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On a Theorem of Kadets and Pełczyński

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Abstract—Necessary and sufficient conditions are found under which a symmetric space X on $[0,1]$ of type 2 has the following property, which was first proved for the spaces L_p, $p>2$, by Kadets and Pełczyński: if $\{u_n\}_{n=1}^\infty$ is an unconditional basic sequence in X such that

$$\|u_n\|_X \asymp \|u_n\|_{L_1}, \qquad n\in\mathbb{N},$$

then the norms of the spaces X and L_1 are equivalent on the closed linear span $[u_n]$ in X. For sequences of martingale differences, this implication holds in *any* symmetric space of type 2.

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1. INTRODUCTION

The following result, which is usually called the *Kadets–Pełczyński alternative*, was proved in the classical paper [1] (see Theorem 3). If Y is an infinite-dimensional subspace of L_p, $p>2$, then either Y is isomorphic to ℓ_2 or Y contains a complemented subspace of L_p isomorphic to ℓ_p. The proof of this important fact in [1] uses the following assertion, which we state in a more general situation (see [2, Theorem 4.1], [3, Proposition 1 and its proof], or [4, Lemma 5.2.1]). In [1], Proposition 1 was proved in the case $X=L_p$, $1<p<\infty$.

Proposition 1. *If Y is a closed subspace of a separable symmetric space X on* $[0,1]$, *then one of the following two assertions holds*:

a) *the norms of the spaces X and L_1 are equivalent on Y*;

b) *Y contains a normalized almost disjoint sequence in X, i.e., there is a sequence $\{y_n\}_{n=1}^\infty\subset Y$, $\|y_n\|_X=1$, such that*

$$\|y_n-x_n\|_X\to 0 \qquad \text{as} \quad n\to\infty$$

for some disjoint sequence $\{x_n\}_{n=1}^\infty\subset X$.

Moreover, every sequence $\{z_n\}_{n=1}^\infty\subset X$, $\|z_n\|_X=1$, satisfies one of the following two conditions:

1) $\|z_n\|_{L_1}\ge c$ *for some $c>0$ and all $n\in\mathbb{N}$*;

2) *there is an almost disjoint sequence $\{z_{n_k}\}\subset\{z_n\}$ in X.*

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Another interesting assertion contained in the same Theorem 3 of [1] (see assertion 3g) is less known: if $\{u_n\}_{n=1}^{\infty}$ is an unconditional basic sequence in L_p, $p > 2$, such that

$$\|u_n\|_{L_p} \asymp \|u_n\|_{L_1}, \qquad n = 1, 2, \dots,$$

then the norms of the spaces L_p and L_1 are equivalent on the entire closed linear span $[u_n]$ in L_p.

The main objective of the present paper is to find out to what extent one can extend the last implication to general symmetric spaces. First of all, we obtain necessary and sufficient conditions under which this implication is valid for symmetric spaces of Rademacher type 2. We shall also show that this implication holds for *all* spaces of type 2 and for sequences of martingale differences. Using these results, we obtain some new properties of weakly convergent sequences of normalized functions in symmetric spaces. In conclusion, we specify a class of symmetric spaces in which any sequence of equimeasurable functions has a subsequence equivalent to the canonical basis of the space ℓ_2.

2. PRELIMINARIES

In what follows, we use some notions and results of the theory of symmetric spaces (for details, see the monographs [3] and [5]–[7]).

A Banach space X of functions measurable on $[0, 1]$ is said to be *symmetric*, or *rearrangement-invariant*, if the following conditions hold:

(1) X is a function lattice, i.e., if $g \in X$ and $|f(t)| \le |g(t)|$ for all $t \in [0, 1]$, then $f \in X$ and $\|f\|_X \le \|g\|_X$;

(2) if functions f and g are *equimeasurable*, i.e.,

$$m(\{t \in [0, 1] : |g(t)| > u\}) = m(\{t \in [0, 1] : |f(t)| > u\}), \qquad u > 0,$$

where $m(A)$ stands for the Lebesgue measure of a set $A \subset \mathbb{R}$, and $g \in X$, then $f \in X$ and $\|f\|_X = \|g\|_X$.

In what follows, without loss of generality, we assume that $\|\chi_{[0,1]}\|_X = 1$ (throughout the paper, by χ_F we denote the characteristic function of a set F). Then [5, Theorem 2.4.1] the following continuous embeddings hold for every symmetric space X on $[0, 1]$:

$$L_\infty \subset X \subset L_1, \tag{2.1}$$

where $\|f\|_{L_1} \le \|f\|_X$, $f \in X$, and $\|f\|_X \le \|f\|_{L_\infty}$, $f \in L_\infty$.

Below we give examples of symmetric spaces. Every convex increasing function $M(u)$, $M(0) = 0$, on $[0, \infty)$ generates the *Orlicz space* L_M, which is a natural generalization of L_p-spaces, with the norm

$$\|f\|_{L_M} = \inf\left\{\lambda > 0 : \int_0^1 M\left(\frac{|f(t)|)}{\lambda}\right) dt \le 1\right\}.$$

As shown in [8], the system of Rademacher functions

$$r_k(t) = \operatorname{sign}\sin(2^k \pi t), \qquad 0 \le t \le 1, \quad k = 1, 2, \dots,$$

is equivalent in a symmetric space X to the canonical basis in ℓ_2 if and only if $X \supset G$, where G is the closure of L_∞ in the Orlicz space L_N for $N(u) = e^{u^2} - 1$.

