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Distance Graphs with Large Chromatic Number and without Cliques of Given Size in the Rational Space

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Abstract—We study distance graphs with exponentially large chromatic number which do not contain cliques of prescribed size in the rational space.

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1. INTRODUCTION

In 1950, Nelson and Hadwiger posed the problem of finding the chromatic number $\chi(\mathbb{R}^n)$ of the space $\mathbb{R}^n$. This quantity is equal to the least number of colors that can be used to paint all points of $\mathbb{R}^n$ so that the distance between points of the same color is distinct from 1 (the distance 1 is said to be *forbidden*). Note that the value of $\chi(\mathbb{R}^n)$ is independent of the value of the positive number taken for the forbidden distance. This problem is now considered as one of the classical problems of combinatorial geometry. The main results for the real space are given, e.g., in [1] and [2]. In fact, the given problem can also be stated for the case of an arbitrary metric space $\mathbb{X}$ with metric ρ and forbidden distance d. Such a chromatic number will be denoted by $\chi((\mathbb{X},\rho),d)$. So, in 1976, Benda and Perles (see [3]) proposed to consider $\mathbb{X}=\mathbb{Q}^n$, $\rho=l_2$, where l_2 is the Euclidean metric. The value of the chromatic number of the space $\mathbb{Q}^n$ depends on the forbidden distance, which, for every pair of points with rational coordinates is either a rational number or a quadratic irrationality. For the chromatic numbers of the rational space many results were obtained.

- $\chi((\mathbb{Q}^1,l_2),1)=2$; this result is obvious;
- $\chi((\mathbb{Q}^2,l_2),1),1)=2$; this result is due to Woodall, 1973 (see [4]);
- $\chi((\mathbb{Q}^3,l_2),1)=2$;
- $\chi((\mathbb{Q}^4,l_2),1)=4$; both results are due to Benda and Perles, 1976 (see [3]);
- $\chi((\mathbb{Q}^5,l_2),1)\geq 8$; this estimate is due to Cibulka (see [5]), 2008; in 1993, Chilakamarri (see [6]) conjectured that here the equality is attained; however, the upper bound has not been established yet;
- $\chi((\mathbb{Q}^6,l_2),1)\geq 10$; this result is due to Mann, 2003, (see [7]);
- $\chi((\mathbb{Q}^7,l_2),1)\geq 15$ (see [5]);
- $\chi((\mathbb{Q}^8,l_2),1)\geq 16$ (see [7]);

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- $\chi((\mathbb{Q}^9, l_2), 1) \geq 22$;
- $\chi((\mathbb{Q}^{10}, l_2), 1) \geq 30$;
- $\chi((\mathbb{Q}^{11}, l_2), 1) \geq 35$;
- $\chi((\mathbb{Q}^{12}, l_2), 1) \geq 37$; these results are due to Cherkashin, Kulikov, and Raigorodskii, 2017 (see [8], [9]);
- in Raigorodskii's 2004 paper (see [10]), the following estimate was obtained for $d \in \mathbb{Q}$:
$$\chi((\mathbb{Q}^n, l_1), d) \geq (\zeta_2 + o(1))^n, \qquad \zeta_2 = 1.365\dots\ ;$$
- it was also proved in [10] that, for all $u \in \mathbb{N}$ and $d \in \mathbb{Q}$, there exists an $\varepsilon = \varepsilon(u) > 0$ such that the following estimate holds:
$$\chi((\mathbb{Q}^n, l_u), d) \geq (1 + \varepsilon + o(1))^n;$$
- the following estimates hold:
$$(1.199 + o(1))^n \leq \chi((\mathbb{Q}^n, l_2), 1) \leq \chi((\mathbb{R}^n, l_2), 1) \leq (3 + o(1))^n;$$
the lower bound is due to Ponomarenko and Raigorodskii (see [11], [12]), 2013, and the upper bound is due to Larman and Rogers, 1972 (see [13]);
- for particular irrational values of d and an increasing number n, a series of estimates for the quantity $\chi((\mathbb{Q}^n, l_u), d)$ was obtained in [14], 2016, for $u \geq 2$ and $d = \sqrt[u]{2p^\alpha}$, where p is a prime and $\alpha \in \mathbb{N}$.

And this is all that is known.

By a *distance graph* in the metric space $\mathbb{X}$ with metric ρ we mean a graph $G = (V, E)$ whose vertex set V is contained in $\mathbb{X}$ and edge set
$$E \subseteq \{\{x, y\} : x, y \in V,\ \rho(x, y) = a\}, \qquad a \in \mathbb{R}_+.$$

Distance graphs are of interest, because they arise in a natural way in connection with the problem of the chromatic number of a space. Indeed, let us take, for example, the graph $G = (V, E)$ with
$$V = \mathbb{R}^n, \qquad E = \{\{\overline{x}, \overline{y}\} : l_2(\overline{x}, \overline{y}) = 1\}.$$
Consider its chromatic number $\chi(G)$ (the least number of colors that can be used to paint all vertices of the graph so that there are no edges with endpoints of one color). Obviously, $\chi(\mathbb{R}^n) = \chi(G)$. By the Erdős–de Bruijn theorem (see, e.g., [2]), to evaluate $\chi(\mathbb{R}^n)$, it suffices to restrict the study to finite distance graphs.

In the present paper, we solve a number of extremal problems for distance graphs in the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{Q}^n$ of increasing dimension.

Let us give the necessary definitions.

Let a'_{-1}, a'_1, and q' be positive real numbers less than 1. For each natural number n, we put $a_1 = [a'_1 n]$ and $a_{-1} = [a'_{-1} n] - 1$ and let q be a natural number such that $q = q' n(1 + o(1))$.

