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Semigroup Classification and Gelfand–Shilov Classification of Systems of Partial Differential Equations

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Abstract—Two approaches to systems of linear partial differential equations are considered: the traditional approach based on the generalized Fourier transform and the semigroup approach, under which the system is considered as a particular case of an operator-differential equation. For these systems, the semigroup classification and the Gelfand–Shilov classification are compared.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The abstract Cauchy problem

$$u'(t) = Au(t) + W(t), \qquad t \in [0;\tau), \quad \tau \le \infty, \quad u(0) = f, \tag{1.1}$$

with operators A and inhomogeneities W requiring the search for the generalized solutions of the problem (in the spaces of distributions, ultradistributions, Gelfand–Shilov spaces, etc.), has been of long-standing interest and is topical to this day. At present, of special interest are problems (1.1) with infinite-dimensional white noise W defined as the generalized derivative of the Wiener process with values in a Hilbert space (see, e.g., [1], [2]). In recent years, such stochastic perturbations have been extensively studied, which enhanced the complexity of the problem under consideration, the problem, even being homogeneous and linear, is not well posed in the classical sense whenever the operator A does not generate a semigroup of class C_0. So if A is the generator of a regularized semigroup of operators (of an integrated, convolutional, or R-semigroup) on a Banach space X, then the solution operators $U(t)$, $t \ge 0$, of the corresponding homogeneous problem

$$u'(t) = Au(t), \quad t \in [0;\tau), \qquad u(0) = f \in \operatorname{dom} A \subset X, \tag{1.2}$$

which are an important part of the solution of problem (1.1):

$$u(t) = U(t)f + (U * W)(t), \qquad t \in [0;\tau),$$

constitute a family of unbounded operators.

Problems (1.2) with differential operators $A = A(i\,\partial/\partial x)$ constitute an independent and well-studied class of Cauchy problems for the systems of partial differential equations

$$\frac{\partial u(t,x)}{\partial t} = A\biggl(i\frac{\partial}{\partial x}\biggr)u(t,x), \qquad t \ge 0, \tag{1.3}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R}^q. \tag{1.4}$$

An effective method for studying the well-posedness of this problem is the Fourier transform, classical or generalized, which allows one to construct classical or generalized solutions, respectively, in the space

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variable. The choice of the space of test functions is determined by the behavior of the matrix exponential $e^{tA(\sigma)}$, the solution operator of the problem dual to (1.3), (1.4) with respect to the Fourier transform $\mathscr{F}$:

$$\frac{\partial v(t,\sigma)}{\partial t} = A(\sigma)v(t,\sigma), \quad t \geq 0, \qquad v(t,\sigma) := \mathscr{F}[u(t,x)], \tag{1.5}$$

$$v(0,\sigma) = \widetilde{f}(\sigma), \quad \sigma \in \mathbb{R}^q, \qquad \widetilde{f}(\sigma) := \mathscr{F}[f(x)]; \tag{1.6}$$

namely, this choice is determined by the growth order of the exponential and the behavior of its analytic continuation. On the basis of the analysis of these characteristics, in [3], all systems were divided into three classes (systems well-posed in the sense of Petrovskii, conditionally well-posed systems, and ill-posed systems) with two subclasses. This classification provides information about whether the system is ill posed, about the required generality for the construction of solutions, and about the types of classes of initial data for regular generalized solutions to exist.

Obviously, along with this approach to the study of problem (1.3), (1.4), one can also apply the powerful apparatus of the semigroup theory of operators, classical and regularized, developed for abstract Cauchy problems (1.2) in Banach spaces, as well as in the spaces of distributions and ultradistributions. In this theory, the Laplace transform (classical and generalized) plays the same role as the Fourier transform in the study of systems of differential equations, namely, it connects the solution operators of problem (1.2) with the resolvent of the operator A: the Laplace transform of the solution operators of problem (1.2) gives the resolvent of the operator A, while the generalized solution operators of problem (1.2) are constructed as the inverse Laplace transform of the resolvent. As a result, problem (1.2) can be classified by the type of the semigroup generated by the operator A, by the behavior of its resolvent, and by the spaces of distributions in which the problem is well posed; naturally, these things are closely related.