Another generalization of the L_p spaces is the family of the spaces $L_{p,q}$, which consist of all measurable functions f on $[0, 1]$ with the following finite norm:

$$\|f\|_{L_{p,q}} := \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^1 (f^*(t)t^{1/p})^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty, \\ \displaystyle\sup_{0<t\le 1} f^*(t)t^{1/p} < \infty, & q = \infty, \end{cases} \qquad 1 < p < \infty.$$

It follows from the definition that $L_{p,p} = L_p$. If $q > p$, then $\|\cdot\|_{L_{p,q}}$ is a quasi-norm equivalent to the norm

$$\left\| \frac{1}{t} \int_0^t f^*(s)\,ds \right\|_{L_{p,q}}.$$

If $1 \le q < \infty$ and $\varphi(t)$ is a concave increasing function on $[0,1]$, $\varphi(0) = 0$, then by $\Lambda_q(\varphi)$ we denote the *Lorentz space* with the norm

$$\|f\|_{\Lambda_q(\varphi)} := \left(\int_0^1 (f^*(t))^q \, d\varphi(t) \right)^{1/q},$$

where $f^*(t)$ is the decreasing rearrangement of $|f(t)|$ [5, Sec. 2.2].

On every symmetric space X, the following *dilation operators* act boundedly:

$$\sigma_\tau x(t) = \begin{cases} x\left(\dfrac{t}{\tau}\right), & 0 \le t \le \min(1,\tau), \\ 0, & \min(1,\tau) < t \le 1, \end{cases}$$

where $\tau > 0$ [5, Theorem 2.4.5]. The numbers

$$\alpha_X = \lim_{\tau \to 0} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}, \qquad \beta_X = \lim_{\tau \to \infty} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}$$

are called the *Boyd indices* of X. We always have $0 \le \alpha_X \le \beta_X \le 1$. For example, $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $p \in [1, \infty]$.

Let X and Y be two symmetric spaces such that $X \subset Y$. The inclusion $X \subset Y$ is said to be *strictly singular* (*disjointly strictly singular*) if, for every sequence of functions (respectively, of disjoint functions) $\{x_k\} \subset X$, the norms of X and Y are not equivalent on the closed linear span $[x_k]$. Obviously, every strictly singular inclusion is disjointly strictly singular. The converse is false: for example, the inclusion $L_p \subset L_q$, $1 \le q < p \le \infty$, is disjointly strictly singular and not strictly singular. Indeed, on the one hand, every sequence of normalized disjoint functions $\{x_n\} \subset L_r[0,1]$ is equivalent in L_r to the canonical basis of the space ℓ_r, $1 \le r \le \infty$ [4, Proposition 6.4.1]. On the other hand, as mentioned above, for every $r < \infty$, the system of Rademacher functions is equivalent in L_r to the canonical basis of the space ℓ_2.

We say that a Banach space E is of *Rademacher type* (*cotype*) p if there is a constant $K > 0$ such that the following inequality holds for every finite family $\{x_i\}_{i=1}^n \subset E$:

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|_E dt \le K \left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p};$$

E is of *Rademacher cotype* p if there is a constant $K > 0$ such that, for every finite $\{x_i\}_{i=1}^n \subset E$,

$$\left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p} \le K \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|_E dt.$$

If a Banach space E has type (cotype) p, then $1 \le p \le 2$ (respectively, $2 \le p \le \infty$); any Banach space has type 1 and cotype ∞. The notions of type and cotype are closely related to the notions of p-convexity and p-concavity.

Let $1 \le p \le \infty$. A Banach function lattice X is said to be *p-convex* if there is a constant $C > 0$ such that, for every finite family $\{x_i\}_{i=1}^n \subset X$, we have

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|_X \le C \left(\sum_{i=1}^n \|x_i\|_X^p \right)^{1/p},$$

and X is said to be *p-concave* if there is a constant $C > 0$ such that, for every finite $\{x_i\}_{i=1}^n \subset X$,

$$\left(\sum_{i=1}^n \|x_i\|_X^p\right)^{1/p} \le C\left\|\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\right\|_X.$$

For $p = \infty$, the expressions in these inequalities are modified in a natural way.

For an arbitrary symmetric space X on $[0,1]$ and any $\delta > 0$, we set

$$U_{X,\delta}(f) := \{t \in [0,1] : |f(t)| > \delta\|f\|_X\}, \qquad f \in X,$$
$$M_{X,\delta} := \{f \in X : m(U_{X,\delta}(f)) \ge \delta\}.$$

In what follows, we repeatedly use the simple fact that condition (a) of Proposition 1 is equivalent to the existence of a $\delta > 0$ such that $Y \subset M_{X,\delta}$ [1, Theorem 1].

Finally, $f \asymp g$ means that $cf \le g \le Cf$ for some constants $c > 0$ and $C > 0$, and these constants do not depend on all or some arguments of the functions (quasi-norms) f and g.

