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On the Theory of Optimal Processes in Discrete Systems

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Abstract—In this paper, by introducing the notion of γ-convex set, we distinguish a wider class of discrete control systems in which the global maximum principle holds. A new type of variation of control for such classes of discrete control systems is proposed and stronger global maximum principle and second-order optimality condition expressed in terms of a singular control of new type are obtained. Generalizing the notion of the relative interior of sets, we obtain an optimality condition for discrete systems in the form of an equality, which we call Pontryagin's equation.

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1. INTRODUCTION

Historically, the study of the optimization problem for discrete systems began soon after the study of the Pontryagin maximum principle [1]. The first discrete analog of the maximum principle for a linear discrete optimal control problem was obtained by Rozonoér [2]. Doubts about the possibility of carrying over the maximum principle to nonlinear discrete systems expressed in [2] were justified later by various examples (see [3]–[5]).

After Butkovskii's paper [3], the problem of deriving necessary optimality conditions for discrete systems has been studied further mostly in the following directions.

1. In the papers of the first direction, on the basis of developed and generalized ideas of Rozonoèr, classes of discrete problems in which the global discrete maximum principle holds are determined (see, e.g., [5]–[8]). The first results in this direction were obtained in [9] and [10] under the assumption of the convexity of the set of admissible velocities.

2. In the papers of the second direction, various variants for weakening convexity requirements are proposed. Only local discrete maximum principles have been obtained (see, e.g., [4], [11]–[17]). In all of these papers, some parts (neighborhoods) of the admissible control domain in which a local discrete maximum principle or its consequences hold are distinguished in explicit or implicit form. Note that such a form of necessary optimality condition, is, in general, not constructive for applications to the solution of concrete problems.

3. The third direction is related to the derivation of necessary optimality conditions of higher order in terms of singular and quasisingular controls. The main results in this direction were obtained in [14], [18]–[22], etc.

An analysis of the now available results shows that the theory of necessary conditions in discrete systems is far from its completion. Therefore, the derivation of constructive and strong optimality conditions of the first and higher order is still important from the theoretical and the practical point

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of view. The present paper is devoted to the study of optimality conditions for admissible processes in this setting.

In this paper, by introducing the notion of γ-convex set, we propose a new variant for weakening convexity requirements for the set of admissible velocities. Using this notion and improving the techniques from [23], we obtain a stronger global discrete maximum principle and a necessary optimality condition of the second order in terms of singular (in the sense of Definition 3 in Sec. 5) controls. In addition, generalizing the notion of the relative interior of a set introduced in [24], we prove the validity of an optimality condition in the form of an equality, which we call *Pontryagin's equation*.

2. STATEMENT OF THE PROBLEM, DEFINITIONS, AND STATEMENTS

Suppose that it is required to minimize the functional

$$S(u(\,\cdot\,)) = \Phi(x(t_1)) \to \min_{u(\,\cdot\,)} \tag{2.1}$$

subject to the constraints

$$x(t+1) = f(x(t), u(t), t), \quad x(t_0) = x_0, \qquad t \in I := \{t_0, t_0+1, \dots, t_1 - 1\}, \tag{2.2}$$

$$u(t) \in U(t) \subseteq \mathbb{R}^r, \qquad t \in I. \tag{2.3}$$

Here $\mathbb{R}^r$ is r-dimensional Euclidean space, $x(\,\cdot\,) \in \mathbb{R}^n$ is the state vector, $u(\,\cdot\,) \in \mathbb{R}^r$ is the control vector, t is (discrete) time, t_0, t_1, x_0 are given points, $\Phi(\,\cdot\,)$ and $f(\,\cdot\,,\,\cdot\,,\cdot\,)$ are given functions, and $U(t)$, $t \in I$, are given sets.

Controls satisfying the constraint (2.3) will be called *admissible*. An admissible control $u(t)$, $t \in I$, minimizing the functional (2.1) subject to the constraint (2.2) will be called *optimal*, and the corresponding solution $x(t)$, $t \in I_1 := I \cup \{t_1\}$, of system (2.2) will be called an *optimal trajectory*. The pair $(u(\,\cdot\,), x(\,\cdot\,))$ will be called an *optimal process*.

Let us introduce some notions and present a statement that will be useful in the study of problem (2.1)–(2.3).

Definition 1. Let $Z \subset \mathbb{R}^m$, let $z_0 \in Z$, and let $\widehat{z} \in Z \setminus \{z_0\} \neq \varnothing$. A point z_0 will be called a *relative interior point of the set Z along the line*

$$l(z_0, \widehat{z}) := \{\widetilde{z} : \widetilde{z} = z_0 + \tau(\widehat{z} - z_0),\ \tau \in R\}$$

if there exists a number $\gamma = \gamma(\widehat{z}) \in (0,1]$ such that, for all $\varepsilon \in (-\gamma, \gamma)$, the inclusion $z_0 + \varepsilon(\widehat{z} - z_0) \in Z$ holds. A point z_0 will be called a *relative interior point of the set* Z *in the wide sense* if the point z_0 is a relative interior point of the set Z along any line from the set $\{\ell(z_0, z) : z \in \mathrm{Z} \setminus \{z_0\}\}$. The set of such points will be called the *relative interior of the set* Z *in the wide sense* and will be denoted by the symbol $Z^{|0|}$. A set Z will be called *relatively open in the wide sense* if $Z^{|0|} = Z$.

Obviously, we have the inclusion $\operatorname{ri} Z \subseteq Z^{|0|}$, where $\operatorname{ri} Z$ is the relative interior of the set Z. However, the converse is not always valid. As an example, consider the set

$$Z_1 = \{(0,0)\} \cup \{(z_1, z_2) \in \mathbb{R}^2 : z_1 z_2 > 0, z_i \in \mathbb{R},\ i = 1, 2\}.$$

It is seen that $\operatorname{ri} Z_1 \subset Z_1^{|0|}$ and $z_0 = (0,0) \in Z_1^{|0|}$, but $z_0 = (0,0) \notin \operatorname{ri} Z_1$. Note also that the set Z_1 is relatively open in the wide sense; however, it is not a relatively open set, i.e., $\operatorname{ri} Z_1 \neq Z_1$. Therefore, we can say that Definition 1 is a generalization of the notion of the relative interior of a set given in [24].

Definition 2. A set $Z \subset E^m$ will be called *γ-convex with respect to a point* $z_0 \in Z$ if, for each point $z \in Z$, there exists a number $\gamma = \gamma(z) \in (0,1]$ such that, for all $\varepsilon \in (0, \gamma]$, the inclusion $z_0 + \varepsilon(z - z_0) \in Z$ holds. If a set Z is γ-convex with respect to each of its points, then we shall call it *γ-convex*.