Definition 1. We introduce the *sequence* $\{G_n(a_1, q)\}_{n\in\mathbb{N}} = \{G_n\}_{n\in\mathbb{N}}$ of graphs $G_n = (V_n, E_n)$ with vertex sets
$$V_n = \{\overline{x} = (x_1, \dots, x_n) : x_i \in \{0, 1\},\ |\{i : x_i = 1\}| = a_1\}$$
and edge sets
$$E_n = \{\{\overline{x}, \overline{y}\} : l_2(\overline{x}, \overline{y}) = \sqrt{2q}\,\}.$$

Definition 2. We introduce the *sequence* $\{G'_n(a_1,q)\}_{n\in\mathbb{N}} = \{G'_n\}_{n\in\mathbb{N}}$ of graphs $G'_n = (V'_n, E'_n)$ with vertex sets

$$V'_n = \left\{\overline{x} = (x_1,\dots,x_n) : x_i \in \left\{0, \frac{1}{\sqrt{2q}}\right\}, \left|\left\{i : x_i = \frac{1}{\sqrt{2q}}\right\}\right| = a_1\right\}$$

and edge sets

$$E'_n = \{\{\overline{x},\overline{y}\} : l_2(\overline{x},\overline{y}) = 1\}.$$

Definition 3. We introduce the *sequence* $\{\widetilde{G}_n(\{a_{-1},a_1\},q)\}_{n\in\mathbb{N}} = \{\widetilde{G}_n\}_{n\in\mathbb{N}}$ of graphs $\widetilde{G}_n = (\widetilde{V}_n, \widetilde{E}_n)$ with vertex sets

$$\widetilde{V}_n = \{\overline{x} = (x_1,\dots,x_n) : x_i \in \{-1,0,1\},\ |\{i : x_i = -1\}| = a_{-1},\ |\{i : x_i = 1\}| = a_1\}$$

and edge sets

$$\widetilde{E}_n = \{\{\overline{x},\overline{y}\} : l_2(\overline{x},\overline{y}) = \sqrt{2q}\,\}.$$

Definition 4. We introduce the *sequence* $\{\widetilde{G}'_n(\{a_{-1},a_1\},q)\}_{n\in\mathbb{N}} = \{\widetilde{G}'_n\}_{n\in\mathbb{N}}$ of graphs $\widetilde{G}'_n = (\widetilde{V}'_n, \widetilde{E}'_n)$ with vertex sets

$$\widetilde{V}'_n = \left\{\overline{x} = (x_1,\dots,x_n) : x_i \in \left\{-\frac{1}{\sqrt{2q}}, 0, \frac{1}{\sqrt{2q}}\right\},\right.$$
$$\left.\left|\left\{i : x_i = -\frac{1}{\sqrt{2q}}\right\}\right| = a_{-1},\ \left|\left\{i : x_i = \frac{1}{\sqrt{2q}}\right\}\right| = a_1\right\}.$$

and edge sets

$$\widetilde{E}'_n = \{\{\overline{x},\overline{y}\} : l_2(\overline{x},\overline{y}) = 1\}.$$

We are interested in the behavior of the chromatic numbers of distance graphs under the additional condition that these graphs do not contain cliques of given size.

Let $\omega(G)$ be the number of vertices in the maximal complete subgraph of the graph $G = (V,E)$:

$$\omega(G) = \max\{|W| : W \subseteq V,\ \forall\, \overline{x},\overline{y} \in W\ \{\overline{x},\overline{y}\} \in E\}.$$

The quantity $\omega(G)$ is called the *clique number* of the graph.

For $\mathbb{X} \in \{\mathbb{R},\mathbb{Q}\}$, we set

$$\zeta_k(\mathbb{X}) = \sup\big\{\zeta : \exists \text{ a function } \delta = \delta(n) \text{ such that } \lim_{n\to\infty} \delta(n) = 0$$
$$\text{and } \forall\, n\ \exists G, \text{ a distance graph in } \mathbb{X}^n$$
$$\text{such that } \omega(G) < k,\ \chi(G) \ge (\zeta + \delta(n))^n\big\}.$$

These quantities were studied in [1] and [15] for the case of the real space. In the present paper, we obtain constraints under which graphs from the sequences $\{G'_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{G}'_n\}_{n\in\mathbb{N}}$ do not contain k-cliques and, further, prove new estimates for the quantity $\zeta_k(\mathbb{Q})$, using an explicit construction. We also refine a probabilistic result from the paper [15], thus improving estimates of $\zeta_k(\mathbb{R})$. In addition, we obtain new probabilistic estimates of $\zeta_k(\mathbb{Q})$ for all k.

Concluding the introduction, we note that the history of the problem of the chromatic number of a space and various results concerning distance graphs can be learned from the surveys, papers, and books [16]–[35].

2. FORMULATION OF THE RESULTS AND COMPARISON OF THE ESTIMATES WITH THE PREVIOUS ONES

This section is divided into two parts.

2.1. Graphs without Cliques

In this subsection, we describe the graphs of the form indicated above that do not contain cliques of given size. For the graphs of the sequencess $\{G_n\}_{n\in\mathbb{N}}$, $\{\widetilde{G}_n\}_{n\in\mathbb{N}}$, Kupavskii [15, Theorems 1 and 2] obtained conditions under which these graphs do not contain k-cliques. Below we state this result not only for such graphs but also for the graphs of the sequencess $\{G'_n\}_{n\in\mathbb{N}}$, $\{\widetilde{G}'_n\}_{n\in\mathbb{N}}$ (in this case, the proof is similar to that of Theorems 1 and 2 from [15]).