3. RESULTS

Proposition 2. *Let X be a symmetric space on $[0,1]$, and let $\{u_n\}_{n=1}^\infty \subset X$ be an unconditional basic sequence in X, $\|u_n\|_{L_1} \asymp \|u_n\|_X = 1$, $n \in \mathbb{N}$. Then the following assertions hold*:

a) *for some $c > 0$ and all $(a_k) \in \ell_2$,*

$$\left\|\sum_{k=1}^\infty a_k u_k\right\|_X \ge c\|(a_k)\|_{\ell_2};$$

b) *if, in addition, the space X has type 2, then $\{u_n\}$ is equivalent in X to the canonical basis of ℓ_2.*

Proof. (a) By virtue of Fubini's theorem, Khinchine's L_1-inequality [9], and Minkowski's inequality, we have

$$\int_0^1 \left\|\sum_{k=1}^\infty r_k(t)a_k u_k\right\|_{L_1} dt = \int_0^1\int_0^1 \left|\sum_{k=1}^\infty r_k(t)a_k u_k(s)\right| dt\,ds \ge \frac{1}{\sqrt{2}}\int_0^1 \left(\sum_{k=1}^\infty a_k^2 u_k(s)^2\right)^{1/2} ds$$
$$\ge \frac{1}{\sqrt{2}}\left(\sum_{k=1}^\infty a_k^2\left(\int_0^1 |u_k(s)|\,ds\right)^2\right)^{1/2} ds \ge \frac{c'}{\sqrt{2}}\left(\sum_{k=1}^\infty a_k^2\right)^{1/2}.$$

Since the sequence $\{u_n\}_{n=1}^\infty$ is unconditional in X, these relations, together with the embeddings (2.1), yield

$$\left\|\sum_{k=1}^\infty a_k u_k\right\|_X \asymp \int_0^1 \left\|\sum_{k=1}^\infty r_k(t)a_k u_k\right\|_X dt \ge \int_0^1 \left\|\sum_{k=1}^\infty r_k(t)a_k u_k\right\|_{L_1} dt \ge \frac{c'}{\sqrt{2}}\left(\sum_{k=1}^\infty a_k^2\right)^{1/2}.$$

(b) It suffices to prove the upper ℓ_2-bound. Since the sequence $\{u_k\}_{k=1}^\infty$ is unconditional in the space X of type 2, it follows that, for arbitrary $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$, we have

$$\left\|\sum_{k=1}^n a_k u_k\right\|_X \asymp \int_0^1 \left\|\sum_{k=1}^n r_k(t)a_k u_k\right\|_X dt \le K\left(\sum_{k=1}^n \|a_k u_k\|_X^2\right)^{1/2}$$
$$\le K \sup_{k=1,2,\dots} \|u_k\|_X \|(a_k)\|_{\ell_2} = K\|(a_k)\|_{\ell_2}.$$

This inequality extends in a standard way to infinite sums, which completes the proof of the proposition. □

Corollary 1. *If a symmetric space X has type 2, $X \neq L_2$, and $\{u_n\}_{n=1}^\infty$ is an unconditional basis in X, then*

$$\|u_n\|_{L_1} \not\asymp \|u_n\|_X, \qquad n \in \mathbb{N}.$$

Proof. If we assume that $\{u_n\}_{n=1}^\infty$ is an unconditional basis in X and $\|u_n\|_{L_1} \asymp \|u_n\|_X$, $n \in \mathbb{N}$, then, by Proposition 2, this basis is equivalent in X to the canonical basis of ℓ_2. Hence X is isomorphic to a separable Hilbert space. Let us show that, in this case, $X = L_2$ (with the equivalence of the norms). Since this contradicts the assumption, we see that the corollary will be proved. First of all, since the parallelogram identity holds in every Hilbert space, it follows that X has also cotype 2. Hence, for some constant $K > 0$ independent of $n \in \mathbb{N}$ and of the functions $f_1, \dots, f_n \in X$, we have

$$K^{-1}\biggl(\sum_{i=1}^n \|f_i\|_X^2\biggr)^{1/2} \le \int_0^1 \biggl\|\sum_{i=1}^n r_i(t) f_i\biggr\|_X \, dt \le K\biggl(\sum_{i=1}^n \|f_i\|_X^2\biggr)^{1/2}. \tag{3.1}$$

Let $f \in X$, $\|f\|_X = 1$. Then it follows from the definition of the operator $\sigma_{1/n}$ that $f = \sum_{i=1}^n f_i$, where the functions f_i are disjoint and, for every $i = 1, 2, \dots, n$, the function f_i is equimeasurable with the function $\sigma_{1/n} f$. Therefore, substituting these functions into (3.1), we obtain

$$K^{-1} n^{-1/2} \le \|\sigma_{1/n} f\|_X \le K n^{-1/2}, \qquad n \in \mathbb{N}.$$