It follows from Definition 2 that each convex, open, or even relatively open set in the wide sense is γ-convex. However, the converse, in general, is not true (it suffices to consider the set $Z_2 = [0, 1) \cup (2, 3]$).

Proposition 1. *Let $Z \subseteq \mathbb{R}^r$, let $z_0 \in Z$, and let $F(z) = Az + b$, where A is an $n \times r$ matrix and $b \in \mathbb{R}^n$. Then*

(α) *the set $F(Z)$ is γ-convex with respect to the point $F(z_0)$ if the set Z is γ-convex with respect to the point z_0;*

(β) *$F(z_0) \in [F(Z)]^{|0|}$ if $z_0 \in Z^{|0|}$.*

Proof. First, let us prove assertion (α). Let $y \in F(Z)$ be an arbitrary point. Then there exists a point $z \in Z$ such that $y = F(z)$. By Proposition 1 and Definition 2, there exists a number $\gamma = \gamma(z) \in (0, 1]$ such that, for all $\varepsilon \in (0, \gamma]$, the inclusion $z_0 + \varepsilon(z - z_0) \in Z$ holds. Therefore,

$$F(z_0 + \varepsilon(z - z_0)) \in F(Z),$$

$$F(z_0 + \varepsilon(z - z_0)) = Az_0 + b + \varepsilon[(Az + b) - (Az_0 + b)] = F(z_0) + \varepsilon[F(z) - F(z_0)].$$

Hence, since $F(z)$ is arbitrary, we obtain assertion (α).

Quite similarly, taking into account Definition 1, we prove assertion (β). The proposition is proved. □

Let $(u^0(\cdot), x^0(\cdot))$ be an admissible process. Just as in [25], we introduce the set

$$U[x^0(\cdot)](t) = \{\widetilde{u} \in U(t) : f(x^0(t), \widetilde{u}, t) - f(x^0(t), u^0(t), t) = 0\}, \qquad t \in I. \tag{2.4}$$

Remark 1. Obviously,

$$U[x^0(\cdot)](t) \neq \varnothing,$$

because $u^0(t) \in U[x^0(\cdot)](t)$, $t \in I$; also, if $\widetilde{u}(t) \in U(t)$, $t \in I$, where $\widetilde{u}(t) \in U[x^0(\cdot)](t)$, $t \in \widetilde{I} \subseteq I$, and $\widetilde{u}(t) = u^0(t)$, $t \in I \setminus \widetilde{I}$, where $\widetilde{I}$ is an arbitrary subset of the set I, then $\widetilde{u}(\cdot)$ is an admissible control and the pair $(\widetilde{u}(\cdot), x^0(\cdot))$ is an admissible process.

It should be noted that, in contrast to continuous control problems, in control problems for discrete systems, the role of the set $U[x^0(\cdot)](t)$, $t \in I$, is substantial enough, because if at least one set $U[x^0(\cdot)](\theta)$ consists of no less than two elements, then it provides additional information about the optimality of the control $u^0(t)$, $t \in I$. Therefore, the set in question allows us to significantly narrow down the set of controls potentially related to optimality [21]. Let us also stress that, in most cases, it is relatively easy to find elements of the set $U[x^0(\cdot)](\theta)$. For example, in problem (2.1)–(2.3), if

$$f(x(t), u(t), t) = g(x(t)) + q(x(t), t)u(t), \qquad t \in I,$$

then finding elements of the set $U[x^0(\cdot)](\theta)$, where $\theta \in I$, can be reduced to solving a linear algebraic system of equations.

In the study of an admissible process $(u^0(\cdot), x^0(\cdot))$, we shall use the following assumptions:

(A1) the functional $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable in $X(x^0(t_1 - 1))$, where

$$X(x^0(t_1 - 1)) \subseteq \mathbb{R}^n$$

is an open set and contains the set

$$\{x : x = f(x^0(t_1 - 1),\ \widehat{u},\ t_1 - 1),\ \widehat{u} \in U(t_1 - 1)\}$$

and, in addition, for each $t \in I$, the function $f(\cdot, \cdot, t)\colon \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ and its partial derivative $f_x(\cdot, \cdot, t)$ are continuous with respect to (x, u) in $Q(t)$, where $Q(t) \subseteq \mathbb{R}^n \times \mathbb{R}^r$ is an open set containing the set $\{x^0(t)\} \times U(t)$;

(A2) the functional $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable in $X(x^0(t_1 - 1))$ and, in addition, for each $t \in I$, the function $f(\cdot, \cdot, t)\colon \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ and its partial derivatives $f_x(\cdot, \cdot, t)$, $f_{xx}(\cdot, \cdot, t)$ are continuous with respect to (x, u) in $Q(t)$;

(A3) for each $t \in I_{-1}$, the set $f(x^0(t), U(t), t)$ is γ-convex with respect to the point $x^0(t + 1)$, where $I_{-1} := I \setminus \{t_1 - 1\}$ (note that the set $f(x^0(t_1 - 1), U(t_1 - 1), t_1 - 1)$ can be arbitrary).

3. FORMULAS FOR THE INCREMENT OF THE QUALITY FUNCTIONAL

Let $(u^0(\cdot), x^0(\cdot))$ be an admissible process, and let assumptions (A2) and (A3) hold.

We consider an arbitrary fixed vector parameter $\xi := (\theta, \upsilon, \widetilde{u}(\cdot))$, where $\theta \in I \setminus \{t_1 - 1\} =: I_{-1}$, $\upsilon \in U(\theta)$, and $\widetilde{u}(\cdot))$ is an admissible control such that

$$\widetilde{u}(t_1 - 1) = \widehat{u} \in U(t_1 - 1) \qquad \text{and} \qquad \widetilde{u}(t) \in U[x^0(\cdot)](t),$$

$t \in I_{-1}$. We define the variation of the control $u^0(t)$, $t \in I$, as follows:

$$u(t;\xi,\varepsilon) = \begin{cases} \widehat{u}, & t = t_1 - 1, \\ \upsilon(\varepsilon), & t = \theta, \\ \widetilde{u}(t), & t \in I \setminus \{\theta, t_1 - 1\}. \end{cases} \tag{3.1}$$

Here the function $\upsilon(\varepsilon)\colon (0,\gamma] \to U(\theta)$ is defined (in implicit form) as a solution of the system

$$f(x^0(\theta), \upsilon(\varepsilon), \theta) - f(\theta) = \varepsilon \Delta_\upsilon f(\theta), \qquad \varepsilon \in (0,\gamma], \tag{3.2}$$

where $(0,\gamma] \subset (0,1]$, the number $\gamma = \gamma(\upsilon)$ is given by Definition 2, and

$$f(\theta) := f(x^0(\theta), u^0(\theta), \theta), \qquad \Delta_\upsilon f(\theta) := f(x^0(\theta), \upsilon, \theta) - f(x^0(\theta), u^0(\theta), \theta). \tag{3.3}$$

In view of Definition 2, such a solution exists, because assumption (A3) holds.