Theorem 1. (1) *Suppose that $a'_1, q' > 0$, $q' < a'_1$, and $a'_1, q' \in \mathbb{R}$. Consider the sequence $\{G_n\}_{n\in\mathbb{N}}$ (the sequence $\{G'_n\}_{n\in\mathbb{N}}$) of distance graphs. Let k be a natural number, $k \geq 3$. If the inequality*

$$a'_1 - \frac{(ka'_1)^2 - \{ka'_1\}^2 - [ka'_1]}{k(k-1)} < q'$$

holds, then, for sufficiently large n, the graphs of the sequence $\{G_n\}_{n\in\mathbb{N}}$ (respectively, $\{G'_n\}_{n\in\mathbb{N}}$) do not contain complete subgraphs (respectively, cliques) on k vertices.

(2) *Suppose that $a'_1, a'_{-1}, q' > 0$, $q' < a'_1$, and $a'_1, a'_{-1}, q' \in \mathbb{R}$. Consider the sequence $\{\widetilde{G}_n\}_{n\in\mathbb{N}}$ (the sequence $\{\widetilde{G}'_n\}_{n\in\mathbb{N}}$) of distance graphs. Let k be a natural number, $k \geq 3$.*

If the inequalities $k(1 - a'_1 - a'_{-1}) + \{ka'_1\} \geq 1$ and

$$a'_1 + a'_{-1} + \frac{k(a'_1 + a'_{-1}) - (k(a'_1 - a'_{-1}))^2 - \{k(a'_1 - a'_{-1})\} + \{k(a'_1 - a'_{-1})\}^2}{k(k-1)} < q'$$

or $k(1 - a'_1 - a'_{-1}) + \{ka'_1\} < 1$ and

$$\frac{(k-k^2)(2a'_1 + 2a'_{-1} - 1) + (4k-4)\{ka'_1\} + 4\{ka'_1\}^2 + 4k(a'_1 + a'_{-1})[ka'_1] - 4(ka'_1)^2}{k(k-1)} + a'_1 + a'_{-1} < q'$$

hold, then, for sufficiently large n, the graphs of the sequence $\{\widetilde{G}_n\}_{n\in\mathbb{N}}$ (respectively, of the $\{\widetilde{G}'_n\}_{n\in\mathbb{N}}$) do not contain k-cliques.

2.2. Chromatic Numbers

In this subsection, we study the dependence of the constants $\zeta_k(\mathbb{R})$ and $\zeta_k(\mathbb{Q})$ on k. Results for the case of the real space were obtained in [1], and [15], where two approaches, constructive and probabilistic, were used.

The first approach is based on the construction of explicit distance graphs without cliques. All pairs of vertices at distance 1 from each other are joined by edges. As the graphs we use the graphs from the sequences $\{G_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{G}_n\}_{n\in\mathbb{N}}$, and optimization with respect to the parameters a'_{-1}, a'_1, and q' is performed.

In the second approach, we begin by taking graphs with cliques of arbitrarily large size and then remove part of the edges of these graphs, eliminating all cliques of the given size, and, from the set of examined graphs, choose the optimal one. Thus, we use subgraphs of the graphs of the sequences $\{G_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{G}_n\}_{n\in\mathbb{N}}$ and also perform optimization with respect to the parameters a'_{-1}, a'_1, and q'.

In the case of the rational space, we shall use the same approaches but based on the sequences $\{G'_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{G}'_n\}_{n\in\mathbb{N}}$, and optimization with respect to q will be more complicated.

2.2.1. Constructive approach. The following theorems are valid.

Theorem 2. *Let $k \geq 3$ be a natural number. Then*

$$\zeta_k(\mathbb{Q}) \geq \max_{x\in(0;1/8)} \min_{q'\in[x;4x]} \max_{a'_1} \frac{q'^{q'}(1-q')^{1-q'}}{(a'_1)^{a'_1}(1-a'_1)^{1-a'_1}},$$

where the maximum is taken over all a'_1 satisfying the constraints

$$a'_1 \in (0,1), \qquad a'_1 < 2q', \qquad q' < a'_1, \qquad a'_1 - \frac{(ka'_1)^2 - \{ka'_1\}^2 - [ka'_1]}{k(k-1)} < q'.$$

Theorem 3. *Let $k \geq 3$ be a natural number. Then*

$$\zeta_k(\mathbb{Q}) \geq \max_{x\in(0;1/8)} \min_{q'\in[x;4x]} \max_{a'_1,a'_{-1}} \frac{(q'-2l)^{q'-2l}(1-q'+l)^{1-q'+l}}{(a'_1)^{a'_1}(a'_{-1})^{a'_{-1}}(1-a'_1-a'_{-1})^{1-a'_1-a'_{-1}}},$$

where the maximum is taken over all a'_{-1} and a'_1 satisfying the constraints

$$a'_{-1} < a'_1, \qquad a'_1 + a'_{-1} < \frac{1}{2}, \qquad 3a'_{-1} + a'_1 = 2q',$$

$$a'_1 + a'_{-1} + \frac{k(a'_1+a'_{-1}) - (k(a'_1-a'_{-1}))^2 - \{k(a'_1-a'_{-1})\} + \{k(a'_1-a'_{-1})\}^2}{k(k-1)} < q',$$

and $l = (3q'+1-\sqrt{1+6q'-3(q')^2}\,)/6$.

2.2.2. Probabilistic method. Using this method, estimates for the case of the real space were obtained in [1], [38]–[40]. Further, Kupavskii [15] succeeded in improving these estimates by studying the structure of the graphs in greater detail. For the probabilistic method and other results obtained by its use, see [41]–[43].

In this paper, we succeeded in improving the probabilistic results from [15] for $\zeta_k(\mathbb{R})$, as well as in obtaining similar results for the space $\mathbb{Q}$.