In particular, if $f = \chi_{[0,1]}$, then this implies (since $\|\chi_{[0,1]}\|_X = 1$)

$$K^{-1} n^{-1/2} \le \|\chi_{[(i-1)/n, i/n]}\|_X \le K n^{-1/2}$$

for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, n$. In conclusion, applying (3.1) once again and using the last inequalities, we see that, for constants independent of $n \in \mathbb{N}$ and $c_i \in \mathbb{R}$, we have

$$\begin{aligned} \biggl\|\sum_{i=1}^n c_i \chi_{[(i-1)/n, i/n]}\biggr\|_X &= \int_0^1 \biggl\|\sum_{i=1}^n r_i(t) c_i \chi_{[(i-1)/n, i/n]}\biggr\|_X \, dt \\ &\asymp \biggl(\sum_{i=1}^n |c_i|^2 \|\chi_{[(i-1)/n, i/n]}\|_X^2\biggr)^{1/2} \asymp n^{-1/2}\biggl(\sum_{i=1}^n |c_i|^2\biggr)^{1/2}. \end{aligned}$$

Since the space X is separable, it follows that the norms of X and L_2 are equivalent on a dense set. As a result, as claimed, $X = L_2$. □

Theorem 1. *Suppose that X is a symmetric space on $[0, 1]$, $X \supset G$. Then the following conditions are equivalent*:

i) *if $\{u_n\}_{n=1}^\infty \subset X$ is an arbitrary basic sequence in X equivalent to the canonical basis of ℓ_2 and $\|u_n\|_{L_1} \asymp \|u_n\|_X = 1$, $n \in \mathbb{N}$, then $[u_n] \subset M_{X,\eta}$ for some $\eta > 0$;*

ii) *there are no sequences of disjoint functions in X equivalent in X to the canonical basis of ℓ_2.*

Proof. (ii) ⇒ (i). Recall (see Sec. 2 and [1, Theorem 1]) that $Y \subset M_{X,\eta}$ for some $\eta > 0$ (Y is a subset of the symmetric space X) if and only if the norms of the spaces X and L_1 are equivalent on Y. Therefore, if (i) fails to hold, then, by Proposition 1, there is a sequence $\{u_n\}_{n=1}^\infty \subset X$ equivalent to the canonical basis of ℓ_2 such that $\|u_n\|_{L_1} \asymp \|u_n\|_X = 1$, $n \in \mathbb{N}$, and the closed linear span $[u_n]$ contains a sequence $\{v_k\}$, $\|v_k\|_X = 1$, equivalent in X to some disjoint sequence $\{x_k\}$. By the principle of small perturbations [4, Theorem 1.3.9], we can assume (passing to a subsequence if necessary) that the sequence $\{v_k\}$ (together with $\{x_k\}$) is unconditional. Therefore, since the subspace $[u_n]$ is isomorphic to ℓ_2 by assumption, it follows that $\{v_k\}$ is equivalent to the canonical basis of ℓ_2 (see, e.g., [10, Proposition 1.1]). Finally, the disjoint sequence $\{x_k\}$ is also equivalent to the ℓ_2-basis, which contradicts condition (ii).

(i) $\Rightarrow$ (ii). Since every normalized disjoint sequence in the space L_1 is equivalent to the canonical basis of ℓ_1 (see, e.g., [4, Proposition 6.4.1]), it follows that, without loss of generality, we may assume that $X \neq L_1$.

Suppose that condition (ii) fails to hold. Then there is a disjoint sequence $\{f_k\}_{k=1}^{\infty} \subset X$, $\|f_k\|_X = 1$, equivalent in X to the canonical basis of ℓ_2. Since X is symmetric, we may assume that $\{t : f_k(t) \neq 0\} \subset [2^{-k-1}, 2^{-k}]$, $k = 1, 2, \dots$. Consider the sequence $\{u_k\}_{k=1}^{\infty}$ defined by

$$u_{2i-1} := f_{2i-1} + r_i \chi_{[1/2,1]} \quad \text{and} \quad u_{2i} := f_{2i} + r_i \chi_{[1/2,1]}, \qquad i = 1, 2, \dots .$$

To prove the theorem, it suffices to show that the sequence $\{u_n\}$ satisfies all conditions in (i) (i.e., $\{u_n\}$ is equivalent in X to the canonical basis of ℓ_2 and $\|u_n\|_{L_1} \asymp \|u_n\|_X = 1$, $n \in \mathbb{N}$) but some its block-basis consists of disjoint functions. Indeed, if this is the case and assertion (i) holds, then the norms of the spaces X and L_1 turn out to be equivalent on some infinite-dimensional subspace generated by a sequence of disjoint functions. This means (see Sec. 2) that the inclusion $X \subset L_1$ is not disjointly strictly singular. However, since $X \neq L_1$, this is impossible (see [11] or [12, Corollary 3]). For completeness, we give a simple proof of this fact. If we assume that, for some sequence $\{g_k\}_{k=1}^{\infty} \subset X$ of pairwise disjoint functions, the norms of X and L_1 are equivalent on the closed linear span $[g_k]$ in X, then $[g_k] \subset M_{X,\eta}$ for some $\eta > 0$. Note that the sets $U_{X,\eta}(g_k)$ are pairwise disjoint and

$$m(U_{X,\eta}(g_k)) \geq \eta, \qquad k = 1, 2, \dots$$

(see the definition of these sets in Sec. 2). Therefore,

$$m\biggl(\bigcup_{k=1}^{\infty} U_{X,\eta}(g_k)\biggr) = \sum_{k=1}^{\infty} m(U_{X,\eta}(g_k)) = \infty,$$

which is impossible by virtue of the inclusion $\bigcup_{k=1}^{\infty} U_{X,\eta}(g_k) \subset [0, 1]$.