Obviously, for all $\varepsilon \in (0,\gamma]$, the function $u(t;\xi,\varepsilon)$ is an admissible control. Note that, apparently, a variation of the form (3.1) is considered here for the first time. This constitutes one of the main aspects of the scheme of study of problem (2.1)–(2.3).

Along with the process $(u^0(\cdot), x^0(\cdot))$, we consider the admissible process $(u(\cdot;\xi,\varepsilon), x(\cdot;\xi,\varepsilon))$. Obviously, the increment $x(t;\xi,\varepsilon) - x^0(t) =: \Delta x(t;\xi,\varepsilon)$, $t \in I_1$, is a solution of the system

$$\begin{cases} \Delta x(t+1;\xi,\varepsilon) = f(x^0(t) + \Delta x(t;\xi,\varepsilon), u(t;\xi,\varepsilon), t) - f(x^0(t), u^0(t), t), \\ \Delta x(t_0;\xi,\varepsilon) = 0, \quad \varepsilon \in (0,\gamma]. \end{cases} \tag{3.4}$$

Since $\widetilde{u}(t) \in U[x^0(\cdot)](t)$, $t \in I_{-1}$, it follows that, taking into account (2.4), (3.1), and (3.2), we can write system (3.4) as

$$\Delta x(t+1;\xi,\varepsilon) = \begin{cases} 0, & t_0 - 1 \le t < \theta, \\ \varepsilon \Delta_\upsilon f(\theta), & t = \theta, \\ \Delta_{x(t;\xi,\varepsilon)} f(x^0(t), \widetilde{u}(t), t), & \theta < t < t_1 - 1, \\ \Delta_{\widehat{u}} f(t_1 - 1) + \Delta_{x(t_1-1;\xi,\varepsilon)} f(x^0(t_1 - 1), \widehat{u}, t_1 - 1), & t = t_1 - 1, \end{cases} \tag{3.5}$$

where $\varepsilon \in (0,\gamma]$, $\Delta_\upsilon f(\theta)$ and $\Delta_{\widehat{u}} f(t_1 - 1)$ are determined from (3.3), and

$$\Delta_{x(t;\xi,\varepsilon)} f(x^0(t), \widetilde{u}(t), t) = f(x(t;\xi,\varepsilon), \widetilde{u}(t), t) - f(x^0(t), \widetilde{u}(t), t), \tag{3.6}$$

$$\Delta_{x(t_1-1;\xi,\varepsilon)} f(x^0(t_1 - 1), \widehat{u}, t_1 - 1) = f(x(t_1 - 1;\xi,\varepsilon), \widehat{u}, t_1 - 1) - f(x^0(t_1 - 1), \widehat{u}, t_1 - 1). \tag{3.7}$$

Using Taylor's formula from (3.5), taking into account (3.6), and applying the method of steps, we can show the validity of the inequality

$$\|\Delta x(t;\xi,\varepsilon)\| \le K\varepsilon, \qquad t \in I, \quad \varepsilon \in (0,\gamma], \quad K = \text{const} > 0, \tag{3.8}$$

where $\|\cdot\|$ is the Euclidean norm.

It follows from (3.5) that the solution $\Delta x(t;\xi,\varepsilon)$ at the point $t = t_1$ is finite with respect to ε: $\|\Delta x(t_1;\xi,\varepsilon)\| \sim \varepsilon^0$. Moreover, taking into account (3.5) and (3.8), we see that it is the second term in the representation $\Delta x(t_1;\xi,\varepsilon)$, i.e., the increment (3.7), which is of the order of ε:

$$\|\Delta_{x(t_1-1;\xi,\varepsilon)} f(x^0(t_1 - 1)\widehat{u}, t_1 - 1)\| \le \widehat{K}\varepsilon, \qquad \varepsilon \in (0,\gamma], \quad \widehat{K} = \text{const} > 0. \tag{3.9}$$

Now, on the basis of (3.5) and estimates (3.8) and (3.9), we shall pass to the calculation of the increment of the second order of the quality functional (2.1).

Employing Taylor's expansion at the point $f(x^0(t_1-1),\widehat{u},t_1-1)$ and using (3.5), (3.7)–(3.9), we can write the increment

$$\Phi(x^0(t_1)+\Delta x(t_1;\xi,\varepsilon))-\Phi(x^0(t_1)) =: \Delta_{\xi,\varepsilon}S(u^0(\cdot))$$

of the functional (2.1) caused by (3.1), in the following form:

$$\begin{aligned}\Delta_{\xi,\varepsilon}S(u^0(\cdot)) &= \Phi(f(x^0(t),\widehat{u},t))|_{t=t_1-1}-\Phi(f(x^0(t),u^0(t),t))|_{t=t_1-1}\\ &\quad+\Delta^{(1)}_{\xi,\varepsilon}S(u^0(\cdot))+\frac{1}{2}\Delta^{(2)}_{\xi,\varepsilon}S(u^0(\cdot))+o(\varepsilon^2),\end{aligned} \tag{3.10}$$

where $\varepsilon\in(0,\gamma]$, $\varepsilon^{-2}o(\varepsilon^2)\to 0$ as $\varepsilon\to 0$, and

$$\Delta^{(1)}_{\xi,\varepsilon}S(u^0(\cdot)) := \Phi_x^T(f(x^0(t),\widehat{u},t))\Delta_{x(t;\xi,\varepsilon)}f(x^0(t),\widehat{u},t)|_{t=t_1-1} \tag{3.11}$$

$$\Delta^{(2)}_{\xi,\varepsilon}S(u^0(\cdot)) := \Delta_{x(t;\xi,\varepsilon)}f(x^0(t),\widehat{u},t)\Phi_{xx}(f(x^0(t),\widehat{u},t))\Delta_{x(t;\xi,\varepsilon)}f(x^0(t),\widehat{u},t)|_{t=t_1-1}. \tag{3.12}$$