First, let us give necessary definitions. We shall consider the sequences of graphs $\{G_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{G}_n\}_{n\in\mathbb{N}}$. Let $G_n = (V,E)$ be a graph belonging to one of these sequences. Let N_n denote the number of its vertices. We introduce the notation $\operatorname{conn}_k^j(G_n, v_1, \dots, v_j)$, $j < k$, for the number of k-cliques in the graph G_n which contain the vertices $v_1, \dots, v_j$. Let

$$\operatorname{conn}_k^j(G_n) = \max_{v_1,\dots,v_j} \operatorname{conn}_k^j(G_n, v_1, \dots, v_j).$$

By $s_k^j(\{G_n\}_{n\in\mathbb{N}})$ and $s_k^j(\{\widetilde{G}_n\}_{n\in\mathbb{N}})$ we denote the following quantities:

$$s_k^j(\{G_n\}_{n\in\mathbb{N}}) = \lim_{n\to\infty} \log_{N_n} \operatorname{conn}_k^j(G_n), \qquad G_n \in \{G_n\}_{n\in\mathbb{N}},$$
$$s_k^j(\{\widetilde{G}_n\}_{n\in\mathbb{N}}) = \lim_{n\to\infty} \log_{N_n} \operatorname{conn}_k^j(G_n), \qquad G_n \in \{\widetilde{G}_n\}_{n\in\mathbb{N}}.$$

For the sequences of graphs specified above, the given limit exists; for details concerning this fact and other properties of the quantities introduced above, see [15]. We set

$$s_k^j(a'_1, a'_1 - q') = s_k^j(\{G_n\}_{n\in\mathbb{N}}) = s_k^j(\{G'_n\}_{n\in\mathbb{N}}),$$
$$s_k^j(a'_1, a'_{-1}, a'_1 + a'_{-1} - q') = s_k^j(\{G_n\}_{n\in\mathbb{N}}) = s_k^j(\{\widetilde{G}'_n\}_{n\in\mathbb{N}}).$$

The existence of the limit implies that it is the same for all $q(n)$ asymptotically equal to one another. Therefore, we write that the quantities in question depend on the values of the positive real numbers q'.

In what follows, we shall need Theorem 6 from [15]; we will cite it below. Using it and taking into account the stronger estimate $s_k^j(\{G_n\}_{n\in\mathbb{N}})$ for $q' = a'_1/2$ in Proposition 2, we have succeeded in strengthening the estimate for $\zeta_k(\mathbb{R})$.

Theorem 4. *Let $k \geq 3$ be a natural number. Consider an arbitrary real number a'_1 satisfying the constraint $a'_1 \in (0, 1/2)$ and the quantities*

$$\tau_0 = \tau_0(a'_1) = \left(\frac{a'_1}{2}\right)^{-a'_1/2}\left(1 - \frac{a'_1}{2}\right)^{-1+a'_1/2}, \qquad \tau_1 = \tau_1(a'_1) = (a'_1)^{-a'_1}(1-a'_1)^{-1+a'_1}.$$

Then

$$\zeta_k(\mathbb{R}) \geq \max_{a'_1\in(0,1/2)} \frac{\tau_1(a'_1)^{1-2s_k^2(a'_1,a'_1/2)/((k-2)(k+1))}}{\tau_0(a'_1)}.$$

Let us now state two more theorems proved in the present paper.

Theorem 5. *Let $k \geq 3$ be a natural number. Let $x \in (0;1/8)$ be an arbitrary number. For each q' from the closed interval $[x;4x]$, consider an arbitrary real number a_1' satisfying the constraints*

$$a_1' \in \left(0, \frac{1}{2}\right), \qquad a_1' < 2q'$$

and the quantities

$$\tau_0 = \tau_0(q') = (q')^{-q'}(1-q')^{-1+q'}, \qquad \tau_1 = \tau_1(a_1') = (a_1')^{-a_1'}(1-a_1')^{-1+a_1'}.$$

Then

$$\zeta_k(\mathbb{Q}) \geq \max_{x\in(0;1/8)} \min_{q'\in[x;4x]} \max_{a_1'} \frac{\tau_1(a_1')^{1-2s_k^2(a_1',a_1'-q')/((k-2)(k+1))}}{\tau_0(q')},$$

where the maximum is taken over a_1' under the constraints on this parameter indicated in the statement of the theorem.

Theorem 6. *Let $k \geq 3$ be a natural number. Let $x \in (0;1/8)$ be an arbitrary number. For each q' from the closed interval $[x;4x]$, consider arbitrary real numbers a_{-1}' and a_1' satisfying the constraints*

$$a_{-1}', a_1' \in (0,1), \qquad a_{-1}' + a_1' \leq \frac{1}{2}, \qquad a_{-1}' \leq a_1', \qquad 3a_{-1}' + a_1' = 2q'$$

and the quantities

$$A = \frac{2 + 9a_{-1}' + 3a_1' - \sqrt{(2 + 9a_{-1}' + 3a_1')^2 - 12(3a_{-1}' + a_1')^2}}{12},$$

$$B = \frac{3a_{-1}' + a_1'}{2} - 2A, \qquad C = 1 + A - \frac{3a_{-1}' + a_1'}{2}.$$

Further, let

$$\rho_0 = \rho_0(a_{-1}', a_1') = A^{-A}B^{-B}C^{-C},$$
$$\rho_1 = \rho_1(a_{-1}', a_1') = (1 - a_1' - a_{-1}')^{-1+a_1'+a_{-1}'}(a_{-1}')^{-a_{-1}'}(a_1')^{-a_1'}.$$

Then

$$\zeta_k(\mathbb{Q}) \geq \max_{x\in(0;1/8)} \min_{q'\in[x;4x]} \max_{a_{-1}',a_1'} \frac{\rho_1(a_{-1}', a_1')^{1-2s_k^2(a_1',a_{-1}',a_1'+a_{-1}'-q')/((k-2)(k+1))}}{\rho_0(a_{-1}', a_1')},$$

where the maximum is taken over a_{-1}' and a_1' under the constraints on these parameters indicated in the statement of the theorem.

