ISSN 0001-4346, Mathematical Notes, 2019, Vol. 106, No. 2, pp. 191–202. © Pleiades Publishing, Ltd., 2019.
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On Estimates in $L_2(\mathbb{R})$ of Mean ν-Widths of Classes of Functions Defined via the Generalized Modulus of Continuity of $\omega_{\mathcal{M}}$

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Received April 22, 2018; in final form, September 9, 2018; accepted November 14, 2018

Abstract—For the classes of functions

$$W^r(\omega_{\mathcal{M}},\Phi):=\{f\in L_2^r(\mathbb{R}):\omega_{\mathcal{M}}(f^{(r)},t)\le\Phi(t)\;\forall\,t\in(0,\infty)\},$$

where Φ is a majorant and $r\in\mathbb{Z}_+$, lower and upper bounds for the Bernstein, Kolmogorov, and linear mean ν-widths in the space $L_2(\mathbb{R})$ are obtained. A condition on the majorant Φ under which the exact values of these widths can be calculated is indicated. Several examples illustrating the results are given.

DOI: 10.1134/S000143461907023X

Keywords: *mean dimension, mean ν-width, majorant, entire function of exponential type, generalized modulus of continuity.*

1. INTRODUCTION

The study of smoothness characteristics of functions attracts special attention in approximation theory. Different approaches aimed at the generalization and improvement of these characteristics are considered. In this connection, in the case of 2π-periodic functions, one should note, first of all, the papers [1] and [2] of Boman and Shapiro, which were followed by further improved results in this direction due to Vasil'ev, Babenko, Kozko and Rozhdestvenskii, Ivanov and Ha Thi Min Hue, Gorbachev, Runovski and Schmeisser, and other authors (see, e.g., [3]–[12]).

In the case of extremal problems of approximation theory on the whole real axis, generalizations of the classical modulus of continuity of functions were considered in the papers [13] of Vasiliev (the multidimensional case) and [14] of Artamonov.

Let $L_2(\mathbb{R})$ denote the space of all measurable functions f on the real axis $\mathbb{R}$ which are Lebesgue square-integrable on any finite interval and whose norm is

$$\|f\|:=\left\{\int_{-\infty}^{\infty}|f(x)|^2\,dx\right\}^{1/2}<\infty.$$

The present paper is devoted to the extension of a result due to A. V. Efimov's student Grigoryan (see [15, Theorem 2 and its Corollary 1]) to the case of the space $L_2(\mathbb{R})$ with the use of the generalized smoothness characteristic from [13] for $N=1$ instead of the usual modulus of continuity of the first order.

Using the notation from [13], we present the definition of the generalized modulus of continuity in the space $L_2(\mathbb{R})$. Let $\mathcal{M}:=\{\mu_j\}_{j\in\mathbb{Z}}$ be a collection of complex numbers satisfying the conditions

$$0<\sum_{j\in\mathbb{Z}}|\mu_j|<\infty,\qquad \sum_{j\in\mathbb{Z}}\mu_j=0. \tag{1.1}$$

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By $\Delta_h^{\mathcal{M}}$, $h\in\mathbb{R}$, we mean the generalized difference operator acting from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$. For an arbitrary function $f\in L_2(\mathbb{R})$, almost everywhere on $\mathbb{R}$, we have

$$\Delta_h^{\mathcal{M}}(f,x):=\sum_{j\in\mathbb{Z}}\mu_j f(x+jh). \tag{1.2}$$

Note, for example, that, for the number set

$$\mathcal{M}_{1,m}:=\left.\begin{cases}\mu_j=(-1)^{m-j}\binom{m}{j} & \text{if } j=0,\dots,m,\\ \mu_j=0 & \text{if } j<0 \text{ is or } j>m\end{cases}\right\}_{j\in\mathbb{Z}}, \qquad m\in\mathbb{N},$$

the operator $\Delta_h^{\mathcal{M}_{1,m}}$ becomes, in view of (1.2), the usual finite-difference operator $\Delta_h^m\colon L_2(\mathbb{R})\to L_2(\mathbb{R})$, which, for almost all $x\in\mathbb{R}$, is of the form

$$\Delta_h^m(f,x)=\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jh).$$

By the *generalized modulus of continuity* of an arbitrary element $f\in L_2(\mathbb{R})$ generated by the number set $\mathcal{M}$ we mean the function

$$\omega_{\mathcal{M}}(f,t):=\sup\{\|\Delta_h^{\mathcal{M}}(f)\|:|h|\le t\},\qquad t\ge 0. \tag{1.3}$$

Setting, for example, $\mathcal{M}=\mathcal{M}_{1,m}$, $m\in\mathbb{N}$, from (1.2), (1.3) we obtain the mth-order modulus of continuity

$$\omega_m(f,t)=\sup\{\|\Delta_h^m(f)\|:|h|\le t\}.$$

It is well known (see, e.g., [16, Chap. III, Sec. 3.11.21]) that any function $f\in L_2(\mathbb{R})$ almost everywhere on $\mathbb{R}$ can be represented as

$$f(x)=\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\mathcal{F}(f,t)\frac{e^{ixt}-1}{it}\,dt, \tag{1.4}$$

where $\mathcal{F}(f)\in L_2(\mathbb{R})$ is its Fourier transform, which, for almost all $x\in\mathbb{R}$, can be written as

$$\mathcal{F}(f,x)=\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}f(t)\frac{e^{-ixt}-1}{-it}\,dt.$$

Further,

$$\int_{-\infty}^{\infty}|\mathcal{F}(f,x)|^2\,dx=\int_{-\infty}^{\infty}|f(x)|^2\,dx. \tag{1.5}$$

Setting

$$w_{\mathcal{M}}(x):=\sum_{j\in\mathbb{Z}}\mu_j e^{ijx} \tag{1.6}$$

and using formulas (1.1), (1.2), and (1.4), almost everywhere on $\mathbb{R}$, we have

$$\begin{aligned}\Delta_h^{\mathcal{M}}(f,x)&=\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\mathcal{F}(f,t)\biggl\{\sum_{j\in\mathbb{Z}}\frac{e^{i(x+jh)t}-1}{it}\mu_j\biggr\}\,dt\\ &=\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\mathcal{F}(f,t)w_{\mathcal{M}}(ht)\frac{e^{ixt}-1}{it}\,dt. \end{aligned}\tag{1.7}$$