Consider auxiliary vectors $\psi(t_1-1;\widehat{u})$ and $\psi(t;\widetilde{u}(t+1))$, $t\in I_{-1}$, and matrices $\Psi(t_1-1;\widehat{u})$ and $\Psi(t;\widetilde{u}(t+1))$, $t\in I_{-1}$, being a solution of linear discrete systems of the form [21]

$$\begin{cases}\psi(t-1;\widetilde{u}(t)) = f_x^T(x^0(t),\widetilde{u}(t),t)\psi(t;\widetilde{u}(t+1)), \quad t\in I_{-1},\\ \psi(t_1-2;\widetilde{u}(t_1-1)) = f_x^T(x^0(t_1-1),\widehat{u},t_1-1)\psi(t_1-1;\widehat{u}),\\ \psi(t_1-1;\widehat{u}) := -\Phi_x(f(x^0(t_1-1),\widehat{u},t_1-1)),\end{cases} \tag{3.13}$$

$$\begin{cases}\Psi(t-1;\widetilde{u}(t)) = f_x^T(x^0(t),\widetilde{u}(t),)\Psi(t;\widetilde{u}(t+1))f_x(x^0(t),\widetilde{u}(t),t)\\ \qquad\qquad + H_{xx}(\psi(t;\widetilde{u}(t+1))),x^0(t),\widetilde{u}(t),t), \quad t\in I_{-1},\\ \Psi(t_1-2;\widetilde{u}(t_1-1)) = f_x^T(x^0(t),\widehat{u},t)\Psi(t;\widehat{u})f_x(x^0(t),\widehat{u},t)|_{t=t_1-1}\\ \qquad\qquad + H_{xx}(\psi(t_1-1;\widehat{u}),x^0(t_1-1),\widehat{u},t_1-1),\\ \Psi(t_1-1;\widehat{u}) := -\Phi_{xx}(f(x^0(t_1-1),\widehat{u},t_1-1)),\end{cases} \tag{3.14}$$

where $\widetilde{u}(t)$, $t\in I$, is an admissible control such that

$$\widetilde{u}(t_1-1)=\widehat{u}\in U(t_1-1), \qquad \widetilde{u}(t)\in U[x^0(\cdot)](t), \quad t\in I_{-1},$$

and $H(\psi,x,u,t)=\psi^T f(x,u,t)$ is the Hamilton–Pontryagin function.

By (3.7), (3.8), (3.11), (3.13) and by Taylor's formula, for $\Delta^{(1)}_{\xi,\varepsilon}S(u^0(\cdot))$, we have the representation

$$\begin{aligned}\Delta^{(1)}_{\xi,\varepsilon}S(u^0(\cdot)) &= -H_x^T(\psi(t;\widehat{u}),x^0(t),\widehat{u},t)\Delta x(t;\xi,\varepsilon)|_{t=t_1-1}\\ &\quad-\frac{1}{2}\Delta x^T(t;\xi,\varepsilon)H_{xx}(\psi(t;\widehat{u}),x^0(t),\widehat{u},t)\Delta x(t;\xi,\varepsilon)|_{t=t_1-1}+o(\varepsilon^2).\end{aligned} \tag{3.15}$$

Similarly, in view of (3.7), (3.8), and (3.14), using (3.12), we obtain

$$\Delta^{(2)}_{\xi,\varepsilon}S(u^0(\cdot)) = -\Delta x^T(t;\xi,\varepsilon)f_x^T(x^0(t),\widehat{u},t)\Psi(t;\widehat{u})f_x(x^0(t),\widehat{u},t)\Delta x(t;\xi,\varepsilon)|_{t=t_1-1}+o(\varepsilon^2). \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.10) and using (3.13) and (3.14), we can write

$$\begin{aligned}\Delta_{\xi,\varepsilon}S(u^0(\cdot)) &= \Delta_{\widehat{u}}\Phi(f(t_1-1))-\psi^T(t_1-2;\widetilde{u}(t_1-1))\Delta x(t_1-1;\xi,\varepsilon)\\ &\quad-\frac{1}{2}\Delta x^T(t_1-1;\xi,\varepsilon)\Psi(t_1-2;\widetilde{u}(t_1-1))\Delta x(t_1-1;\xi,\varepsilon)+o_\Sigma(\varepsilon^2).\end{aligned} \tag{3.17}$$

Here and in what follows, the symbol $o_\Sigma(\varepsilon^2)$ denotes the total remainder.

In view of (3.5) and (3.13), we can prove the validity of the following equality for the second term of formula (3.17):

$$p_1(\varepsilon) := \psi^T(t_1-2;\widetilde{u}(t_1-1))\Delta x(t_1-1;\xi,\varepsilon)$$

$$= \varepsilon\psi^T(\theta;\widetilde{u}(\theta+1)\Delta_\upsilon f(\theta) + \sum_{t=\theta_1}^{t_1-2}\psi^T(t;\widetilde{u}(t+1))\Delta_{x(t;\xi,\varepsilon)}f(x^0(t),\widetilde{u}(t),t)$$
$$- \sum_{t=\theta_1}^{t_1-2}\psi^T(t-1;\widetilde{u}(t))\Delta x(t;\xi,\varepsilon).$$

Hence, taking into account (3.6), (3.8), and (3.13), by Taylor's formula for $p_1(\varepsilon)$, we obtain a representation of the following form:

$$p_1(\varepsilon) = \varepsilon\psi^T(\theta;\widetilde{u}(\theta+1))\Delta_\upsilon f(\theta)$$
$$+ \frac{1}{2}\sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;\xi,\varepsilon)H_{xx}(\psi(t;\widetilde{u}(t+1)),x^0(t),\widetilde{u}(t),t)\Delta x(t;\xi,\varepsilon) + o_\Sigma(\varepsilon^2), \qquad \varepsilon\in(0,\gamma]. \tag{3.18}$$

Let us now calculate the third term in (3.17). Using (3.5) and (3.14), it is easy to verify the validity of the identity

$$p_2(\varepsilon) := \Delta x^T(t_1-1;\xi,\varepsilon)\Psi(t_1-2;\widetilde{u}(t_1-1))\Delta x(t_1-1;\xi,\varepsilon)$$
$$= \varepsilon^2\Delta_\upsilon f^T(\theta)\Psi(\theta;\widetilde{u}(\theta+1))\Delta_\upsilon f(\theta) - \sum_{t=\theta}^{t_1-2}\Delta x^T(t;\xi,\varepsilon)\Psi(t-1;\widetilde{u}(t))\Delta x(t,\xi,\varepsilon)$$
$$+ \sum_{t=\theta_1}^{t_1-2}\Delta_{x(t;\xi,\varepsilon)}f^T(x^0(t),\widetilde{u}(t),t)\Psi(t;\widetilde{u}(t+1))\Delta_{x(t;\xi,\varepsilon)}f(x^0(t),\widetilde{u}(t),t).$$

In view of (3.6) and (3.14), again by Taylor's formula for $p_2(\varepsilon)$, we have

$$p_2(\varepsilon) = \varepsilon^2\Delta_\upsilon f^T(\theta)\Psi(\theta;\widetilde{u}(\theta+1))\Delta_\upsilon f(\theta)$$
$$- \sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;\xi,\varepsilon)H_{xx}(\psi(t;\widetilde{u}(\theta+1),x^0(t),u(t),t))\Delta x(t,\xi,\varepsilon) + o_\Sigma(\varepsilon^2). \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17), we obtain

$$\Delta_{\xi,\varepsilon}S(u^0(\,\cdot\,))$$
$$= \Phi(f(x^0(t),\widehat{u},t))|_{t=t_1-1} - \Phi(f(x^0(t),u^0(t),t))|_{t=t_1-1} - \varepsilon\psi^T(\theta;\widetilde{u}(\theta+1))\Delta_\upsilon f(\theta)$$
$$- \frac{\varepsilon^2}{2}\Delta_\upsilon f^T(\theta)\Psi(\theta;\widetilde{u}(\theta+1))\Delta_\upsilon f(\theta) + o_\Sigma(\varepsilon^2), \qquad \varepsilon\in(0,\gamma], \tag{3.20}$$

where $\Delta_\upsilon f(\theta)$, $\psi(\theta;\widetilde{u}(\theta+1))$, and $\Psi(\theta;\widetilde{u}(\theta+1))$ for $\theta\in I_{-1}$ are determined from (3.3), (3.13), and (3.14), respectively.