To prove Theorems 4–6, we need some auxiliary propositions.

Proposition 1. *For all $k \geq i \geq j$ and the sequences of graphs indicated above, the following inequalities hold:*

$$s_k^j(a_1', a_1' - q') \leq s_i^j(a_1', a_1' - q') + (k-i)s_{i+1}^i(a_1', a_1' - q'), \tag{1}$$
$$s_k^j(a_1', a_{-1}', a_1' + a_{-1}' - q') \leq s_i^j(a_1', a_{-1}', a_1' + a_{-1}' - q') + (k-i)s_{i+1}^i(a_1', a_{-1}', a_1' + a_{-1}' - q').$$

Proposition 2. *The following estimates hold:*

$$s_k^2(a_1', a_{-1}', a_1' + a_{-1}' - q') \leq k - 2, \tag{2}$$
$$s_k^2(a_1', a_1' - q') \leq s_3^2(a_1', a_1' - q') + (k-3)s_4^3(a_1', a_1' - q'), \tag{3}$$

$$s_3^2(a_1', a_1' - q') = \max_t \log_{1/((a_1')^{a_1'}(1-a_1')^{1-a_1'})} \frac{P}{Q}, \quad (4)$$

$$s_4^3(a_1', a_1' - q') = \max_{t,l,m,p,r} \log_{1/((a_1')^{a_1'}(1-a_1')^{1-a_1'})} \frac{L}{M}, \quad (5)$$

as well as

$$\begin{aligned}
P &= P(a_1', b) = b^b (a_1' - b)^{2(a_1'-b)} (1 - 2a_1' + b)^{1-2a_1'+b}, \\
Q &= Q(a_1', b, t) = t^{2t} (b-t)^{3(b-t)} (a_1' - 2b + t)^{2(a_1'-2b+t)} (1 - 2a_1' + b - t)^{1-2a_1'+b-t}, \\
L &= L(a_1', b, t, l, m, p, r) \\
&= (t+l)^{(t+l)} (b-t-l)^{3(b-t-l)} (a_1' - 2b + t + l)^{3(a_1'-2b+t+l)} \\
&\qquad \times (1 - 3a_1' + 3b - t - l)^{1-3a_1'+3b-t-l}, \\
M &= M(a_1', b, t, l, m, p, r) \\
&= t^t m^m l^l p^p r^r (b-t-l-m)^{b-t-l-m} (b-t-l-p)^{b-t-l-p} \\
&\qquad \times (b-t-l-r)^{b-t-l-r} (b-t-m-p)^{b-t-m-p} \\
&\qquad \times (b-t-m-r)^{b-t-m-r} (b-t-p-r)^{b-t-p-r} \\
&\qquad \times (a_1' - 3b + 2t + l + m + p)^{a_1'-3b+2t+l+m+p} \\
&\qquad \times (a_1' - 3b + 2t + l + m + r)^{a_1'-3b+2t+l+m+r} \\
&\qquad \times (a_1' - 3b + 2t + l + p + r)^{a_1'-3b+2t+l+p+r} \\
&\qquad \times (a_1' - 3b + 2t + m + p + r)^{a_1'-3b+2t+m+p+r} \\
&\qquad \times (1 - 4a_1' + 6b - 3t - l - m - p - r)^{1-4a_1'+6b-3t-l-m-p-r},
\end{aligned}$$

where $b = a_1' - q'$. *It is assumed that the parameters* a_1', b, t, l, m, p, *and* r *take only those values for which the functions* P, Q, L, *and* M *are well defined.*

2.2.3. Comments and tables of estimates. Here we present the numerical values of lower bounds for the quantities $\zeta_k(\mathbb{R})$ and $\zeta_k(\mathbb{Q})$. In Table 1, the first column contains the values of k, and in columns 2–5, we give the values of $\zeta_k(\mathbb{R})$ obtained in [15] for each k. The last column contains the constants obtained in Theorem 4 of this paper. The sharpest bounds for all values of k are highlighted in bold.

Theorem 4 is a refinement of Theorem 6 from [15] and improves the results for k from 11 to 19. By using distance graphs with vertices from $\{0,1\}^n$, the following estimate for the chromatic number of the space $\mathbb{R}^n$ was obtained in [19]:

$$\chi(\mathbb{R}^n) \geq \left(\frac{1+\sqrt{2}}{2} + o(1)\right)^n = (1.207\ldots + o(1))^n.$$

It is seen from the table that, as k increases, the constant $\zeta_k(\mathbb{R})$ becomes closer to $(1+\sqrt{2})/2$. This means that, in a certain sense, our results are consistent with those in [19] and that, in a certain sense, Theorem 4 provides the optimal (within the framework of the given method) result for each k and in the asymptotics in k.

In Table 2, the first column contains the values of k, and in columns 2–5, for each k, we give the lower bounds for $\zeta_k(\mathbb{Q})$ obtained in Theorems 2, 3, 5, and 6, respectively. The sharpest bounds are highlighted in bold.

It is seen that, for certain values of k, Theorems 2 and 3 yield values less than 1. This means that Theorems 2 and 3 yield weak results in the case of the rational space. It should be noted that, for the real space, the explicit approach used in [1] and [15] yielded the best results for small k. In addition, it is seen that, in the fourth column, $\zeta_k(\mathbb{Q})$ tends to $1.150\ldots$, and in the fourth column, $\zeta_k(\mathbb{Q})$ approaches $1.199\ldots$ These are precisely the constants obtained in the papers [14] and [11], [12], respectively, in which the best now available estimates for $\chi(\mathbb{Q}^n)$ were obtained. Thus, for each k and in the asymptotics in k, Theorems 5 and 6 yield the optimal (in a certain sense) results within the framework of the methods used.