It follows from (1.4) and (1.7) that, for almost all $x\in\mathbb{R}$,

$$\mathcal{F}(\Delta_h^{\mathcal{M}}(f),x)=w_{\mathcal{M}}(hx)\mathcal{F}(f,x). \tag{1.8}$$

Then, in view of (1.8) from (1.5), we obtain the relation

$$\|\Delta_h^{\mathcal{M}}(f)\|^2 = \int_{-\infty}^{\infty} |\mathcal{F}(\Delta_h^{\mathcal{M}}(f), x)|^2\,dx = \int_{-\infty}^{\infty} |\mathcal{F}(f, x)|^2 |w_{\mathcal{M}}(hx)|^2\,dx. \tag{1.9}$$

In what follows, we shall always assume that all elements of the number set $\mathcal{M}$ are real. Then it follows from (1.6) and (1.1) that the function $|w_{\mathcal{M}}(x)|^2$ is 2π-periodic, continuous, and even and, at the point $x = 0$, it takes the zero value.

Using relation (1.9), we rewrite formula (1.3) as

$$\omega_{\mathcal{M}}(f, t) = \sup\biggl\{\biggl(\int_{-\infty}^{\infty} |\mathcal{F}(f, x)|^2 |w_{\mathcal{M}}(hx)|^2\,dx\biggr)^{1/2} : 0 \le h \le t\biggr\}, \qquad t \ge 0. \tag{1.10}$$

If, for example, $\mathcal{M} = \mathcal{M}_{1,m}$, $m \in \mathbb{N}$, then, using (1.10) and (1.6), we obtain the well-known relation

$$\omega_m(f, t) = \sup\biggl\{\biggl(2^m \int_{-\infty}^{\infty} |\mathcal{F}(f, x)|^2 (1 - \cos(hx))^m\,dx\biggr)^{1/2} : 0 \le h \le t\biggr\}, \qquad t \ge 0,$$

because $|w_{\mathcal{M}_{1,m}}(x)|^2 = 2^m(1 - \cos x)^m$.

2. MEAN ν-WIDTHS OF CLASSES OF FUNCTIONS IN THE SPACE $L_2(\mathbb{R})$

By the symbol $\mathbb{B}_{\sigma,2}$, $\sigma \in (0, \infty)$, we shall denote the subspace of functions that are restrictions to $\mathbb{R}$ of entire functions of exponential type σ if these restrictions belong to the space $L_2(\mathbb{R})$.

Of particular note is that only after the introduction by Magaril-Il'yaev of an appropriate definition [17], [18], it became possible to calculate asymptotic characteristics of classes of functions in $L_2(\mathbb{R})$ similar, for example, to the n-widths in the 2π-periodic case, but with the usual dimension of a finite-dimensional subspace replaced by mean dimension. Magaril-Il'yaev's definition of mean dimension was a modification of the notion introduced earlier by Tikhomirov. As a result, it became possible to compare the approximation properties of the subspace $\mathbb{B}_{\sigma,2}$ with similar characteristics of other subspaces of $L_2(\mathbb{R})$ of the same mean dimension and solve extremal problems of approximation theory dealing with optimization.

Before introducing the necessary extremal characteristics, we present some notions and definitions from [17], [18]. Let $BL_2(\mathbb{R})$ be the unit ball in $L_2(\mathbb{R})$, let $\operatorname{Lin}(L_2(\mathbb{R}))$ be the collection of all linear subspaces in $L_2(\mathbb{R})$, and let

$$\operatorname{Lin}_n(L_2(\mathbb{R})) := \{\mathcal{L} \in \operatorname{Lin}(L_2(\mathbb{R})) : \dim \mathcal{L} \le n\}, \qquad n \in \mathbb{Z}_+;$$
$$d(Q, A, L_2(\mathbb{R})) := \sup\{\inf\{\|x - y\| : y \in A\} : x \in Q\}$$

be the best approximation of the set $Q \subset L_2(\mathbb{R})$ by a set $A \subset L_2(\mathbb{R})$. By A_T, where $T > 0$, we mean the restriction of the set $A \subset L_2(\mathbb{R})$ to the closed interval $[-T, T]$ and by $\operatorname{Lin}_C L_2(\mathbb{R})$ we denote the collection of subspaces $\mathcal{L} \in \operatorname{Lin}(L_2(\mathbb{R}))$ for which the set $(\mathcal{L} \cap BL_2(\mathbb{R}))_T$ is precompact in $L_2([-T, T])$ for any $T > 0$.

If $\mathcal{L} \in \operatorname{Lin}_C(L_2(\mathbb{R}))$ and $T, \varepsilon > 0$, then there exists an $n \in \mathbb{Z}_+$ and a $\mathcal{K} \in \operatorname{Lin}_n(L_2(\mathbb{R}))$ such that

$$d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{K}, L_2([-T, T])) < \varepsilon.$$

Let

$$D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R})) := \min\bigl\{n \in \mathbb{Z}_+ : \exists \mathcal{K} \in \operatorname{Lin}_n(L_2([-T, T]))\ d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{K}, L_2([-T, T])) < \varepsilon\bigr\}.$$

This function is nondecreasing in T and nonincreasing in ε. The quantity

$$\overline{\dim}(\mathcal{L}, L_2(\mathbb{R})) := \lim\biggl\{\liminf\biggl\{\frac{D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R}))}{2T} : T \to \infty\biggr\} : \varepsilon \to 0\biggr\},$$

where $\mathcal{L} \in \operatorname{Lin}_C(L_2(\mathbb{R}))$, is called the *mean dimension* of the subspace $\mathcal{L}$ in $L_2(\mathbb{R})$. In [18], it was shown that

$$\overline{\dim}(\mathbb{B}_{\sigma,2}; L_2(\mathbb{R})) = \frac{\sigma}{\pi}. \tag{2.1}$$

Let Q be a centrally symmetric subset of $L_2(\mathbb{R})$, and let $\nu > 0$ be an arbitrary number. Then by the *mean ν-width in the sense of Kolmogorov* of the set Q in $L_2(\mathbb{R})$ we mean the quantity

$$\overline{d}_\nu(Q, L_2(\mathbb{R})) := \inf\bigl\{\sup\{\inf\{\|f-\varphi\| : \varphi \in \mathcal{L}\} : f \in Q\} : \mathcal{L} \in \mathrm{Lin}_C(L_2(\mathbb{R})),\ \overline{\dim}(\mathcal{L}, L_2(\mathbb{R})) \le \nu\bigr\}.$$

A subspace on which the outer infimum is attained is said to be *extremal*.