Therefore, the following proposition holds.

Proposition 2. *Let $(u^0(\,\cdot\,),x^0(\,\cdot\,))$ be an admissible process, and let assumptions* ($A2$) *and* ($A3$) *hold. Then, for each vector parameter $\xi=(\theta,\upsilon,\widetilde{u}(\,\cdot\,))$, where $\theta\in I_{-1}$, $\upsilon\in U(\theta)$, $\widetilde{u}(\,\cdot\,)\in U(\,\cdot\,)$, $\widetilde{u}(t_1-1)=\widehat{u}\in U(t_1-1)$, and $\widetilde{u}(t)\in U[x^0(\,\cdot\,)](t)$ for $t\in I_{-1}$, equality* (3.20) *holds.*

In exactly the same way as in the scheme of derivation of formula (3.20), we also prove the following proposition.

Proposition 3. *Let $(u^0(\,\cdot\,),x^0(\,\cdot\,))$ be an admissible process and assumptions* (A1) *and* (A3) *hold. Then, for each vector parameter $\xi=(\theta,\upsilon,\widetilde{u}(\,\cdot\,))$, where $\theta\in I_{-1}$, $\upsilon\in U(\theta)$, $\widetilde{u}(\,\cdot\,)\in U(\,\cdot\,)$, $\widetilde{u}(t_1-1)=\widehat{u}\in U(t_1-1)$, and $\widetilde{u}(t)\in U[x^0(\,\cdot\,)](t)$ for $t\in I_{-1}$, the following expansion is valid*:

$$\Delta_{\xi,\varepsilon}S(u^0(\,\cdot\,)) = \Phi(f(x^0(t),\widehat{u},t))|_{t=t_1-1} - \Phi(f(x^0(t),u^0(t),t))|_{t=t_1-1}$$
$$- \varepsilon\psi^T(\theta;\widetilde{u}(\theta+1))\Delta_\upsilon f(\theta) + o_\Sigma(\varepsilon), \qquad \varepsilon\in(0,\gamma], \tag{3.21}$$

where $\Delta_\upsilon f(\theta)$ and $\psi(\theta;\widetilde{u}(\theta+1))$ are defined by (3.3) *and* (3.13)*, respectively.*

In the following sections, using the obtained expansions (3.20) and (3.21), we obtain optimality conditions of the first and second order.

4. NECESSARY OPTIMALITY CONDITION OF THE FIRST ORDER

Let $(u^0(\,\cdot\,), x^0(\,\cdot\,))$ be an admissible process. We introduce the following sets:

$$U_0(t_1-1) = \big\{\widehat{u} \in U(t_1-1) : \Phi(f(x^0(t),\widehat{u},t))|_{t=t_1-1} - \Phi(f(x^0(t),u^0(t),t))|_{t=t_1-1} = 0\big\}, \qquad (4.1)$$

$$\begin{aligned}
&Q[x^0(\,\cdot\,)](t_1-2)\\
&\quad = \big\{\psi(t_1-2) : \psi(t_1-2) = -f_x^T(x^0(t),\widehat{u},t)\Phi_x(f(x^0(t),\widehat{u},t))|_{t=t_1-1},\ \widehat{u} \in U_0(t_1-1)\big\},\\
&Q[x^0(\,\cdot\,)](t) = \big\{\psi(t) : \psi(t) = \psi(t;\widetilde{u}(t+1)),\\
&\qquad\qquad \widetilde{u}(t+1) \in U[x^0(\,\cdot\,)](t+1),\ \widetilde{u}(t_1-1) = \widehat{u} \in U_0(t_1-1),\\
&\qquad\qquad \psi(\,\cdot\,;\widetilde{u}(t+1)) \text{ is a solution of system (3.13)}\big\}, \qquad t \in \{t_1-3, t_1-4, \dots, t_0\},\\
&\Lambda[x^0(\,\cdot\,)](t_1-2)\\
&\quad = \big\{\lambda(t_1-2) : \lambda(t_1-2) = -f_x^T(x^0(t),\widehat{u},t)\Phi_x(f(x^0(t),\widehat{u},t))|_{t=t_1-1},\ \widehat{u} \in U_0(t_1-1)\big\},\\
&\Lambda[x^0(\,\cdot\,)](t) = \big\{\lambda(t) : \lambda(t) = f_x^T(x^0(t+1),\widetilde{u},t+1)\lambda(t+1),\ \widetilde{u} \in U[x^0(\,\cdot\,)](t+1),\\
&\qquad\qquad \lambda(t+1) \in \Lambda[x^0(\,\cdot\,)](t+1)\big\}, \qquad t \in \{t_1-3, t_1-4, \dots, t_0\},
\end{aligned} \qquad (4.2)$$

where the sets $U[x^0(\,\cdot\,)](t)$, $t \in I$, are defined by (2.4).

Taking into account Remark 1 and the definitions of the sets $Q[x^0(\,\cdot\,)](t)$, $t \in I_{-1}$, and $\Lambda[x^0(\,\cdot\,)](t)$, $t \in I_{-1}$, and using the method of steps (successively with respect to t: $t = t_1-2, t_1-3, \dots, t_0$), we prove the validity of the following lemma.

Lemma 1. *For each $t \in I_{-1}$, the following equality holds*:

$$Q[x^0(\,\cdot\,)](t) = \Lambda[x^0(\,\cdot\,)](t).$$

Theorem 1. *Let assumptions* (A1) *and* (A3) *hold, and let, for each $t \in I_{-1}$, $\Lambda[x^0(\,\cdot\,)](t)$ be the set defined by* (4.2). *Then, for the process $(u^0(\,\cdot\,), x^0(\,\cdot\,))$ to be optimal, it is necessary that the following inequalities hold*:

$$\Phi(f(x^0(t),\widehat{u},t))|_{t=t_1-1} - \Phi(f(x^0(t),u^0(t),t))|_{t=t_1-1} \ge 0 \qquad \textit{for all} \quad \widehat{u} \in U(t_1-1), \qquad (4.3)$$

$$\Delta_\upsilon H(\lambda(t),x^0(t),u^0(t),t) \le 0 \qquad \textit{for all} \quad (t,\upsilon,\lambda(t)) \in I_{-1} \times U(t) \times \Lambda[x^0(\,\cdot\,)](t). \qquad (4.4)$$

where $\Delta_\upsilon H(\lambda(t),x^0(t),u^0(t),t) = \lambda^T(t)\Delta_\upsilon f(t)$ and $\Delta_\upsilon f(t)$ are defined by (3.3).