Table 1

k	T4 [15]	T5 [15]	T6 [15]	T7 [15]	T4
,	$\zeta_k(\mathbb{R}) \geq$				
3	**1.0582**	is	1.0147	is	1.0147
4	**1.0663**	1.0374	1.0321	1.0028	1.0365
5	**1.0857**	1.0601	1.0491	1.0169	1.0529
6	**1.0898**	1.0754	1.0641	1.0339	1.0683
7	**1.0995**	1.0865	1.0771	1.0501	1.0812
8	**1.1019**	1.0948	1.0881	1.0646	1.0918
9	**1.1077**	1.1013	1.0976	1.0773	1.1008
10	**1.1093**	1.1066	1.1057	1.0886	1.1088
11	1.1131	1.1109	1.1128	1.0985	**1.1157**
12	1.1142	1.1145	1.1190	1.1073	**1.1218**
13	1.1170	1.1175	1.1245	1.1151	**1.1271**
14	1.1178	1.1201	1.1293	1.1220	**1.1317**
15	1.1198	1.1224	1.1336	1.1283	**1.1358**
16	1.1205	1.1225	1.1375	1.1339	**1.1396**
17	1.1220	1.1241	1.1409	1.1390	**1.1430**
18	1.1226	1.1254	1.1441	1.1437	**1.1461**
19	1.1239	1.1266	1.1470	1.1479	**1.1488**
20	1.1243	1.1278	1.1496	**1.1518**	1.1513
100	1.1366	1.1446	1.1947	**1.2197**	1.1945
1000	1.1394	1.1491	1.2058	**1.2375**	1.2058
1000000	1.1394	1.1542	1.2071	**1.2395**	1.2071

3. PROOFS. CHROMATIC NUMBERS

3.1. Proof of Theorem 2

Let $x \in (0, 1/8)$ be an arbitrary number. For each n (the dimension), we set

$$q = 2^{2[\log_2(2xn)/2]+1}.$$

Consider the graph G'_n from the sequence of graphs $\{G'_n\}_{n\in\mathbb{N}}$. In view of such a choice of q, the sets V'_n are embedded in $\mathbb{Q}^n$.

By the Dirichlet principle, it be easy to obtain the following estimate[1] for the chromatic number of the graph:

$$\chi(G'_n) \geq \frac{|V'_n|}{\alpha(G'_n)} = \frac{C_n^{a_1}}{\alpha(G'_n)},$$

where $\alpha(G'_n)$ is the independence number of the graph, i.e., the maximal cardinality of a subset of the vertex set in which there are no edges.

[1] *Translator's note*. Here and elsewhere, C_n^m stands for the binomial coefficient $\binom{n}{m}$.

Table 2

k	T2	T3	T5	T6
,	$\zeta_k(\mathbb{Q}) \geq$			
3	0.9980	0.9843	**1.0088**	0.9972
4	0.9980	0.9887	**1.0176**	1.0019
5	0.9980	0.9926	**1.0270**	1.0131
6	0.9980	0.9961	**1.0405**	1.0267
7	0.9980	0.9995	**1.0468**	1.0398
8	0.9980	1.0027	1.0517	**1.0519**
9	1.0020	1.0058	1.0555	**1.0626**
10	1.0062	1.0088	1.0614	**1.0722**
11	1.0083	1.0117	1.0715	**1.0808**
12	1.0104	1.0146	1.0801	**1.0881**
13	1.0106	1.0174	1.0875	**1.0950**
14	1.0125	1.0202	1.0939	**1.1007**
15	1.0126	1.0229	1.0996	**1.1061**
16	1.0126	1.0256	1.1046	**1.1107**
17	1.0126	1.0283	1.1083	**1.1151**
18	1.0127	1.0309	1.1097	**1.1096**
19	1.0143	1.0335	1.1110	**1.1225**
20	1.0149	1.0347	1.1121	**1.1262**
100	1.0220	1.0692	1.1445	**1.1828**
1000	1.0242	1.0721	1.1498	**1.1968**
1000000	1.0256	1.0721	1.1504	**1.1989**

Lemma 1. *The following estimate holds*:

$$\alpha(G'_n) \leq \sum_{i \leq q-1} C_n^i.$$

This lemma is precisely Lemma 2 from the paper [14], which also contains its proof. This lemma uses the condition $a'_1 < 2q'$. Then, using the lemma and optimizing the expression, we obtain

$$\chi(G'_n) \geq \max_{x \in (0;1/8)} \max_{a'_1} \frac{C_n^{[a'_1 n]}}{n C_n^q}.$$

In view of the constraints on the parameters in Theorem 1, we have a graph without k-cliques for sufficiently large n. The value of q depends on n. Note that q' takes values in the closed interval $[x; 4x]$. Therefore, to obtain a lower bound for the quantity $\zeta_k(\mathbb{Q})$, we may take, for example, its minimum over all q' from the closed interval $[x; 4x]$. Now, applying Stirling's formula, for $\zeta_k(\mathbb{Q})$, we obtain

$$\zeta_k(\mathbb{Q}) \geq \max_{x \in (0;1/8)} \min_{q' \in [x;4x]} \max_{a'_1} \frac{(q')^{q'}(1-q')^{1-q'}}{(a'_1)^{a'_1}(1-a'_1)^{1-a'_1}}.$$

3.2. Proof of Theorem 3

Let $x \in (0,1/8)$ be an arbitrary number. For each n (the dimension), we set

$$q = 2^{2[\log_2(2xn)/2]+1}.$$

Consider the graph $\widetilde{G}'_n$ from the sequence of graphs $\{\widetilde{G}'_n\}_{n\in\mathbb{N}}$. By the choice of q, the sets V'_n are embedded in $\mathbb{Q}^n$. Just as in the proof of Theorem 1, we shall use the inequality

$$\chi(\widetilde{G}'_n) \ge \frac{|V'_n|}{\alpha(\widetilde{G}'_n)} = \frac{C_n^{a_1} C_{n-a_1}^{a-1}}{\alpha(\widetilde{G}'_n)}.$$

Lemma 2. *The following estimate holds*:

$$\alpha(\widetilde{G}'_n) \le \sum_{j=0}^{[(q-1)/2]} \sum_{i=0}^{q-1-2j} C_n^j C_{n-j}^i.$$

This lemma is precisely Lemma 1 from the paper [14], in which its proof can be found.

