)} \sup_{z\in\mathbb{R}^2_+} \int_{B(z,r)\cap\mathbb{R}^2_+} |f(w)|^p\, dA(w) = 0, \tag{5.3}$$

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}^2_+} \int_{B(x,1)} |f(y)|^p \chi_{\mathbb{R}^2_+ \setminus B(0,N)}(y)\, dy = 0, \tag{5.4}$$

as introduced in [12], [13]. The corresponding subspaces of holomorphic functions will be denoted by $V^{(*)}_{0,\infty} \mathcal{A}^{p,\psi}(\mathbb{R}^2_+)$.

Theorem 9. *If $1 < p < \infty$, $\psi \in \Phi$, and $f \in V^{(*)}_{0,\infty} \mathcal{A}^{p,\psi}(\mathbb{R}^2_+)$, then*

$$\lim_{\varepsilon\to 0} \|U_\varepsilon f - f\|_{L^{p,\psi}(\mathbb{R}^2_+)} = 0.$$

Proof. By Minkowski's integral inequality, we have

$$\|U_\varepsilon f - f\|_{L^{p,\psi}(\mathbb{R}^2_+)} \le \int_e |\eta(w)| \|f(z - \varepsilon w) - f(z)\|_{L^{p,\psi}(\mathbb{R}^2_+)}\, dA(w).$$

It is easy to check that $\|f(z - \varepsilon w)\|_{L^{p,\psi}(\mathbb{R}^2_+)} \le \|f\|_{L^{p,\psi}(\{z:\Im(z)>\varepsilon\})} \le \|f\|_{L^{p,\psi}(\mathbb{R}^2_+)}$, so that we can apply the Lebesgue dominated convergence theorem. Then it remains to use the fact that the functions in $V^{(*)}_{0,\infty} L^{p,\psi}(\mathbb{R}^2_+)$ have the property

$$\lim_{h\to 0} \|f(z+h) - f(z)\|_{L^{p,\psi}(\mathbb{R}^2_+)} = 0$$

(see [13, Theorem 5.10]). This completes the proof of the theorem. □

FUNDING

The work of A. N. Karapetyants and S. G. Samko was supported in part by the Russian Foundation for Basic Research under grant 18-01-00094. The work of A. N. Karapetyants was also supported in part by the Russian Foundation for Basic Research under grant 18-51-05009 Arm-a and by Visiting Fulbright Scholar Program. The research of H. Rafeiro was supported by a Research Start-up Grant of United Arab Emirates University, Al Ain, United Arab Emirates, via grant no. G00002994. The research of S. Samko was supported by the Russian Foundation for Basic Research under grant 19-01-00223.

REFERENCES

1. H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces* (Springer-Verlag, New York, 2000).
2. P. Duren and A. Schuster, *Bergman Spaces*, in *Mathematical Surveys and Monographs* (Amer. Math. Soc., Providence, RI, 2004), Vol. 100.
3. K. Zhu, Spaces of holomorphic functions in the unit ball, Springer, Graduate texts in Math., 2004
4. K. Zhu, *Operator Theory in Function Spaces*, in *AMS Mathematical Surveys and Monographs* (Amer. Math. Soc., Providence, RI, 2007), Vol. 138.
5. A. Aleman, S. Pott, and M. C. Reguera, "Sarason conjecture on the Bergman space," Int. Mathematics. Research. Notices, **14**, 4320–4349 (2016).
6. D. Bekolle and A. Bonami, "Inegalités à poids pour le noyau de Bergman," CR Acad. Sci. Paris Ser. **AB 286** (18), 775–778 (1978).
7. D. Milutin, "Boundedness of the Bergman projections on L^p spaces with radial weights," Publications de l'Institut Mathematique **86 (100)**, 5–20 (2009).
8. V. S. Guliyev and S. G. Samko, "Maximal, potential, and singular operators in the generalized variable-exponent Morrey spaces on unbounded sets," J. Math. Sci. **193** (2) 228–248 (2013).
9. N. Samko, "Maximal, potential, and singular operators in vanishing generalized Morrey spaces," J. Global Optim. **57** (4), 1385–1399 (2013).
10. A. Karapetyants and S. Samko, "On Boundedness of Bergman Projection Operators in Banach Spaces of Holomorphic Functions in Half-Plane and Harmonic Functions in Half-Space," J. Math. Sci. **226** (4), 344–354 (2017).
11. A. Karapetyants, H. Rafeiro, and S. Samko, *Boundedness of the Bergman Projection and Some Properties of Bergman-Type Spaces*, in *Complex Analysis and Operator Theory* (1-15) (2018); `https://doi.org/10.1007/s11785-018-0780-y`.
12. A. Almeida and S. Samko, "Approximation in Morrey spaces," J. Funct. Anal. **272**, 2392–2411 (2017).
13. A. Almeida and S. Samko, "Approximation in generalized Morrey spaces," Georgian Math. J., 2018 (in press).
14. V. P. Zaharyuta and V. I. Yudovich, "The general form of a linear functional in H_p'," Uspekhi Mat. Nauk **19** (2(116)), 139–142 (1964).
15. V. S. Guliyev, J. J. Hasanov, and S. G. Samko, "Boundedness of the maximal, potential and singular operators in the generalized variable-exponent Morrey spaces," Math. Scand. **107**, 285–304 (2010).
16. L. Diening and M. Ruzichka, "Calderón–Zygmund operators on generalized Lebesgues spaces $L^{p(\cdot)}$ and problems related to fluid dynamics," J. Reine Angew. Math. **563**, 197–220 (2003).
17. G. R. Chacón and H. Rafeiro, "Variable exponent Bergman spaces," Nonlinear Analysis: Theory, Methods, and Applications **105**, 41–49 (2014).
18. G. R. Chacón and H. Rafeiro, "Toeplitz operators on variable exponent Bergman spaces," Mediterr. J. Math. **13** (5), 3525–3536 (2014).
19. A. Almeida, J. Hasanov, and S. Samko, "Maximal and potential operators in variable-exponent Morrey spaces," Georgian Math. J. **15**, 195–208 (2008).
20. V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Vol. I: *Variable Exponent Lebesgue and Amalgam Spaces*, in *Operator Theory: Advances and Applications* (Birkhäuser, Basel, 2016).
21. V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko, *Integral Operators in Non-Standard Function Spaces*, Vol. II: *Variable Exponent Hölder, Morrey–Campanato and Grand Spaces*, in *Operator Theory: Advances and Applications* (Birkhäuser, Basel, 2016).
22. C. Bennett and R. Sharpley, *Interpolation of Operators* (Academic Press, Boston, 1988).
23. V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, "Necessary and sufficient conditions for boundedness of the fractional maximal operator in the local Morrey-type spaces," J. Comput. Appl. Math. **208** (1), 280–301 (2007).