Proof. Let $(u^0(\,\cdot\,), x^0(\,\cdot\,))$ be an optimal process. Then, taking into account (A1) and (A3) and using Proposition 3, for each vector parameter $\xi = (\theta, \upsilon, \widetilde{u}(\,\cdot\,))$, where $\theta \in I_{-1}$, $\upsilon \in U(\theta)$, $\widetilde{u}(\,\cdot\,) \in U(\,\cdot\,)$, $\widetilde{u}(t_1-1) = \widehat{u} \in U(t_1-1)$, and $\widetilde{u}(t) \in U[x^0(\,\cdot\,)](t)$ for $t \in I_{-1}$, for sufficiently small $\varepsilon > 0$, the following inequality holds:

$$\begin{aligned}
\Delta_{\xi,\varepsilon} S(u^0(\,\cdot\,)) = {}&\Phi(f(x^0(t),\widehat{u},t))|_{t=t_1-1} - \Phi(f(x^0(t),u^0(t),t))|_{t=t_1-1}\\
&- \varepsilon\psi^T(\theta;\widetilde{u}(\theta+1))\Delta_\upsilon f(\theta) + o_\Sigma(\varepsilon) \ge 0.
\end{aligned} \qquad (4.5)$$

Hence, setting $\upsilon = u^0(\theta)$ and taking into account the fact that $\widehat{u} \in U(t_1-1)$ is arbitrary, we immediately see that inequality (4.3) holds. Further, let $\widetilde{u}(t_1-1) = \widehat{u} \in U_0(t_1-1)$ be an arbitrary point. Then, taking into account (4.1) and the fact that $\varepsilon > 0$ is sufficiently small and using (4.5), for each vector parameter $\xi = (\theta; \upsilon, \widetilde{u}(\,\cdot\,))$, we obtain

$$\psi^T(\theta;\widetilde{u}(\theta+1))\Delta_\upsilon f(\theta) \le 0, \qquad (4.6)$$

where $\theta \in I_{-1}$, $\upsilon \in U(\theta)$, and $\widetilde{u}(\theta+1) \in U[x^0(\,\cdot\,)](\theta+1)$ are arbitrary point such that

$$\psi(\theta;\widetilde{u}(\theta+1)) \in Q[x^0(\,\cdot\,)](\theta).$$

Therefore, by Lemma 1, using the Hamilton–Pontryagin function and (4.6), we obtain

$$\Delta_v H(\lambda(\theta), x^0(\theta), u^0(\theta), \theta) \le 0$$

for all $(\theta, v, \lambda(\theta)) \in I_{-1} \times U(\theta) \times \Lambda[x^0(\,\cdot\,)](\theta)$. Therefore, inequality (4.4) is proved, and so is the theorem. □

Let us give more effective (in the sense of verification and computation) corollaries of Theorem 1.

Corollary 1. *Let assumption* (A1) *hold, and let, for each $t \in I_{-1}$, the following inclusion hold*: $f(x^0(t), u^0(t), t) \in [f(x^0(t), U(t), t)]^{|0|}$. *Then, for the process $(u^0(\,\cdot\,), x^0(\,\cdot\,))$ to be optimal, it is necessary that the following equality be valid*:

$$\Delta_v H(\lambda(t), x^0(t), u^0(t), t) = 0 \qquad \text{for all} \quad (t, v, \lambda(t)) \in I_{-1} \times U(t) \times \Lambda[x^0(\,\cdot\,)](t). \tag{4.7}$$

Proof. Let $(u^0(\,\cdot\,), x^0(\,\cdot\,))$ be an optimal process. Since the inclusion

$$x^0(t+1) = f(x^0(t), u^0(t), t) \in [f(x^0(t), U(t), t)]^{|0|}, \qquad t \in I_{-1}$$

holds, in view of Definition 1, we see that assumption (A3) is valid. Combining this with the validity of (A1), by Theorem 1, we have (4.4). In addition, in view of Definition 1, for any $t \in I_{-1}$ and $v \in U(t)$, there exists a number $\gamma > 0$ such that, for all $\varepsilon \in (-\gamma, \gamma)$, the following inclusion holds:

$$f(t) + \varepsilon(f(x^0(t), v, t) - f(t)) \in f(x^0(t), U(t), t),$$

where $f(t)$ is defined by (3.3).

Therefore, for each $\varepsilon \in (-\gamma, \gamma)$, there exists a vector $u(\varepsilon) \in U(t)$ such that

$$f(x^0(t), u(\varepsilon), t) = f(t) + \varepsilon(f(x^0(t), v, t) - f(t))$$

and inequality (4.4) holds for all $u(\varepsilon) \in U(t)$, $\varepsilon \in (-\gamma, \gamma)$. Then, taking into account the fact that $\varepsilon \in (-\gamma, \gamma)$ is arbitrary, we see that (4.7) holds. The corollary is proved. □

Note that, in most cases, it is easy to verify the γ-convexity condition for the set $f(x^0(t), U(t), t)$, $t \in I_{-1}$, and the validity of the inclusion $x^0(t+1) \in [f(x^0(t), U(t), t)]^{|0|}$, $t \in I_{-1}$.

The following result shows that this verification can be reduced to that of the γ-convexity of the sets $U(t)$, $t \in I_{-1}$, and the inclusion $u^0(t) \in [U(t)]^{|0|}$, $t \in I_{-1}$.

Corollary 2. *Let assumption* (A1) *hold, and let, in problem* (2.1)–(2.3),

$$f(x, u, t) = g(x, t) + B(x, t)u, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r, \quad t \in I,$$

where $g(\,\cdot\,)$ is an n-vector and $B(\,\cdot\,)$ is an $n \times r$ matrix. Also let $(u^0(\,\cdot\,), x^0(\,\cdot\,))$ be an optimal process. Then

(α) *inequality* (4.4) *holds if, for each $t \in I_{-1}$, the set $U(t)$ is γ-convex with respect to the points $u^0(t)$*;

(β) *equality* (4.7) *holds if, for each $t \in I_{-1}$, the inclusion $u^0(t) \in [U(t)]^{|0|}$ is valid.*

Proof. The proof of assertion (α) follows from Theorem 1 in view of Proposition 1, and the proof of assertion (β) follows from Corollary 1 in view of Proposition 1. □

In the conclusion of Sec. 4, we note that the assertion of Theorem 1 is a new discrete analog of the Pontryagin maximum principle. It is easy to verify that it generalizes and strengthens the corresponding result from [9] and [10] (see Example 1 below). The optimality condition (4.3) was first obtained in [23]; condition (4.4), in contrast to the earlier known conditions (see, e.g., [8]–[10], [22]), is obtained in terms of the set $\Lambda[x^0(\,\cdot\,)](t)$, $t \in I_{-1}$ (i.e., the set of linearly independent Lagrange multipliers). The effectiveness of the set $\Lambda[x^0(\,\cdot\,)](t)$, $t \in I_{-1}$, is shown by the following example.