Using the lemma, we obtain

$$\alpha(\widetilde{G}'_n) \le \sum_{j=0}^{[(q-1)/2]} \sum_{i=0}^{q-1-2j} C_n^j C_{n-j}^i \le n \sum_{j=0}^{[(q-1)/2]} C_n^j C_{n-j}^{q-1-2j} = nF.$$

Let $f = C_n^{[\kappa n]} C_{n-[\kappa n]}^{q-2[\kappa n]}$, where κ is a real number from the interval $(0,1/2)$. We take l from the assumption of Theorem 3. At $\kappa = l$, the function f attains its maximum, which will be denoted by $f_{\max}$ (for details, see [15], [16]). We see that the inequality $F < nf_{\max}$ holds. Hence

$$\chi(\widetilde{G}'_n) \ge \max_{x\in(0;1/8)} \max_{a'_1,a'_{-1}} \frac{C_n^{[a'_1 n]} C_{n-[a'_1 n]}^{[a'_{-1} n]}}{n^2 f_{\max}}.$$

For sufficiently large n, the graph will not contain k-cliques due to the choice of the parameters in Theorem 3. Just as in the proof of Theorem 3, to obtain a lower bound for the quantity $\zeta_k(\mathbb{Q})$, we may take, for example, its minimum over all q' from the closed interval $[x;4x]$. Now, applying Stirling's formula, for $\zeta_k(\mathbb{Q})$, we obtain

$$\zeta_k(\mathbb{Q}) \ge \max_{x\in(0;1/8)} \min_{q'\in[x;4x]} \max_{a'_1,a'_{-1}} \frac{(q'-2l)^{q'-2l}(1-q'+l)^{1-q'+l}}{(a'_1)^{a'_1}(a'_{-1})^{a'_{-1}}(1-a'_1-a'_{-1})^{1-a'_1-a'_{-1}}}.$$

3.3. Proof of Theorem 5

Let $x \in (0,1/8)$ be an arbitrary number. For each n (the dimension), we set

$$q = 2^{2[\log_2(2xn)/2]+1}.$$

Let

$$c(a'_1, q') = \frac{\tau_1(a'_1)^{1-2s_k^2(a'_1,a'_1-q')/((k-2)(k+1))}}{\tau_0(q')}.$$

Let us fix k, x, q', and a'_1 and, thereby, τ_0, τ_1, and c. If $c \le 1$, then the assertion of Theorem 5 is trivial. Consider the case in which $c > 1$.

Let $c' \in (1,c)$ be an arbitrary number. Here it is important that c' is strictly less than c, although it can be arbitrarily close to c. If we show that $\zeta_k(\mathbb{Q}) \ge c'$, then, taking the supremum with respect to c' on both sides of the inequality, we obtain the required inequality $\zeta_k(\mathbb{Q}) \ge c$.

Thus, we need to verify the existence of a function $\delta(n) = o(1)$ such that, for all n, there exists a distance graph $G' = (V', E')$ in $\mathbb{Q}^n$ for which, simultaneously, $\omega(G') < k$ and $\chi(G') \ge (c' + \delta(n))^n$.

Let a_1 satisfy the inequality $a_1 - 2q < 0$. For each sufficiently large $n \in \mathbb{N}$, consider the graph $G'_n = (V'_n, E'_n) \in \{G'_n\}_{n\in\mathbb{N}}$. Just as in Lemma 2 from [12], we perform a renormalization and find that $G_n = (V_n, E_n) \in \{G_n\}_{n\in\mathbb{N}}$.

By using Stirling's formula, it is easy to show that

$$N = |V_n| = (\tau_1 + o(1))^n.$$

The estimate

$$\alpha = \alpha(G_n) \le nC_n^q$$

was obtained in Lemma 2 from [14] Since $q = q'n$, it follows from Stirling's formula that, as $n \to \infty$, $\alpha \le (\tau_0 + \delta_1)^n$ with some $\delta_1 = o(1)$.

Now, using the standard estimate $\chi(G) \ge |V|/\alpha(G)$, we obtain

$$\chi(G_n) \ge \left(\frac{\tau_1}{\tau_0} + \delta_2(n)\right)^n$$

with some $\delta_2 = o(1)$; this result is better than that in Theorem 5. However, the distance graph G_n contains cliques of size greater than k.

In what follows, we shall use the probabilistic method.

We set $G = (V_n, E) \in \{G_n\}_{n\in\mathbb{N}}$, including each edge from E_n in E with probability $p = \gamma^n$ without regard to the other edges; here $\gamma \in (\tau_0 c'/\tau_1, 1)$ (such an edge exists, because $c' < c \le \tau_1/\tau_0$). We obtain a probability space $(\Omega_n, \mathcal{B}_n, P_n)$ in which

$$\Omega_n = \{G = (V_n, E),\ E \subseteq E_n\}, \qquad \mathcal{B}_n = 2^{\Omega_n},$$
$$P_n(G) = p^{|E|}(1-p)^{|E_n|-|E|}, \qquad G = (V_n, E).$$

We set $l = [(\tau_1/c')^n]$. On Ω_n, we define two families of events. Let us number all l-element subsets of V_n and introduce the events

$$X_i = \{\text{the } i\text{th } l\text{-element subset does not contain edges}\}, \qquad i = 1, \dots, C_N^l.$$

Further, let us number all k-cliques of the graph G_n. We denote their number by $cl_k(G_n)$ and introduce the events

$$Y_j = \{\text{the } j\text{th } k\text{-element subset is a clique}\}, \qquad i = 1, \dots, cl_k(G_n).$$

Since $c' > 1$, it follows that, for large n, we have $l < N = |V_n|$; therefore, the events X_i are well defined.