Thus, let us prove that the sequence $\{u_k\}$ satisfies the conditions in (i). First of all, since $\|\chi_{[0,1]}\|_X = 1$ (see Sec. 2), it follows that

$$\|u_{2i}\|_X \leq \|f_{2i}\|_X + \|r_i \chi_{[1/2,1]}\|_X \leq 2 = 4\|r_i \chi_{[1/2,1]}\|_{L_1} \leq 4\|u_{2i}\|_{L_1}.$$

In the same way, we obtain

$$\|u_{2i-1}\|_X \leq 4\|u_{2i-1}\|_{L_1}, \qquad i = 1, 2, \dots .$$

Further, since $X \supset G$, it follows from Khinchine's inequality (see, e.g., [13, Theorem 5.8.7] or [7, Remark 2.1]) and the equivalence of the sequence $\{f_k\}$ in X to the canonical basis of ℓ_2 that, for arbitrary $a_k \in \mathbb{R}$, we have

$$\biggl\|\sum_{k=1}^{\infty} a_k u_k\biggr\|_X \leq \biggl\|\sum_{k=1}^{\infty} a_k f_k\biggr\|_X + \biggl\|\sum_{i=1}^{\infty} (a_{2i-1} + a_{2i}) r_i\biggr\|_X \leq C\|(a_k)\|_{\ell_2}.$$

On the other hand,

$$\biggl\|\sum_{k=1}^{\infty} a_k u_k\biggr\|_X \geq \biggl\|\biggl(\sum_{k=1}^{\infty} a_k u_k\biggr)\chi_{[0,1/2]}\biggr\|_X = \biggl\|\sum_{k=1}^{\infty} a_k f_k\biggr\|_X \asymp \|(a_k)\|_{\ell_2}.$$

Therefore, the sequence $\{u_k\}$ is equivalent to the canonical basis of ℓ_2. At the same time, the functions

$$u_{2i} - u_{2i-1} = f_{2i} - f_{2i-1}, \qquad i = 1, 2, \dots ,$$

form a disjoint sequence. Thus, the theorem is proved. □

Remark 1. If X is a symmetric space and $X \not\supset G$, then X contains no sequence $\{u_n\}_{n=1}^{\infty} \subset X$ equivalent to the canonical basis ℓ_2 and such that

$$\|u_n\|_{L_1} \asymp \|u_n\|_X = 1, \qquad n \in \mathbb{N}.$$

Indeed, suppose that such a sequence $\{u_n\}$ exists. Then, since it is equivalent to the canonical basis of ℓ_2, it follows that $u_n \to 0$ weakly in X and so also in L_1. Therefore (since $\|u_n\|_{L_1} \asymp 1$, $n \in \mathbb{N}$),

$\{u_n\}$ contains no subsequences convergent in L_1. Thus, applying the well-known Aldous–Fremlin theorem [14], one can extract a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that

$$\Big\|\sum_{k=1}^{\infty} a_k u_{n_k}\Big\|_{L_1} \ge c\|(a_k)\|_{\ell_2}$$

for some constant $c > 0$ and all $a_k \in \mathbb{R}$. This, together with (2.1) and properties of the sequence $\{u_n\}$, implies that $\{u_{n_k}\}$ is equivalent to the canonical basis of ℓ_2 both in X and in L_1. Thus, the inclusion $X \subset L_1$ is not strictly singular. Since $X \not\supset G$, it follows from a well-known characterization of strictly singular inclusions of symmetric spaces (see Theorem 2 of [15]) that this inclusion is not even disjointly strictly singular. The last property fails to hold (see the proof of the implication (i) $\Longrightarrow$ (ii) of Theorem 1) and, therefore, our assertion is proved.

Thus, in the case where $X \not\supset G$, condition (i) always formally holds. Therefore, if such a space contains a sequence of disjoint functions that is equivalent in X to the canonical basis of ℓ_2, then the implication (i) $\Longrightarrow$ (ii) of Theorem 1 fails to hold. For X one can take, for example, the Lorentz space $\Lambda_2(\varphi)$ such that[1] $\Lambda_2(\varphi) \not\supset G$ [2, Theorem 5.1].

Suppose that a symmetric space X has Rademacher type 2. Then X is 2-convex and q-concave for some $q < \infty$ [3, Proposition 1.f.17]. Applying the definitions of these properties to disjoint families of equimeasurable functions, we readily see that the Boyd indices of the space X satisfy the inequalities $0 < \alpha_X \le \beta_X \le 1/2$. In particular, this implies $X \supset G$. Finally, applying Theorem 1 together with Proposition 2, we obtain the following result.