By the mean linear ν-width of the set Q in $L_2(\mathbb{R})$ we mean the quantity

$$\overline{\delta}_\nu(Q, L_2(\mathbb{R})) := \inf\{\sup\{\|f - V(f)\| : f \in Q\} : (X, V)\},$$

where $Q \subset X$ and the infimum is taken over all pairs (X, V) such that X is a normed space directly embedded in $L_2(\mathbb{R})$ and $V\colon X \to L_2(\mathbb{R})$ is a continuous linear operator for which $\mathrm{Im}\, V \in \mathrm{Lin}_C(L_2(\mathbb{R}))$ and the inequality $\overline{\dim}(\mathrm{Im}\, V, L_2(\mathbb{R})) \le \nu$ holds. Here $\mathrm{Im}\, V$ is the image of the operator V. A pair on which the infimum is attained is said to be *extremal*.

The quantity

$$\overline{b}_\nu(Q, L_2(\mathbb{R})) := \sup\bigl\{\sup\{\rho > 0 : \mathcal{L} \cap \rho B L_2(\mathbb{R}) \subset Q\} \\ : \mathcal{L} \in \mathrm{Lin}_C(L_2(\mathbb{R})), \overline{\dim}(\mathcal{L}, L_2(\mathbb{R})) > \nu,\ \overline{d}_\nu(\mathcal{L} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1\bigr\}$$

is called the *mean ν-width in the sense of Bernstein* of the set Q in $L_2(\mathbb{R})$. The last condition imposed on $\mathcal{L}$ in the calculation of the outer supremum means that we consider only subspaces for which the analog of Tikhomirov's theorem on the width of the ball holds. This requirement is satisfied, for example, by the subspace $\mathbb{B}_{\sigma,2}$ with $\sigma > \nu\pi$, i.e.,

$$\overline{d}_\nu(\mathbb{B}_{\sigma,2} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1.$$

The following inequalities between the given extremal characteristics of a set $Q \subset L_2(\mathbb{R})$ hold:

$$\overline{b}_\nu(Q, L_2(\mathbb{R})) \le \overline{d}_\nu(Q, L_2(\mathbb{R})) \le \overline{\delta}_\nu(Q, L_2(\mathbb{R})). \tag{2.2}$$

Note that, in the space $L_2(\mathbb{R})$, the exact value of these mean ν-widths of classes of functions defined by their smoothness characteristics ω_m, $m \in \mathbb{N}$, were calculated, for example, in [19]–[21].

3. ESTIMATES OF THE MEAN ν-WIDTHS OF SOME CLASSES OF FUNCTIONS DEFINED BY THEIR SMOOTHNESS CHARACTERISTICS $\omega_{\mathcal{M}}$

Let the symbol $L_2^r(\mathbb{R})$, $r \in \mathbb{N}$, denote the class of functions $f \in L_2(\mathbb{R})$ whose $(r-1)$th derivatives $f^{(r-1)}$ ($f^{(0)} \equiv f$) are locally absolutely continuous and the rth derivatives $f^{(r)}$ belong to the space $L_2(\mathbb{R})$. Note that $L_2^r(\mathbb{R})$ is a Banach space with norm

$$\|f\| + \|f^{(r)}\|.$$

Let $\Phi(t)$, $t \in [0, \infty)$, be a continuous increasing function such that $\Phi(0) = 0$. In what follows, Φ will be called a *majorant*. Let $W^r(\omega_{\mathcal{M}}, \Phi)$, where $r \in \mathbb{Z}_+$, denote the class of functions $f \in L_2^r(\mathbb{R})$ ($L_2^0(\mathbb{R}) \equiv L_2(\mathbb{R})$) for which the following inequality holds for any $t \in (0, \infty)$:

$$\omega_{\mathcal{M}}(f^{(r)}, t) \le \Phi(t).$$

For the function $w_{\mathcal{M}}$ defined by (1.6), let $t_* \in (0, 2\pi)$ ($t_* = t_*(\omega_{\mathcal{M}})$) denote a value of its argument for which the following equality holds:

$$|w_{\mathcal{M}}(t_*)| = \max\{|w_{\mathcal{M}}(x)| : 0 < x < 2\pi\}.$$

If there are more than one such values, then, for t_*, we take the least of them. We say that $w_{\mathcal{M}}$ satisfies *property* A if, on the closed interval $[0, t_*]$, the function $|w_{\mathcal{M}}|$ is monotonically increasing. So, for example, in the case $\mathcal{M} = \mathcal{M}_{1,m}$, $m \in \mathbb{N}$, using (1.6), we obtain

$$w_{\mathcal{M}_{1,m}}(x) = (e^{ix} - 1)^m.$$

Therefore, the function

$$|w_{\mathcal{M}_{1,m}}(x)| = \{2(1 - \cos x)\}^{m/2}$$

satisfies *property* A and, for it, $t_* = \pi$. Note that this property has turned out to be very useful in the 2π-periodic case for the derivation of sharp Jackson-type inequalities with generalized modulus of continuity, as well as in the calculation of exact values of n-widths for classes of (ψ, β)-differentiable functions [10], [11].