24. P. D. Liu, Y. L. Hou, and M. F. Wang, "Weak Orlicz space and its applications to the martingale theory," Sci. China Math. **53** (4), 905–916 (2010).
25. D. Gallardo, "Orlicz spaces for which the Hardy-Littlewood maximal operator is bounded," Publ. Mat. Barc. **32** (2), 261–266 (1988).
26. E. Nakai, "A characterization of pointwise multipliers on the Morrey spaces," Sci. Math. **3** (3), 445–454 (2000).
27. L. Diening, P. Harjulehto, P. Hasto, and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents,* in *Lecture Notes in Math.* (Springer-Verlag, Heidelberg, 2011), Vol. 2017.
28. D. Cruz-Uribe and A. Fiorenza *Variable Lebesgue Spaces. Foundations and Harmonic Analysis,* in *Applied and Numerical Harmonic Analysis* (Springer, Basel, 2013).
29. F. Deringoz, V. S. Guliyev, and S. Samko, "Vanishing generalized Orlicz–Morrey spaces and fractional maximal operator," Publ. Math. Debrecen **90** (1-2), 125–147, (2017).
30. H. Triebel, *Local Function Spaces, Heat and Navier–Stokes Equations*, in *EMS Tracts in Math.* (2013), Vol. 20.
31. L. Pick, A. Kufner, O. John, and S. Fucik, *Function Spaces,* I, in *De Gruyter Series in Nonlinear Analysis and Applications* (2013), Vol. 14.
32. H. Rafeiro, N. Samko, and S. Samko, *Morrey–Campanato Spaces: an Overview*, in *Operator Theory: Advances and Applications* (Springer, Basel, 2013), Vol. 228, pp. 293–323.
33. A. Kufner, O. John, and S. Fucik, *Function Spaces*, Monographs and Textbooks on Mechanics of Solids and Fluids. Mechanics: Analysis (Noordhoff Int. Publ., Leyden, 1977).
34. M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, in *Annals of Math. Studies* (Princenton University Press, Princenton, NJ, 1977), Vol. 105.
35. M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Math. (Marcel Dekker, 1991), Vol. 146.
36. V. Kokilashvili and M. M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces* (World Scientific, Singapore, 1991).
37. M. A. Krasnoselskii and Ya. B. Rutitski, *Convex Functions and Orlicz Spaces*, Translated from the first Russian edition (Noordhoff, Groningen, 1961).
38. C. Vitanza, "Functions with vanishing Morrey norm and elliptic partial differential equations," in *Proceedings of Methods of Real Analysis and Partial Differential Equations*, Capri (Springer, 1990), pp. 147–150.
39. J. Duoandikoetxea. *Fourier Analysis*, Graduate Studies (Amer. Math. Soc., 2001), Vol. 29.