If we show that

$$P\biggl(\bigwedge_{i=1}^{C_N^l} \overline{X_i} \wedge \bigwedge_{j=1}^{cl_k(G_n)} \overline{Y_j}\biggr) > 0,$$

then this will imply the existence of a subgraph G in G_n not containing k-cliques and such that $\alpha(G)$ does not exceed l. This inequality holds for γ close to $\tau_0 c/\tau 1$ and

$$\gamma < \tau_1^{1-2s_k^2(a'_1, a'_1 - q')/((k-2)(k+1))}.$$

The detailed proof of this inequality was described in Theorem 6 from [15]. Given the graph G, we perform the inverse renormalization, obtaining a graph G' in the space $\mathbb{Q}^n$.

As a result, we have

$$\chi(G') = \chi(G) \ge \max_{x\in(0;1/8)} \min_{q'\in[x;4x]} \max_{a'_1} \left(\frac{\tau_1(a'_1)^{1-2s_k^2(a'_1, a'_1-q')/((k-2)(k+1))}}{\tau_0(q')} + \delta(n)\right)^n,$$

and the theorem is proved.

3.4. Proof of Theorem 6

The scheme of proof is practically the same for Theorem 6 as in its analog from the previous section. The essential difference is only in the construction of the graph $\widetilde{G}_n$. For each sufficiently large n, we consider the graph $\widetilde{G}'_n = (\widetilde{V}'_n, \widetilde{E}'_n) \in \{\widetilde{G}'_n\}_{n\in\mathbb{N}}$. Just as in Lemma 1 from [14], we perform renormalization and obtain the graph $\widetilde{G}_n = (\widetilde{V}_n, \widetilde{E}_n) \in \{\widetilde{G}_n\}_{n\in\mathbb{N}}$.

Let a'_{-1} and a'_1 be real numbers satisfying the condition

$$[a'_{-1}n] + [a'_1 n] - 2q < -2[a'_{-1}n]$$

(required for the applicability of the linear algebraic method).

By Stirling's formula, we have $|\widetilde{V}_n| = (\rho_1 + o(1))^n$. In addition, it is known that

$$\alpha = \alpha(\widetilde{G}_n) \le (\rho_0 + o(1))^n.$$

Subsequent arguments are obvious, and the theorem is proved.

3.5. Proofs of the Propositions

Proposition 1 is precisely Statement 1 from [15], where its proof is given. Inequality (2) and equality (4) are analogs of inequality (7) and equality (8) of assertion 2 of Proposition 2 of [15] with $q' = (a'_1 + 3a'_{-1})/2$ and $q' = a'_1/2$, respectively. The proof is carried out in a similar way. Inequality (3) follows from inequality (1) of Proposition 1 with $i = 3$ and $j = 2$.

To obtain equality (5), we must calculate the number of ways needed to complete a prescribed triangle on vertices u, v, and w of the graph G_n to a 4-clique on vertices u, v, w, and y of this graph. We shall denote by $[tn]$ the number of unit coordinates in which all the four vectors intersect, by $[ln]$ the number of unit coordinates in which only u, v, and w intersect, by $[mn]$ the number of unit coordinates in which only u, v, and y intersect, by $[pn]$ the number of unit coordinates in which only u, w, and y intersect, and by $[rn]$ the number of unit coordinates in which only v, w, and y intersect; here t, l, m, p, and r are nonnegative real numbers. Let us find the maximum of this quantity with respect to t, l, m, p, and r (we denote it by $R(a'_1, b)$); then the number of all vectors generating the 4-clique containing the given triangle will not exceed

$$(n)^4 R(a'_1, x) = R(a'_1, x)(1 + o(1))^n.$$

It is easy to obtain the following expression:

$$\begin{aligned} R(a'_1, b, t, l, m, p, r) &= C^{[tn]}_{[tn]+[ln]} C^{[mn]}_{[bn]-[tn]-[ln]} C^{[pn]}_{[bn]-[tn]-[ln]} C^{[rn]}_{[bn]-[tn]-[ln]} \\ &\quad \times C^{[bn]-[tn]-[mn]-[pn]}_{[a'_1 n]-2[bn]+[tn]+[ln]} C^{[bn]-[tn]-[mn]-[rn]}_{[a'_1 n]-2[bn]+[tn]+[ln]} C^{[bn]-[tn]-[pn]-[rn]}_{[a'_1 n]-2[bn]+[tn]+[ln]} \\ &\quad \times C^{[a'_1 n]-3[bn]+2[tn]+[mn]+[pn]+[rn]}_{n-3[a'_1 n]+3[bn]-[tn]-[ln]} \\ &= \left(\frac{L}{M} + o(1)\right)^n. \end{aligned}$$

It is easy to see that

$$s^3_4(\{G_n\}_{n\in\mathbb{N}}) = \max_{t,l,m,p,r} \log_{1/((a'_1)^{a'_1}(1-a'_1)^{1-a'_1})} \frac{L}{M}.$$

The maximum of this expression with respect to the variables t, l, m, p, and r was found by computer, with constraints on them for each value of the parameters a'_1 and q' taken into account.

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