Example 1. Consider $S(u(\cdot)) = x_2^2(2) - x_2(2) \to \min_{u(\cdot)}$,

$$\begin{cases} x_1(t+1) = x_1(t) - u^2(t), \\ x_2(t+1) = -3x_1(t)|u(t)-1| + x_2^2(t)u^2(t) - 1, \end{cases} \qquad \begin{cases} x_1(0) = 0, \\ x_2(0) = -1, \end{cases}$$

$$u(t) \in U(t), \quad t \in I = \{0; 1\}, \qquad U(0) = U(1) = [-1, 1].$$

Let us study the optimality of the admissible control $u^0(t)$, $t \in \{0; 1\}$, defined by $u^0(0) = 0$ and $u^0(1) = 1$. Let us apply Theorem 1. By (4.1) and (4.2), we have

$$x^0(t) = (x_1^0(t), x_2^0(t))^T, \quad t \in \{0, 1, 2\}, \qquad \text{where} \quad x_1^0(0) = x_1^0(1) = 0,$$

$$x_1^0(2) = -1, \qquad x_2^0(0) = x_2^0(1) = -1, \qquad x_2^0(2) = 0,$$

$$f(x^0(1), u^0(1), 1) = (-1, 0)^T, \qquad f(x^0(1), \widehat{u}, 1) = \begin{pmatrix} -\widehat{u}^2 \\ \widehat{u}^2 - 1 \end{pmatrix},$$

$$f_x^T(x^0(t), \widehat{u}, 1) = \begin{pmatrix} 1 & -3|\widehat{u} - 1| \\ 0 & -2\widehat{u}^2 \end{pmatrix}, \qquad \Phi_x(f(x^0(1), \widehat{u}, 1)) = (0; 2(\widehat{u}^2 - 1) - 1)^T,$$

$$\Phi(f(x^0(1), \widehat{u}, 1)) - \Phi(f(x^0(1), u^0(1), 1)) = (\widehat{u}^2 - 1)(\widehat{u}^2 - 2), \qquad U_0(1) = \{1; -1\},$$

$$\Lambda[x^0(\cdot)](0) = \left\{ \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -6 \\ -2 \end{pmatrix} \right\}, \qquad \Delta_v f(0) = \begin{pmatrix} -v^2 \\ v^2 \end{pmatrix},$$

$$f(x^0(0), U(0), 0) = f(x^0(1), U(1), 1) = \left\{ \begin{pmatrix} -v^2 \\ v^2 - 1 \end{pmatrix} : v \in [-1, 1] \right\}.$$

Obviously, the sets $f(x^0(t\cdot), U(t), t)$, $t \in \{0; 1\}$ are convex. This fact and the above calculations imply inequality (4.3):

$$(\widehat{u}^2 - 1)(\widehat{u}^2 - 2) \geq 0 \qquad \text{for all} \quad \widehat{u} \in [-1, 1].$$

However, for $\lambda(0) = (-6, -2)^T$, inequality (4.4) takes the form $4u^2 \leq 0$, $v \in [-1, 1]$, i.e., it does not hold. Therefore, the control $u^0(t)$, $t \in \{0; 1\}$, with $u^0(0) = 0$ and $u^0(1) = 1$ cannot be optimal by Theorem 1.

Note that the results of [9]–[12], [14], [22] leave the control $u^0(t)$, $t \in \{0; 1\}$, among the canditates for being optimal. Moreover, other known results (see [4], [21], [23]) do not apply to this example, because the function f is not differentiable with respect to the variable u at the point $u = 1$.

5. NECESSARY OPTIMALITY CONDITION OF THE SECOND ORDER

Note that Theorem 1 may turn out to be insufficiently informative, i.e., there exist problems in which an admissible process $(u^0(\cdot), x^0(\cdot))$ is not optimal; however, as a necessary optimality condition, the assertion of Theorem 1 holds and even degenerates (see Example 2).

Definition 3. Let an admissible process $(u^0(\cdot), x^0(\cdot))$ satisfy conditions (4.2) and (4.3). A control $u^0(t)$, $t \in I$, will be called *singular at a point $\theta \in I_{-1}$ with respect to an admissible control $\widetilde{u}(t)$* $t \in I$, where $\widetilde{u}(t_1 - 1) \in U_0(t_1 - 1)$ and $\widetilde{u}(t) \in U[x^0(\cdot)](t)$, $t \in I_{-1}$, over a set $U_0(\theta) \subseteq U(\theta)$ if, for all $v \in U_0(\theta)$, the following equality holds:

$$\psi^T(\theta; \widetilde{u}(\theta + 1))\Delta_v f(\theta) = 0, \tag{5.1}$$

where $U_0(\theta) \setminus U[x^0(\cdot)](\theta) \neq \varnothing$ and $\psi(\theta; \widetilde{u}(\theta + 1))$ is the value of the solution of system (3.13).

Remark 2. If $\widetilde{u}(t) = u^0(t)$, $t \in I$, then a singular (in the sense of Definition 3) control $u^0(\cdot)$ will be called *singular at the point θ over the set $U_0(\theta)$*.

Remark 3. Definition 3 defines a new class of singular controls. In other words, a singular control in the sense of Definition 3 is not necessarily singular in the sense of [12], [14], [18]–[21] (see Example 2).

Theorem 2. *Let* $(u^0(\cdot), x^0(\cdot))$ *be an optimal process, and let assumptions* (A2) *and* (A3) *hold. Also, let a control* $u^0(t)$, $t \in I$, *be singular at a point* $\theta \in I_{-1}$, *with respect to an admissible control* $\widetilde{u}(t)$, $t \in I$, *where* $\widetilde{u}(t_1 - 1) \in U_0(t_1 - 1)$ *and* $\widetilde{u}(t) \in U[x^0(\cdot)](t)$, $t \in I_{-1}$, *over a set* $U_0(\theta)$. *Then the following inequality holds*:

$$\Delta_v f^T(\theta)\Psi(\theta; \widetilde{u}(\theta+1))\Delta_v f(\theta) \le 0 \qquad \text{for all} \quad v \in U_0(\theta), \tag{5.2}$$

where $\Delta_v f(\theta)$ *is defined by* (3.3) *and* $\Psi(\theta; \widetilde{u}(\theta+1))$ *is the value of the solution of system* (3.14) *at the point* θ.