On the basis of the preliminary comparison of very similar (in form) estimates of the norm of the resolvent of the operator A generating different semigroups and estimates of the matrix functions $e^{tA(\sigma)}$ for different classes of differential systems, the authors conjectured that the generators of C_0-semigroups and those of integrated semigroups yield systems well-posed in the sense of Petrovskii, the generators of convolutional semigroups yield conditionally well-posed systems, and generators of R-semigroups yield ill-posed systems. But, as the results obtained below show, the situation is not that simple.

In the present paper, we compare two types of classification of the Cauchy problem (1.3), (1.4): the semigroup classification and the Gelfand–Shilov classification. We consider the case of an $m \times m$ matrix differential operator $A(i\,\partial/\partial x)$ whose components are linear differential operators with constant coefficients of maximal order p. To make such a comparison possible, we study problem (1.2) in the function space

$$X = L_2^m(\mathbb{R}^q) = L_2(\mathbb{R}^q) \times \cdots \times L_2(\mathbb{R}^q)$$

of vector functions $f(x) = (f_1(x), \dots, f_m(x))$, $x \in \mathbb{R}^q$, with norm

$$\|f\| = \sqrt{\sum_{j=1}^{m} \|f_j\|^2_{L_2(\mathbb{R}^q)}}\,.$$

In this space, the domain $\operatorname{dom} A = \{f \in X : f^{(p)} \in X\}$ of the operator $A = A(i\,\partial/\partial x)$ is composed of vector functions f whose components $f_j(x)$ are functions p-times differentiable in the space $L_2(\mathbb{R}^q)$.

2. SEMIGROUP CLASSIFICATION OF AN ABSTRACT CAUCHY PROBLEM AND ITS GENERALIZED SOLUTIONS

Let us give a short summary of the main results in the semigroup theory of operators and those related to the well-posedness of problem (1.2). In what follows, we rely mainly on [4]. For additional information on the subject, see, e.g., [5]–[9].

Let X be a Banach space, and let A be a closed linear operator on X with domain $\operatorname{dom} A$. By a *solution of problem* (1.2) on the closed interval $[0;T] \subset [0;\tau)$ we shall mean

$$u \in C([0;T], \operatorname{dom} A) \cap C^1([0;T], X).$$

2.1. Let A be an operator densely defined on the space X. The operator A generates a C_0-semigroup of operators on X if and only if any one of the following (equivalent) conditions hold:

- problem (1.2) is uniformly well posed on $\operatorname{dom} A$ for $t \geq 0$, i.e., for all $T > 0$ and $f \in \operatorname{dom} A$,
  - (a) there exists a unique solution of problem (1.2) on $[0; T]$;
  - (b) the solution is stable with respect to the variation of the initial data, uniformly in $t \in [0; T]$:

$$\sup_{t\in[0;T]} \|u(t)\| \leq C_T \|f\|;$$

- $\mathscr{R}(\lambda)$ is the resolvent of the operator A defined on some domain of the right half-plane of the complex plane $\operatorname{Re}\lambda > \omega$ and, for some $C > 0$,

$$\|\mathscr{R}^{(k)}(\lambda)\|_{\mathscr{L}(X)} \leq \frac{Ck!}{(\operatorname{Re}\lambda - \omega)^{k+1}}, \qquad \operatorname{Re}\lambda > \omega, \quad k \in \mathbb{N}_0. \tag{2.1}$$

Remark 1. Note that the estimate

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} \leq \frac{1}{\operatorname{Re}\lambda - \omega}, \qquad \operatorname{Re}\lambda > \omega, \tag{2.2}$$

ensures the validity of the series of estimates (2.1).