Further, we set

$$|w_{\mathcal{M}}(x)|_* := \begin{cases} |w_{\mathcal{M}}(x)| & \text{if } 0 \le x \le t_*, \\ |w_{\mathcal{M}}(t_*)| & \text{if } t_* \le x \le 2\pi. \end{cases} \tag{3.1}$$

For an arbitrary function $f \in L_2(\mathbb{R})$, we let $\mathcal{A}_\sigma(f)$, $\sigma \in (0, \infty)$, denote its mean-square approximation by elements of the subspace $\mathbb{B}_{\sigma,2}$, i.e.,

$$\mathcal{A}_\sigma(f) := \inf\{\|f - g\| : g \in \mathbb{B}_{\sigma,2}\}.$$

In [22], it was noted that, for a function $f \in L_2(\mathbb{R})$, the entire function

$$\mathcal{L}_\sigma(f, x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \mathcal{F}(f, t) e^{ixt}\, dt, \tag{3.2}$$

which belongs to the space $\mathbb{B}_{\sigma,2}$, has the least deviation from f in the sense of the metric $L_2(\mathbb{R})$, i.e.,

$$\mathcal{A}_\sigma(f) = \|f - \mathcal{L}_\sigma(f)\| = \biggl\{\int_{|t|\ge\sigma} |\mathcal{F}(f, t)|^2\, dt\biggr\}^{1/2}. \tag{3.3}$$

The best approximation of a class $\mathfrak{M} \subset L_2(\mathbb{R})$ by the subspace $\mathbb{B}_{\sigma,2}$ is defined as follows:

$$\mathcal{A}_\sigma(\mathfrak{M}) = \sup\{\mathcal{A}_\sigma(f) : f \in \mathfrak{M}\}.$$

Theorem 1. *Let the function $w_{\mathcal{M}}$ satisfy property* A, *let $r \in \mathbb{Z}_+$, let Φ be an arbitrary majorant, and let $\nu \in (0, \infty)$. Then the following relation holds*:

$$\begin{aligned} &\frac{1}{(\nu\pi)^r} \inf\biggl\{\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|} : 0 < \tau \le \frac{t_*}{\nu\pi}\biggr\} \\ &\qquad \le \overline{\Pi}_\nu(W^r(\omega_{\mathcal{M}}, \Phi); L_2(\mathbb{R})) \le \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in W^r(\omega_{\mathcal{M}}, \Phi)\} \\ &\qquad = \mathcal{A}_{\nu\pi}(W^r(\omega_{\mathcal{M}}, \Phi)) \le \frac{1}{(\nu\pi)^r} \varlimsup_{\tau\to 0+} \frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|}, \end{aligned} \tag{3.4}$$

where $\overline{\Pi}_\nu(\,\cdot\,)$ is any of the mean ν-widths: *the Bernstein ν-width $\overline{b}_\nu(\,\cdot\,)$, the Kolmogorov ν-width $\overline{d}_\nu(\,\cdot\,)$, or the linear ν-width $\overline{\delta}_\nu(\,\cdot\,)$.*

Proof. By formula (2.1), for the mean dimension of the subspace $\mathbb{B}_{\sigma,2}$, we have

$$\overline{\dim}(\mathbb{B}_{\sigma,2}; L_2(\mathbb{R})) = \frac{\sigma}{\pi}.$$

In view of this, setting $\sigma = \nu\pi$, for an arbitrary function $f \in L_2^r(\mathbb{R})$, from relation (3.3) we obtain

$$\mathcal{A}^2_{\nu\pi}(f^{(r)}) = \int_{|t|\ge\nu\pi} |\mathcal{F}(f^{(r)}, t)|^2\, dt.$$

Next, we consider arbitrarily small positive numbers ε for each of which there exists a value $K_{\varepsilon,f} \in (\nu\pi, \infty)$ depending on f and ε such that the following relation holds:

$$\mathcal{A}^2_{\nu\pi}(f^{(r)}) \le \int_{\nu\pi\le|t|\le K_{\varepsilon,f}} |\mathcal{F}(f^{(r)}, t)|^2\, dt + \varepsilon. \tag{3.5}$$

Since the function $|w_{\mathcal{M}}|$ satisfies *property* A, for an arbitrary number $\tau \in (0, t_*/K_{\varepsilon,f}]$, using relations (1.10), we obtain

$$\int_{\nu\pi \le |t| \le K_{\varepsilon,f}} |\mathcal{F}(f^{(r)},t)|^2\,dt \le \frac{1}{|w_{\mathcal{M}}(\tau\nu\pi)|^2} \int_{\nu\pi \le |t| \le K_{\varepsilon,f}} |\mathcal{F}(f^{(r)},t)|^2 |w_{\mathcal{M}}(\tau t)|^2\,dt$$
$$\le \frac{\omega^2_{\mathcal{M}}(f^{(r)},\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|^2}. \tag{3.6}$$

Since, for any element $f \in L_2^r(\mathbb{R})$, we have [22]

$$\mathcal{A}_{\nu\pi}(f) \le \frac{1}{(\nu\pi)^r}\,\mathcal{A}_{\nu\pi}(f^{(r)}),$$

it follows from (3.5) (3.6) that, for $0 < \tau \le t_*/K_{\varepsilon,f}$,

$$\mathcal{A}^2_{\nu\pi}(f) \le \frac{\omega^2_{\mathcal{M}}(f^{(r)},\tau)}{(\nu\pi)^{2r}|w_{\mathcal{M}}(\tau\nu\pi)|^2} + \frac{\varepsilon}{(\nu\pi)^{2r}}. \tag{3.7}$$

It follows from inequality (3.5) that, as $\varepsilon \to 0+$, we have $K_{\varepsilon,f} \to +\infty$, and hence $\tau \to 0+$. In view of the above, passing to the upper limit as $\varepsilon \to 0+$ on the right-hand side of equality (3.7), for an arbitrary function $f \in W^r(\omega_{\mathcal{M}}, \Phi)$, we can write

$$\mathcal{A}_{\nu\pi}(f) \le \frac{1}{(\nu\pi)^r} \overline{\lim_{\tau\to 0+}} \frac{\omega_{\mathcal{M}}(f^{(r)},\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|} \le \frac{1}{(\nu\pi)^r} \overline{\lim_{\tau\to 0+}} \frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|}.$$