Corollary 2. *If X is a symmetric space of type 2 on $[0,1]$, then the following conditions are equivalent*:

i) *if $\{u_n\}_{n=1}^{\infty} \subset X$ is an arbitrary unconditional basic sequence in X and $\|u_n\|_{L_1} \asymp \|u_n\|_X = 1$, $n \in \mathbb{N}$, then $[u_n] \subset M_{X,\eta}$ for some $\eta > 0$;*

ii) *there is no sequence of disjoint functions in X that is equivalent in X to the canonical basis of ℓ_2.*

As mentioned in Sec. 1, in the case of L_p spaces, $p > 2$, the last result was obtained in the paper [1] (see Theorem 3g). In the same paper (see Corollary 5), the following assertion was proved (also for $X = L_p$, $p > 2$).

Corollary 3. *Let X be a symmetric space of type 2 on $[0,1]$. Then every sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\|x_n\|_X = 1$ and $x_n \to 0$ weakly in X contains a subsequence which is equivalent in X either to a sequence of pairwise disjoint functions or to the canonical basis of ℓ_2. The second possibility is realized in the case where*

$$\liminf_{n\to\infty} \frac{\|x_n\|_X}{\|x_n\|_{L_1}} < \infty. \tag{3.2}$$

Proof. Since the space X has type 2, it is q-concave for some $q < \infty$. Hence it follows from a well-known connection between the q-concavity and the separability of Banach lattices (see, e.g., [3, Theorem 1.f.12(ii)] and [3, Proposition 1.a.7]) that X is separable. Therefore, by Proposition 1, we may assume that (3.2) holds. Moreover, without loss of generality, we also assume that $\|x_n\|_{L_1} \ge c$ for some $c > 0$ and all $n \in \mathbb{N}$. We claim that, in this case, $\{x_n\}$ contains a subsequence equivalent in X to the canonical basis of ℓ_2.

As above, we have $0 < \alpha_X \le \beta_X \le 1/2$. Hence there is an unconditional basis in X (for example, the Haar system; see [3, Theorem 2.c.6]), and according to the Bessaga–Pełczyński selection principle [4, Proposition 1.3.10], since x_n weakly converges to 0, we may assume (passing to a subsequence if needed) that $\{x_n\}$ is an unconditional basis sequence in X. Finally, applying Proposition 2(b), we obtain the desired result. ☐

[1] The condition $\Lambda_2(\varphi) \not\supset G$ is equivalent to the condition $\sup_{0<t\le 1} \varphi(t)\log(e/t) = \infty$.

Let us show that, for sequences of martingale differences, condition (i) of Corollary 2 holds in any symmetric space of type 2 without any assumptions concerning properties of disjoint sequences.

Theorem 2. *Let X be a symmetric space on $[0,1]$ of type 2 such that $X \neq L_2$, and let $\{u_n\}_{n=1}^{\infty} \subset X$ be a sequence of martingale differences such that $\|u_n\|_{L_1} \asymp \|u_n\|_X = 1$, $n \in \mathbb{N}$. Then $[u_n] \subset M_{X,\delta}$ for some $\delta > 0$.*

In particular, if Y is a symmetric space and $Y \supset X$, then the sequence $\{u_n\}$ is equivalent in Y to the canonical basis of ℓ_2.

We present two proofs of this result. The first of them uses the unconditionality of sequences of martingale differences in a symmetric space with nontrivial Boyd indices. The other proof is substantially shorter, because some known results are used.

First proof. First of all, recall (see [16] or [17]) that, for every symmetric space Y with nontrivial Boyd indices (i.e., such that $0 < \alpha_Y \leq \beta_Y < 1$) and an arbitrary sequence $\{u_n\}_{n=1}^{\infty} \subset Y$ of martingale differences, we have

$$\Big\|\sum_{k=1}^{n} a_k u_k\Big\|_Y \asymp \Big\|\Big(\sum_{k=1}^{n} a_k^2 u_k^2\Big)^{1/2}\Big\|_Y \tag{3.3}$$

with constants independent of $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$. Therefore, since the space X has type 2 (and hence $0 < \alpha_X \leq \beta_X \leq 1/2$), it follows that the previous equivalence holds, in particular, for X. Thus, the sequence $\{u_k\}_{k=1}^{\infty}$ is unconditional in X, and by Proposition 2 (b) it is equivalent in X to the canonical basis of ℓ_2.

Further, since X has type 2, it follows that X is 2-convex and hence the norm on X can be represented in the form $\|x\|_X = \|x^2\|_Y^{1/2}$, where Y is a symmetric space on $[0,1]$ [3, Sec. 1.d, p. 53]. Therefore, by (2.1),

$$\|x\|_X \geq \|x^2\|_{L_1}^{1/2} = \|x\|_{L_2},$$

i.e., $X \subset L_2$. Thus, $\{u_k\} \subset L_2$ and $\|u_k\|_{L_1} \asymp \|u_k\|_{L_2}$, $k \in \mathbb{N}$. Moreover, the sequence $\{u_k\}$ is equivalent to the canonical basis of ℓ_2 also in L_2 (this follows, e.g., from (3.3) in the case of $Y = L_2$). Thus, the norms of the spaces X and L_2 are equivalent on the closed linear span $[u_k]$ in X. Therefore, in particular, the inclusion $X \subset L_2$ is not strictly singular. Let us show that, at the same time, it is disjointly strictly singular.