2.2. Let A be an operator densely defined on the space X, and let the set of its regular points be not empty: $\rho(A) \neq \varnothing$. The operator A generates a (nondegenerate) n-times integrated exponentially bounded semigroup of operators on the space X if and only if any one of the following conditions holds:

- problem (1.2) is uniformly (n, ω)-well posed for $t \geq 0$, i.e., for all $T > 0$ and $f \in \operatorname{dom} A^{n+1}$,
  - (a) there exists a unique solution of problem (1.2) on $[0; T]$;
  - (b) the solution is stable with respect to the variation of the initial data in the graph norm of the operator A:

$$\|u(t)\| \leq Ce^{\omega t}\|f\|_n, \qquad t \geq 0,$$

  where $\|f\|_n = \|f\| + \|Af\| + \dots + \|A^n f\|$;

- the resolvent of the operator A is defined on the right half-plane of the complex plane $\operatorname{Re}\lambda > \omega$ and, for some $C > 0$,

$$\left\| \frac{d^k}{d\lambda^k}\left(\frac{\mathscr{R}(\lambda)}{\lambda^n}\right)\right\|_{\mathscr{L}(X)} \leq \frac{Ck!}{(\operatorname{Re}\lambda - \omega)^{k+1}}, \qquad \operatorname{Re}\lambda > \omega, \quad k \in \mathbb{N}_0; \tag{2.3}$$

- problem (1.2) is well posed in the space of X-valued exponentially bounded distributions $\mathscr{S}'_\omega(X)$, which is defined as follows:

$$f \in \mathscr{S}'_\omega(X) \qquad \text{if and only if} \quad fe^{\omega t} \in \mathscr{S}'(X) := \mathscr{L}(\mathscr{S}, X).$$

2.3. Let A be an operator densely defined in X, and let $\rho(A) \neq \varnothing$. The operator A generates a local (on $[0; \tau)$) n-times integrated semigroup of operators in the space X if and only if any of the following conditions holds:

- problem (1.2) is n-well posed for $t \in [0; \tau)$, i.e., for all $f \in \operatorname{dom} A^{n+1}$ and $T < \tau$,
  - (a) there exists a unique solution of problem (1.2) on $[0; T]$;
  - (b) there exists a $C_T > 0$ such that

$$\sup_{t\in[0;T]} \|u(t)\| \leq C_T \|f\|_n;$$

- problem (1.2) is well posed in the space of distributions $\mathscr{D}'(X) := \mathscr{L}(\mathscr{D}, X)$.

It follows from these equivalent conditions that the resolvent of the operator A is defined on the domain

$$\Lambda^{\ln}_{n,\gamma,\omega} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > n\gamma \ln|\lambda| + \omega\}, \qquad n \in \mathbb{N}, \quad \gamma > 0, \quad \omega \in \mathbb{R},$$

and, for some $C > 0$,

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} \le C|\lambda|^n, \qquad \lambda \in \Lambda^{\ln}_{n,\gamma,\omega}. \tag{2.4}$$

In turn, using estimate (2.4), we can prove in the general case that the operator A generates a local $n+2$-times integrated semigroup.

2.4. Let $K(t)$, $t \ge 0$, be an exponentially bounded function such that

$$\exists\, \theta \in \mathbb{R} \qquad |K(t)| \le Ce^{\theta t}, \quad t \ge 0, \tag{2.5}$$

and let, as $|\lambda| \to \infty$, its Laplace transform $\mathscr{L}[K](\lambda)$, $\operatorname{Re}\lambda > \theta$, satisfy the condition

$$|\mathscr{L}[K](\lambda)| = \mathscr{O}(e^{-M(\kappa|\lambda|)}),$$

where $M(\xi)$ is a positive function of the variable $\xi \ge 0$ increasing as $\xi \to \infty$ not faster than ξ^p, $p < 1$.