Hence, using relation (2.2), we obtain the following upper bounds for the extremal characteristics of the classes $W^r(\omega_{\mathcal{M}}, \Phi)$:

$$\overline{b}_\nu(W^r(\omega_{\mathcal{M}},\Phi); L_2(\mathbb{R})) \le \overline{d}_\nu(W^r(\omega_{\mathcal{M}},\Phi); L_2(\mathbb{R})) \le \overline{\delta}_\nu(W^r(\omega_{\mathcal{M}},\Phi); L_2(\mathbb{R}))$$
$$\le \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in W^r(w_{\mathcal{M}},\Phi)\}$$
$$= \mathcal{A}_{\nu\pi}(W^r(\omega_{\mathcal{M}},\Phi)) \le \frac{1}{(\nu\pi)^r} \overline{\lim_{\tau\to 0+}} \frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|}. \tag{3.8}$$

Let us now derive lower bounds for the extremal characteristics of the classes $W^r(\omega_{\mathcal{M}}, \Phi)$. Let $\widehat{\sigma} := \nu\pi(1+\varepsilon)$, where $\varepsilon \in (0, \widetilde{\nu})$ is an arbitrary number, and let $\widetilde{\nu} := \min(\nu, 1/\nu)$. By (2.1), the mean dimension of the subspace $\mathbb{B}_{\widehat{\sigma},2}$ is

$$\overline{\dim}(\mathbb{B}_{\widehat{\sigma},2}; L_2(\mathbb{R})) = \nu(1+\varepsilon).$$

We consider the set of entire functions

$$\mathcal{B}_{\widehat{\sigma}}(\rho) := \mathbb{B}_{\widehat{\sigma},2} \cap \rho B L_2(\mathbb{R}) = \{g \in \mathbb{B}_{\widehat{\sigma},2} : \|g\| \le \rho\},$$

where

$$\rho := \frac{1}{\widehat{\sigma}^{\,r}} \inf\left\{\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\widehat{\sigma})|_*} : 0 < \tau \le \frac{t_*}{\nu\pi}\right\} \tag{3.9}$$

and the function $|w_{\mathcal{M}}|_*$ is defined by relation (3.1).

By the Wiener–Paley theorem (see, e.g., [16, Chap. IV, Sec. 4.6.1]), for an arbitrary entire function $q \in \mathbb{B}_{\widehat{\sigma},2}$ on the real axis $\mathbb{R}$, we have the representation

$$q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\widehat{\sigma}}^{\widehat{\sigma}} \mathcal{F}(q,t) e^{ixt}\,dt,$$

where its Fourier transform

$$\mathcal{F}(q,x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} q(t) \frac{e^{-itx} - 1}{-it}\,dt$$

is zero for almost all $x \in \mathbb{R}$ such that $|x| > \widehat{\sigma}$. Note that if $q \in \mathbb{B}_{\widehat{\sigma},2}$, then the rth derivative $q^{(r)}$ will also belong to the subspace $\mathbb{B}_{\widehat{\sigma},2}$. Using formulas (1.10) and (3.9) and taking into account the fact that, for an arbitrary function $q \in \mathbb{B}_{\widehat{\sigma},2}$, the inequality $\|q^{(r)}\| \le \widehat{\sigma}^r \|q\|$ holds (see, e.g., [16, Chap. IV, Sec. 4.8.61]), for any element g from the set $\mathcal{B}_{\widehat{\sigma}}(\rho)$, we obtain

$$\begin{aligned}\omega_{\mathcal{M}}(g^{(r)}, t) &= \sup\biggl\{\biggl(\int_{-\widehat{\sigma}}^{\widehat{\sigma}} |\mathcal{F}(g^{(r)}, x)|^2\, |w_{\mathcal{M}}(hx)|^2\, dx\biggr)^{1/2} : 0 < h \le t\biggr\}\\ &\le |w_{\mathcal{M}}(\widehat{\sigma}t)|_* \biggl\{\int_{-\widehat{\sigma}}^{\widehat{\sigma}} |\mathcal{F}(g^{(r)}, x)|^2\, dx\biggr\}^{1/2} = |w_{\mathcal{M}}(\widehat{\sigma}t)|_* \|g^{(r)}\| \le |w_{\mathcal{M}}(\widehat{\sigma}t)|_*\, \rho\, \widehat{\sigma}^r\\ &= |w_{\mathcal{M}}(\widehat{\sigma}t)|_* \inf\biggl\{\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\widehat{\sigma})|_*} : 0 < \tau \le \frac{t_*}{\nu\pi}\biggr\}. \end{aligned} \tag{3.10}$$

Let us show that the set $\mathcal{B}_{\widehat{\sigma}}(\rho)$ belongs to the class $W^r(\omega_{\mathcal{M}}, \Phi)$, i.e., for an arbitrary element $g \in \mathcal{B}_{\widehat{\sigma}}(\rho)$ and any $t \in (0, \infty)$, the following inequality holds:

$$\omega_{\mathcal{M}}(g^{(r)}, t) \le \Phi(t). \tag{3.11}$$

To this end, we consider the following two cases: $0 < t < t_*/\widehat{\sigma}$ and $t_*/\widehat{\sigma} \le t < \infty$. In the first case, taking into account the inequality $\widehat{\sigma} > \nu\pi$, and setting $\tau = t$ on the right-hand side of relation (3.10), we obtain $\omega_{\mathcal{M}}(g^{(r)}, t) \le \Phi(t)$. In the second case, in view of (3.1), we have $|w_{\mathcal{M}}(\widehat{\sigma}t)|_* = |w_{\mathcal{M}}(t_*)|$, and hence, from (3.10), for $\tau = t_*/\widehat{\sigma}$, we obtain

$$\omega_{\mathcal{M}}(g^{(r)}, t) \le \Phi\biggl(\frac{t_*}{\widehat{\sigma}}\biggr) \le \Phi(t).$$

Therefore, inequality (3.11) holds and $\mathcal{B}_{\widehat{\sigma}}(\rho) \subset W^r(\omega_{\mathcal{M}}, \Phi)$.