Proof. Since assumptions (A2) and (A3) hold, it follows from Proposition 2 that the formula for the expansion (3.20) is valid. Then, by Definition 3, in view of $\widetilde{u}(t_1 - 1) = \widehat{u} \in U_0(t_1 - 1)$ and (5.1), for all $v \in U_0(\theta)$, from (3.20) we obtain

$$\Delta_{\xi,\varepsilon} S(u^0(\cdot)) = -\frac{\varepsilon^2}{2}\Delta_v f^T(\theta)\Psi(\theta; \widetilde{u}(0+1))\Delta_v f(\theta) + o_\Sigma(\varepsilon^2), \qquad \varepsilon \in (0, \gamma],$$

where $\xi = (\theta, v, \widetilde{u}(\cdot))$.

This implies the validity of inequality (5.2), because, for a sufficiently small $\varepsilon > 0$, the inequality $\Delta_{\xi,\varepsilon} S(u^0(\cdot)) \ge 0$ holds along the optimal process $(u^0(\cdot), x^0(\cdot))$. The theorem is proved. □

Consider an example illustrating the meaningfulness of Theorem 2.

Example 2. Consider $S(u(\cdot)) = x_2^2(2) - x_2(2) \to \min_{u(\cdot)}$,

$$\begin{cases} x_1(t+1) = x_1(t) - u^2(t), \\ x_2(t+1) = -x_1(t)|u(t) - 1| + x_2^2(t)u^2(t) - 1, \ t \in \{0; 1\}, \end{cases}$$

$$x_1(0) = 0, \quad x_2(0) = -1, \qquad u(t) \in U(t), \quad t \in \{0; 1\},$$

$$U(0) = [-1, 1], \qquad U(1) = [\pm 1; \pm 2].$$

Let us study the optimality of the admissible control $u^0(t)$, $t \in \{0; 1\}$, defined by $u^0(0) = 0$ and $u^0(1) = 1$. It follows from (2.4), (4.1), and (4.2) that

$$x^0(t) = (x_1^0(t), x_2^0(t))^T, \quad t \in \{0; 1; 2\}, \qquad \text{where} \quad x_1^0(0) = x_1^0(1) = 0, \quad x_1^0(2) = -1,$$

$$x_2^0(0) = x_2^0(1) = -1, \quad x_2^0(2) = 0, \qquad f(x^0(1), u^0(1), 1) = (-1, 0)^T,$$

$$f(x^0(1), \widehat{u}, 1) = (-\widehat{u}^2, \widehat{u}^2 - 1)^T, \qquad U[x^0(\cdot)](0) = \{0\}, \quad U[x^0(\cdot)](1) = \{-1; 1\},$$

$$\Phi(f(x^0(1), \widehat{u}, 1)) - \Phi(f(x^0(1), u^0(1), 1)) = (\widehat{u}^2 - 1)(\widehat{u}^2 - 2),$$

$$U_0(1) = \{-1; 1\}, \qquad f_x^T(x^0(1), \widehat{u}, 1)\Phi_x(f(x^0(1), \widehat{u}, 1))|_{\widehat{u}=\pm 1} = \{(0, 2)^T, (2, 2)^T\},$$

$$\Lambda[x^0(\cdot)](0) = \{(0, -2)^T; (-2, -2)^T\}, \qquad \Delta_v f(0) = (-v^2, v^2)^T, \quad v \in [-1, 1],$$

$$f(x^0(0), U(0), 0) = \{(-v^2, v^2 - 1)^T : v \in [-1, 1]\}$$

(obviously, this set is convex).

Taking into account the above calculations, we can readily see that the assertion of Theorem 1 holds:

$$(\widehat{u}^2 - 1)(\widehat{u}^2 - 2) \ge 0 \qquad \text{for all} \quad \widehat{u} \in [-1, 1],$$

$$-2v^2 \le 0, \qquad v \in [-1, 1] \quad \text{for} \quad \lambda(0) = (0, -2)^T, \qquad 0 \le 0 \quad \text{for} \quad \lambda(0) = (-2, -2)^T.$$

Therefore, Theorem 1 leaves the control $u^0(t)$, $t \in \{0, 1\}$, among the candidates for being optimal.

In addition, the control $u^0(t)$, $t \in \{0, 1\}$, is singular at the point $\theta = 0$ with respect to the admissible control $\widehat{u}(t)$, $t \in \{0, 1\}$, defined by $\widetilde{u}(0) = 0$ and $\widetilde{u}(1) = -1$, over the set $U_0(0) = [-1, 1]$, because the process $(u^0(\cdot), x^0(\cdot))$ satisfies conditions (4.2), (4.3) and

$$\psi(0, \widetilde{u}(1)) = (-2, 2)^T, \qquad \Delta_v f(0) = (-v^2, v^2)^T, \quad v \in [-1, 1].$$

Note that the control $u^0(\,\cdot\,)$ is not singular in the sense of [12], [14], [18]–[21]. Let us apply Theorem 2. From system (3.14), we obtain

$$\Psi(0;\widetilde{u}(1)) = (\psi_{ij}), \qquad \text{where} \quad \psi_{11} = \psi_{12} = \psi_{21} = -8, \quad \psi_{22} = -6;$$

therefore, inequality (5.2) is of the form $2\upsilon^4 \le 0$ for all $\upsilon \in [-1, 1]$, which is not true. Therefore, the process $(u^0(\,\cdot\,), x^0(\,\cdot\,))$ is not optimal.

Note that the known results of [26]–[28] do not apply to this example, because the function $f(\,\cdot\,)$ is not differentiable with respect to the variable u at the point $u = 1$ and the set $U(1)$ is neither convex nor open. Note also that the optimality conditions given in [12], [14]–[22] leave the control $u^0(\,\cdot\,)$ among the candidates for being optimal.

6. CONCLUSIONS

As can be seen, problem (2.1)–(2.3) is not the most general one among discrete optimization problems. We chose it solely in order to demonstrate the main aspects of the approach proposed in the paper. However, the assertions of Theorems 1 and 2 can also be generalized to the case of more general discrete control problems.

In addition, on the basis of our scheme of study, we note the following.

1. The necessary optimality conditions in the form of the assertions of Theorem 1 and 2 are characteristic for discrete control systems.

2. In obtaining a number of well-known necessary optimality conditions of the first and higher order (see, e.g., [5], [7], [18], [21], [29], [30]), the assumption on the convexity of the set of admissible velocities or the set of admissible controls can be replaced by the weaker condition of the γ-convexity of the set.

3. The notion of γ-convex set (as well as the notion of the interior of a set in the wide sense) can also be very effective in the study of continuous-time optimal control problems.

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