Assuming that this is not the case, we find pairwise disjoint functions v_k, $k = 1, 2, \dots$, such that

$$\Big\|\sum_{k=1}^{\infty} a_k v_k\Big\|_X \asymp \Big\|\sum_{k=1}^{\infty} a_k v_k\Big\|_{L_2}, \qquad a_k \in \mathbb{R}.$$

Since $\|y\|_Y = \||y|^{1/2}\|_X^2$, we see from the previous relation and the disjointness of v_k, $k = 1, 2, \dots$, that

$$\Big\|\sum_{k=1}^{\infty} a_k^2 v_k^2\Big\|_Y \asymp \Big\|\sum_{k=1}^{\infty} a_k^2 v_k^2\Big\|_{L_1}, \qquad a_k \in \mathbb{R},$$

or, equivalently,

$$\Big\|\sum_{k=1}^{\infty} b_k v_k^2\Big\|_Y \asymp \Big\|\sum_{k=1}^{\infty} b_k v_k^2\Big\|_{L_1}, \qquad b_k \in \mathbb{R}.$$

Thus, the inclusion $Y \subset L_1$, where $Y \neq L_1$, is not disjointly strictly singular either. As was already repeatedly mentioned several times, this is false, and so the disjoint strict singularity of the inclusion $X \subset L_2$ is established.

Thus, all conditions of Theorem 1 in [18] are satisfied; applying the theorem, we conclude that the norms of the spaces L_2 and L_1 are equivalent on the subspace $[u_n]$. Obviously, the norms of the spaces X and L_1 have the same property. Hence (see Sec. 2), $[u_n] \subset M_{X,\delta}$ for some $\delta > 0$.

The other assertion of the theorem follows from the embeddings $X \subset Y \subset L_1$ and the fact that, as proved above, the sequence $\{u_n\}$ is equivalent in X and in L_1 to the canonical basis of ℓ_2. □

Second proof. First of all, note that the sequence $\{u_n\}$ is absolutely equi-integrable, i.e.,

$$\lim_{m(E)\to 0}\sup_{n\in\mathbb{N}}\int_E |u_n(s)|\,ds = 0.$$

Indeed, if this is not the case, then, by the well-known Dunford–Pettis criterion (see, e.g., [4, Theorem 5.2.9]), one can extract a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which is equivalent in L_1 to the canonical basis of ℓ_1. However, this contradicts the fact that $\{u_n\}$ is equivalent in X to the canonical basis of ℓ_2 (see the first proof).

By virtue of the property of $\{u_n\}$ mentioned above and the assumption $\inf_{n\in\mathbb{N}}\|u_n\|_{L_1} > 0$, we can apply to this sequence the lemma on martingale differences from [14], according to which

$$\bigg\|\sum_{n=1}^{\infty} a_n u_n\bigg\|_{L_1} \ge c\|(a_n)\|_{\ell_2}$$

for some $c > 0$ and all $a_n \in \mathbb{R}$. Thus, the norms of the spaces X and L_1 are equivalent on the subspace $[u_n]$ (we again apply the embeddings (2.1)). Finally, we again have $[u_n] \subset M_{X,\delta}$ for some $\delta > 0$, and the theorem is proved. □

In conclusion, we find sufficient conditions on a symmetric space under which every sequence of equimeasurable functions in this space has a subsequence equivalent to the canonical basis of ℓ_2.

Theorem 3. *Suppose that a symmetric space X is q-concave for some $q < \infty$ and $\beta_X < 1/2$. Then an arbitrary sequence $\{u_n\}_{n=1}^{\infty} \subset X$ such that $u_n \to 0$ weakly in X and $u_n^* = u_1^*$ for all $n \ge 2$ contains a subsequence equivalent to the canonical basis of ℓ_2.*

Proof. As above, the q-concavity with $q < \infty$ ensures that X is separable and $\alpha_X > 0$. This, together with the inequality $\beta_X < 1/2$, implies that X has an unconditional basis [3, Theorem 2.c.6]. Hence, as well as in the proof of Corollary 3, we may assume that $\{u_n\}_{n=1}^{\infty}$ is an unconditional basic sequence in X. Therefore, taking into account the relations

$$\|u_n\|_X = \|u_1\|_X, \quad \|u_n\|_{L_1} = \|u_1\|_{L_1}, \qquad n = 2, 3, \dots,$$

and using Proposition 2(a), we immediately obtain the following lower bound for some $c > 0$ and all $a_n \in \mathbb{R}$:

$$\bigg\|\sum_{n=1}^{\infty} a_n u_n\bigg\|_X \ge c\|(a_n)\|_{\ell_2}.$$

Below we use an idea of the paper [19] (see the proof of Proposition 3.1). Since $u_n^* = u_1^*$ for $n \ge 2$, it follows from Theorem 2.7.5 in [6] that, for every $n = 1, 2, \dots$, there is a measure-preserving mapping $\omega_n\colon [0,1] \to [0,1]$ such that $|u_n(t)| = u_1^*(\omega_n(t))$. Let a sequence $(a_n)_{n=1}^{\infty} \in \ell_2$ be fixed. For every $k = 1, 2, \dots$, consider the sublinear operator defined by

$$A_k x(t) := \bigg(\sum_{n=1}^{k}(a_n x(\omega_n(t)))^2\bigg)^{1/2}.$$

The 2-convexity (with constant 1) of the space L_r for $r \ge 2$ (see, e.g., [3, Proposition 1.d.5]) implies