Using the definition of the mean ν-width in the sense of Bernstein, we can write

$$\overline{b}_\nu(W^r(\omega_{\mathcal{M}}, \Phi); L_2(\mathbb{R})) \ge \overline{b}_\nu(\mathcal{B}_{\widehat{\sigma}}(\rho), L_2(\mathbb{R})) \ge \rho,$$

or, in view of (3.9),

$$\overline{b}_\nu(W^r(\omega_{\mathcal{M}}, \Phi); L_2(\mathbb{R})) \ge \frac{1}{(\nu\pi)^r} \inf\biggl\{\frac{\Phi(\tau)}{\mathfrak{N}_{\nu,r,\tau}(\varepsilon)} : 0 < \tau \le \frac{t_*}{\nu\pi}\biggr\}, \tag{3.12}$$

where

$$\mathfrak{N}_{\nu,r,\tau}(\varepsilon) := (1+\varepsilon)^r |w_{\mathcal{M}}(\tau\nu\pi(1+\varepsilon))|_*. \tag{3.13}$$

The quantity (3.13) is monotonically increasing in ε for fixed values of the other parameters ν, r, and τ, and, in view of (3.1),

$$\lim\{\mathfrak{N}_{\nu,r,\tau}(\varepsilon) : \varepsilon \to 0+\} = |w_{\mathcal{M}}(\tau\nu\pi)|.$$

It follows that, for an arbitrary, but fixed, number $\tau \in (0, t_*/(\nu\pi)]$ and an arbitrarily small number $\lambda \in (0, \Phi(t_*/(\nu\pi)))$, there exists a value of $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\lambda, \tau) \in (0, \widetilde{\nu})$ for which the following inequality holds:

$$\frac{1}{\mathfrak{N}_{\nu,r,\tau}(\widetilde{\varepsilon})} > \frac{1}{|w_{\mathcal{M}}(\tau\nu\pi)|} - \frac{\lambda}{\Phi(t_*/(\nu\pi))}.$$

Multiplying both sides of this inequality by $\Phi(\tau)$ and taking into account the fact that Φ is an increasing function, we obtain

$$\frac{\Phi(\tau)}{\mathfrak{N}_{\nu,r,\tau}(\widetilde{\varepsilon})} > \frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|} - \lambda.$$

Then

$$\inf\biggl\{\frac{\Phi(\tau)}{\mathfrak{N}_{\nu,r,\tau}(\widetilde{\varepsilon})} : 0 < \tau \le \frac{t_*}{\nu\pi}\biggr\} > \inf\biggl\{\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|} : 0 < \tau \le \frac{t_*}{\nu\pi}\biggr\} - \lambda. \tag{3.14}$$

Using the definition of the supremum of a number set, from (3.14) we obtain

$$\sup\left\{\inf\left\{\frac{\Phi(\tau)}{\mathfrak{N}_{\nu,r,\tau}(\varepsilon)} : 0<\tau\le\frac{t_*}{\nu\pi}\right\} : 0<\varepsilon<\widetilde{\nu}\right\} = \inf\left\{\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|} : 0<\tau\le\frac{t_*}{\nu\pi}\right\}. \tag{3.15}$$

Calculating the infimum of the right-hand side of inequality (3.12) over $\varepsilon\in(0,\widetilde{\nu})$ and using equality (3.15), we write

$$\overline{b}_\nu(W^r(\omega_{\mathcal{M}},\Phi);L_2(\mathbb{R})) \ge \frac{1}{(\nu\pi)^r}\inf\left\{\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|} : 0<\tau\le\frac{t_*}{\nu\pi}\right\}. \tag{3.16}$$

The required result (3.4) is obtained from relations (3.8) and (3.16), which concludes the proof of the theorem. □

Corollary 1. *If the majorant* Φ *satisfies the condition*

$$\inf\left\{\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|} : 0<\tau\le\frac{t_*}{\nu\pi}\right\} = \varlimsup_{\tau\to0+}\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|}, \tag{3.17}$$

then the following equalities hold:

$$\begin{aligned}\overline{\Pi}_\nu(W^r(\omega_{\mathcal{M}},\Phi);L_2(\mathbb{R})) &= \mathcal{A}_{\nu\pi}(W^r(\omega_{\mathcal{M}},\Phi)) = \sup\{\|f-\mathcal{L}_{\nu\pi}(f)\| : f\in W^r(\omega_{\mathcal{M}},\Phi)\}\\ &= \frac{1}{(\nu\pi)^r}\inf\left\{\frac{\Phi(\tau)}{|w_{\mathcal{M}}(\tau\nu\pi)|} : 0<\tau\le\frac{t_*}{\nu\pi}\right\},\end{aligned} \tag{3.18}$$

where $\nu\in(0,\infty)$, $r\in\mathbb{Z}_+$, *and* $\overline{\Pi}_\nu(\,\cdot\,)$ *is any one of the mean* ν*-widths considered above. Moreover, the pair* $(L_2^r(\mathbb{R}),\mathcal{L}_{\nu\pi})$, *where the operator* $\mathcal{L}_{\nu\pi}$ *is defined by* (3.2) *for* $\sigma=\nu\pi$, *is extremal for the mean linear* ν*-width* $\overline{\delta}_\nu(W^r(\omega_{\mathcal{M}},\Phi);L_2(\mathbb{R}))$ *and the subspace* $\mathbb{B}_{\nu\pi,2}$ *is extremal for the mean Kolmogorov* ν*-width* $\overline{d}_\nu(W^r(\omega_{\mathcal{M}},\Phi);L_2(\mathbb{R}))$.

4. APPLICATIONS OF THE OBTAINED RESULTS

Further, we consider, for example, the following majorants:

$$\Phi_{1,m}(t) := t^m, \qquad m\in\mathbb{N}.$$

4.1. Consider the number set $\mathcal{M}=\mathcal{M}_{1,m}$, $m\in\mathbb{N}$. Since, in this case,

$$|w_{\mathcal{M}_{1,m}}(x)| = 2^m\sin^m\left(\frac{x}{2}\right), \qquad |x|\le\pi,$$

and $t_*=\pi$, it follows that condition (3.17) holds; namely,

$$\inf\left\{\left(\frac{\tau}{\sin(\tau\nu\pi/2)}\right)^m : 0<\tau\le\frac{1}{\nu}\right\} = \varlimsup_{\tau\to0+}\left(\frac{\tau}{2\sin(\tau\nu\pi/2)}\right)^m = \frac{1}{(\nu\pi)^m}.$$

Taking into account the equality $\omega_{\mathcal{M}_{1,m}}=\omega_m$ and using (3.18), we obtain

$$\begin{aligned}\overline{\Pi}_\nu(W^r(\omega_m,\Phi_{1,m});L_2(\mathbb{R})) &= \mathscr{A}_{\nu\pi}(W^r(\omega_m,\Phi_{1,m}))\\ &= \sup\{\|f-\mathcal{L}_{\nu\pi}(f)\| : f\in W^r(\omega_m,\Phi_{1,m})\} = \frac{1}{(\nu\pi)^{r+m}}.\end{aligned}$$

4.2. Consider the number set

$$\mathcal{M}_2 := \left\{\begin{matrix}\mu_j=\dfrac{4}{(\pi j)^2} & \text{if } j=2\nu+1,\ \nu\in\mathbb{Z},\\ \mu_j=0 & \text{if } j=2\nu,\ \nu\in\mathbb{Z}\setminus\{0\},\\ \mu_j=-1 & \text{if } j=0\end{matrix}\right\}_{j\in\mathbb{Z}}.$$

By formula (1.2), to this set there corresponds the generalized difference operator

$$\widehat{\Delta}_h := \Delta_h^{\mathcal{M}_2} = \widehat{T}_h - \mathbb{I},$$

where $\mathbb{I}$ is the identity operator on the space $L_2(\mathbb{R})$ and

$$\widehat{T}_h(f,x) := \frac{4}{\pi^2}\sum_{\nu\in\mathbb{Z}}\frac{f(x+(2\nu+1)h)}{(2\nu+1)^2}$$

almost everywhere on $\mathbb{R}$. Using formula (1.3), we obtain the following smoothness characteristic in $L_2(\mathbb{R})$:

$$\widehat{\omega}(f,t) := \omega_{\mathcal{M}_2}(f,t) = \sup\{\|\widehat{\Delta}_h(f)\| : |h| \le t\}, \qquad t \ge 0,$$

which, in the case of 2π-periodic functions from the spaces $L_p([0,2\pi])$, $1 \le p \le \infty$, was first studied in [8].

Using (1.6) and [23, Sec. 5.4.6.5], for $0 \le x \le \pi$, we have

$$w_{\mathcal{M}_2}(x) = -1 + \frac{8}{\pi^2}\sum_{\nu\in\mathbb{Z}_+}\frac{\cos((2\nu+1)x)}{(2\nu+1)^2} = -\frac{2x}{\pi}.$$

Therefore, for a 2π-periodic even function $|w_{\mathcal{M}_2}|$, we can write

$$|w_{\mathcal{M}_2}(x)| = \frac{2|x|}{\pi}, \qquad \text{where} \quad |x| \le \pi,$$

and $t_* = \pi$. For the majorant $\Phi_{1,1}$, we obtain

$$\inf\biggl\{\frac{\Phi_{1,1}(\tau)}{|w_{\mathcal{M}_2}(\tau\nu\pi)|} : 0 < \tau \le \frac{1}{\nu}\biggr\} = \varlimsup_{\tau\to 0+}\frac{\Phi_{1,1}(\tau)}{|w_{\mathcal{M}_2}(\tau\nu\pi)|} = \frac{1}{2\nu},$$

and hence, using relation (3.18), we can write

$$\overline{\Pi}_\nu(W^r(\widehat{\omega},\Phi_{1,1}); L_2(\mathbb{R})) = \mathcal{A}_{\nu\pi}(W^r(\widehat{\omega},\Phi_{1,1})) = \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in W^r(\widehat{\omega},\Phi_{1,1})\} = \frac{1}{2\pi^r\nu^{r+1}}.$$

4.3. Consider the following number set:

$$\mathcal{M}_3 := \left\{\begin{matrix} \mu_j = \dfrac{3}{(\pi j)^2} & \text{if } j \ne 0, \\ \mu_j = -1 & \text{if } j = 0 \end{matrix}\right\}_{j\in\mathbb{Z}}.$$

In view of (1.2), to this set the generalized difference operator $\overline{\Delta}_h := \overline{T}_h - \mathbb{I}$ corresponds, where, for $f \in L_2(\mathbb{R})$, we have

$$\overline{T}_h(f,x) := (3/\pi^2)\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{f(x+jh)}{j^2}$$

almost everywhere on $\mathbb{R}$.

Using (1.3), we obtain the following smoothness characteristic in the space $L_2(\mathbb{R})$:

$$\omega_{\langle\prime\rangle}(f,t) := \omega_{\mathcal{M}_3}(f,t) = \sup\{\|\overline{\Delta}_h(f)\| : |h| \le t\}, \qquad t \ge 0,$$

which was considered earlier in [14].

Using formula (1.6) and [23, Sec. 5.4.2.7], for $0 \le x \le \pi$, we can write

$$w_{\mathcal{M}_3}(x) = -1 + \frac{6}{\pi^2}\sum_{j\in\mathbb{N}}\frac{\cos(jx)}{j^2} = \frac{3x}{\pi}\biggl(\frac{x}{2\pi} - 1\biggr).$$

Hence, for the even 2π-periodic function $|w_{\mathcal{M}_3}|$, we have

$$|w_{\mathcal{M}_3}(x)| = \frac{3|x|(1-|x|/(2\pi))}{\pi}, \qquad |x| \le \pi,$$

and $t_* = \pi$.