$$\|A_k x\|_{L_r} \le \bigg(\sum_{n=1}^{k} a_n^2 \|x(\omega_n(t))\|_{L_r}^2\bigg)^{1/2} = \bigg(\sum_{n=1}^{k} a_n^2\bigg)^{1/2}\|x\|_{L_r};$$

it follows that the operator A_k is bounded in L_r with norm not exceeding $(\sum_{n=1}^{k} a_n^2)^{1/2}$. The assumption $0 < \alpha_X \le \beta_X < 1/2$ and the above bounds for the norm of the operator A_k in L_r enable us to apply the Marcinkiewicz interpolation theorem [3, Theorem 2.b.15], according to which, for some $C' > 0$ independent of $k = 1, 2, \dots$,

$$\left\|\left(\sum_{n=1}^{k}|a_n u_n|^2\right)^{1/2}\right\|_X = \|A_k u_1\|_X \le C'\left(\sum_{n=1}^{k} a_n^2\right)^{1/2}\|u_1\|_X.$$

On the other hand, since $\{u_n\}$ is unconditional, applying Maurey's inequality [3, Theorem 1.d.6 (i)] (and taking into account the q-concavity of X with $q < \infty$), we obtain

$$\left\|\sum_{n=1}^{k} a_n u_n\right\|_X \asymp \int_0^1\left\|\sum_{n=1}^{k} a_n u_n r_n(s)\right\|_X ds \le C''\left\|\left(\sum_{n=1}^{k}|a_n u_n|^2\right)^{1/2}\right\|_X.$$

It follows from the above bounds that

$$\left\|\sum_{n=1}^{k} a_n u_n\right\|_X \le C\left(\sum_{n=1}^{k} a_n^2\right)^{1/2}\|u_1\|_X,$$

where $C > 0$ does not depend on $k = 1, 2, \dots$ and on the sequence $(a_n)_{n=1}^{\infty} \in \ell_2$. Since X is separable, this inequality can be extended in the standard way to infinite sums. Thus, the theorem is proved. □

Remark 2. The condition that X has type 2 in Proposition 2 (b), Corollary 3, and Theorem 2 (and the condition $\beta_X < 1/2$ in Theorem 3) is exact in the sense that each of these assertions fails to hold even for sequences of equally distributed martingale differences if, instead of this condition, we assume that X is of type p for some $p < 2$ and $X \subset L_2$ (respectively, $\beta_X \le 1/2$).

First, recall that the space $\ell_{p,q}$, $1 < p < \infty$, $1 \le q < \infty$, consists of all sequences of reals $(a_k)_{k=1}^{\infty}$ such that

$$\|(a_k)\|_{\ell_{p,q}} := \left(\sum_{k=1}^{\infty}(a_k^*)^q(k^{q/p} - (k-1)^{q/p})\right)^{1/q} < \infty,$$

where (a_k^*) is the nonincreasing permutation of the sequence $(|a_k|)$. Therefore, by Corollary 3.6 of [20], for every q, $1 \le q < 2$, the space $L_{2,q}$ (see Sec. 2) contains a sequence of independent identically distributed functions $\{x_k\}_{k=1}^{\infty}$ such that $\int_0^1 x_k(s)\,ds = 0$, $k = 1, 2, \dots$, and $\{x_k\}$ contains no subsequence equivalent to the canonical basis of ℓ_2. Moreover, every such sequence satisfies the inequality

$$\left\|\sum_{k=1}^{\infty} a_k x_k\right\|_{L_{2,q}} \le C\|(a_k)\|_{\ell_{2,q}}$$

for some $C > 0$ [19, Corollary 3.13]. Since an arbitrary disjoint sequence of normalized functions in $L_{p,q}$ contains a subsequence equivalent in $L_{p,q}$ to the canonical basis of ℓ_q [21, Lemma 3.1] and $\ell_q \overset{\neq}{\subset} \ell_{2,q}$ for $q < 2$, it obviously follows that $\{x_n\}$ has no subsequence equivalent to a disjoint sequence in $L_{2,q}$. At the same time, the functions x_k are independent,

$$\int_0^1 x_k(s)\,ds = 0, \qquad k = 1, 2, \dots,$$

and hence form an orthogonal system in the space L_2; since they are identically distributed, we have

$$\|x_k\|_{L_2} = \|x_1\|_{L_2} \qquad \text{for all} \quad k \ge 2.$$

Therefore,

$$\int_0^1 x_k(s)y(s)\,ds \to 0 \qquad \text{as} \quad k \to \infty$$

for every function $y \in L_2$. Suppose that $1 < q < 2$. In this case, the space L_2 is dense in the space $(L_{2,q})^* = L_{2,q'}$, $1/q + 1/q' = 1$; hence $x_k \to 0$ weakly in $L_{2,q}$.

Thus, the space $X = L_{2,q}$, $1 < q < 2$, and the sequence $\{x_n\}$ satisfy all assumptions of Proposition 2 (b), Corollary 3, and Theorem 2 (resp., Theorem 3), except the condition that X has type 2 (resp., $\beta_X < 1/2$), but none of these assertions holds in this case.

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