Taking into account the equality

$$\inf\left\{\frac{\Phi_{1,1}(\tau)}{|w_{\mathcal{M}_3}(\tau\nu\pi)|} : 0 < \tau \le \frac{1}{\nu}\right\} = \overline{\lim_{\tau\to 0+}} \frac{\Phi_{1,1}(\tau)}{|w_{\mathcal{M}_3}(\tau\nu\pi)|} = \frac{1}{3\nu},$$

from (3.18) we obtain

$$\begin{aligned}\overline{\Pi}_\nu(W^r(\omega_{\langle\prime\rangle}, \Phi_{1,1}); L_2(\mathbb{R})) &= \mathcal{A}_{\nu\pi}(W^r(\omega_{\langle\prime\rangle}, \Phi_{1,1})) \\ &= \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in W^r(\omega_{\langle\prime\rangle}, \Phi_{1,1})\} = \frac{1}{3\pi^r\nu^{r+1}}.\end{aligned}$$

4.4. In the papers [24] of Kozko and Rozhdestvenskii and [7] of Gorbachev, the Thue–Morse generalized difference and the modulus of continuity were considered. By [7], for the Thue–Morse difference operator, we write the function

$$\widetilde{\mathcal{M}}(z) := \sum_{j\in\mathbb{Z}} \mu_j z^j = (-1)^m \prod_{j=0}^{m-1}(1 - z^{k^j}), \qquad m, k \in \mathbb{N}. \tag{4.1}$$

Setting, for example, $k = 1$ in (4.1), we obtain

$$\widetilde{\mathcal{M}}(z) = (z-1)^m = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} z^j,$$

and the number set $\mathcal{M}_{1,m}$ corresponds to the function $\widetilde{\mathcal{M}}$.

Let the symbol $\mathcal{M}_{4,m}(k)$ denote the number set $\{\mu_j\}_{j\in\mathbb{Z}}$ corresponding to $\widetilde{\mathcal{M}}$ in the general case (4.1). Obviously, $\mathcal{M}_{1,m} = \mathcal{M}_{4,m}(1)$. Using formula (1.2), for the generalized Thue–Morse difference $\widetilde{\Delta}_h^{k,m}(f) := \Delta_h^{\mathcal{M}_{4,m}(k)}(f)$, we can write [24]

$$\begin{aligned}\widetilde{\Delta}_h^{k,1}(f,x) &= \Delta_h^1(f,x) := f(x) - f(x+h), \\ \widetilde{\Delta}_h^{k,2}(f,x) &:= \widetilde{\Delta}_h^{k,1}(f,x) - \widetilde{\Delta}_h^{k,1}(f,x+kh) \\ &= f(x) - f(x+h) - f(x+kh) + f(x+(k+1)h), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \widetilde{\Delta}_h^{k,m}(f,x) &:= \widetilde{\Delta}_h^{k,m-1}(f,x) - \widetilde{\Delta}_h^{k,m-1}(f,x+k^{m-1}h),\end{aligned} \tag{4.2}$$

or

$$\widetilde{\Delta}_h^{k,m} = \Delta_h^1 \circ \Delta_{kh}^1 \circ \cdots \circ \Delta_{k^{m-1}h}^1.$$

We define the Thue–Morse modulus of continuity in the space $L_2(\mathbb{R})$ by formula (1.3), i.e.,

$$\widetilde{\omega}_{k,m}(f,t) := \omega_{\mathcal{M}_{4,m}(k)}(f,t), \qquad t \ge 0. \tag{4.3}$$

Let us consider the particular case $m = 2$ that corresponds to relation (4.2). We have

$$\mathcal{M}_{4,2}(k) = \left\{\begin{matrix}\mu_j = 1 & \text{if } j = 0 \text{ or } j = k+1, \\ \mu_j = -1 & \text{if } j = 1 \text{ or } j = k, \\ \mu_j = 0 & \text{if } j \ne 0, 1, k, k+1\end{matrix}\right\}_{j\in\mathbb{Z}}.$$

Then, using (1.6), we obtain

$$w_{\mathcal{M}_{4,2}(k)}(x) = (1 - e^{ix})(1 - e^{ikx}). \tag{4.4}$$

Further, setting $k = 2$ in (4.4), we can write

$$|w_{\mathcal{M}_{4,2}(2)}(x)| = 4\sin\left(\frac{x}{2}\right)\sin(x), \qquad |x| \le \pi. \tag{4.5}$$

Since the function (4.5) attains its maximum for x satisfying the equation $\cos^2(x/2) = 1/3$, it follows from the definition of the quantity t_* given above that $t_* = 2\arccos(1/\sqrt{3}\,)$. Here, the even 2π-periodic function (4.5) also satisfies *property* A. Since

$$\inf\left\{\frac{\Phi_{1,2}(\tau)}{|w_{\mathcal{M}_{4,2}(2)}(\tau\nu\pi)|} : 0 < \tau \le \frac{2}{\nu\pi}\arccos\left(\frac{1}{\sqrt{3}}\right)\right\} = \overline{\lim_{\tau\to 0+}}\,\frac{\Phi_{1,2}(\tau)}{|w_{\mathcal{M}_{4,2}(2)}(\tau\nu\pi)|} = \frac{1}{2(\nu\pi)^2}\,,$$

using Corollaries 1 and (4.3), we obtain

$$\begin{aligned}\overline{\Pi}_\nu(W^r(\widetilde{\omega}_{2,2};\Phi_{1,2});L_2(\mathbb{R})) &= \mathcal{A}_{\nu\pi}(W^r(\widetilde{\omega}_{2,2};\Phi_{1,2}))\\ &= \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in W^r(\widetilde{\omega}_{2,2};\Phi_{1,2})\} = \frac{1}{2(\pi\nu)^{r+2}}\,.\end{aligned}$$

In conclusion, note that, in Sec. 4.1, the majorants satisfying condition (3.17) are not limited solely to the function $\Phi_{1,m}$, and in Secs. 4.2, 4.3, and 4.4, they are not limited to the functions $\Phi_{1,1}$ and $\Phi_{1,2}$, respectively. We can show that the majorant

$$\widetilde{\Phi}_{2,m}(t) := t^m\psi(t), \qquad m \in \mathbb{N},$$

where ψ is an arbitrary continuous positive nondecreasing function such that $\lim\{\psi(t)\colon t \to 0+\} \ne 0$, satisfies condition (3.17) in Sec. 4.1, and the majorants $\widetilde{\Phi}_{2,1}$ and $\widetilde{\Phi}_{2,2}$ satisfy the condition specified in Secs. 4.2, 4.3, and 4.4, respectively.

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