On $[0;\tau)$, $\tau \le \infty$, the operator A generates a K-convolutional semigroup of operators in the space X if and only if any of the following conditions holds:

- the resolvent of the operator A is defined on some domain of the right half-plane

$$\Lambda^M_{\alpha,\gamma,\omega} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \alpha M(\gamma|\lambda|) + \omega\} \tag{2.6}$$

  and satisfies the condition

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} \le Ce^{\beta M(\gamma|\lambda|)}, \qquad \lambda \in \Lambda^M_{\alpha,\gamma,\omega} \tag{2.7}$$

  on this domain;

- if $M(\xi)$ is the Borel transform of the sequence $\{M_k\}$, then problem (1.2) is well posed in the space of abstract Roumier ultradistributions

$$(\mathscr{D}^{\{M_k\},B}_a)'(X) := \mathscr{L}(\mathscr{D}^{\{M_k\},B}_a, X).$$

2.5. Let R be a bounded operator in the space X. Let A be an operator densely defined in X, commuting with R on its domain, and satisfying the condition

$$\overline{A|_{R(\operatorname{dom} A)}} = A.$$

The operator A generates a local (on $[0;\tau)$) R-semigroup of operators on the space X if and only if any of the following conditions holds:

- problem (1.2) is R-well posed on $[0;\tau)$, i.e., for all $f \in R(\operatorname{dom} A)$ and $T < \tau$,
  - (a) there exists a unique solution of problem (1.2) on the closed interval $[0;T]$;
  - (b) there exists a $C_T > 0$ such that

$$\sup_{t\in[0;T]} \|u(t)\| \le C_T\|R^{-1}f\|;$$

- for any $t \in [0;\tau)$, there exists an asymptotic R-resolvent $\mathscr{R}_t(\lambda)$ of the operator A satisfying the following condition for some $C_t > 0$:

$$\left\|\frac{d^k}{d\lambda^k}\mathscr{R}_t(\lambda)\right\|_{\mathscr{L}(X)} \le \frac{C_t k!}{|\lambda|^{k+1}}, \qquad \frac{k}{\lambda} \in [0;t], \quad \lambda > 0, \quad k \in \mathbb{N}_0.$$

3. CLASSIFICATION OF DIFFERENTIAL SYSTEMS

The classification of the systems of linear differential equations (1.3) is based on the behavior of the solution operators of the dual system (1.5) obtained by applying the generalized Fourier transform to system (1.3) [3]. The solution operator of the Cauchy problem for system (1.5) is the matrix exponential $e^{tA(\sigma)}$ whose growth is determined by the condition

$$e^{t\Lambda(s)} \le \|e^{tA(s)}\|_m \le C(1+|s|)^{p(m-1)}e^{t\Lambda(s)}, \qquad t \ge 0, \quad s = \sigma + i\tau \in \mathbb{C}^q,$$

where $\|\cdot\|_m$ is the norm of the linear operator on m-dimensional Euclidean space endowed with the standard inner product,

$$\Lambda(s) := \max_{1\le k\le m} \operatorname{Re}\lambda_k(s), \qquad s \in \mathbb{C}^q,$$

and the functions $\lambda_1(s),\dots,\lambda_m(s)$ are the roots of the characteristic equation

$$\det(\lambda I - A(s)) = 0, \qquad s \in \mathbb{C}^q.$$

The number

$$p_0 = \inf\{\rho : |\Lambda(s)| \le C_\rho(1+|s|)^\rho,\ s \in \mathbb{C}^q\},$$

called the *sharp power growth order* of the function $\Lambda(\cdot)$ and the *reduced order of system* (1.3), determines the growth order of the exponential globally in the complex plane:

$$\|e^{tA(s)}\|_m \le C(1+|s|)^{p(m-1)}e^{bt\cdot|s|^{p_0}}, \qquad b \in \mathbb{R}, \quad t \ge 0, \quad s \in \mathbb{C}^q. \tag{3.1}$$

Estimate (3.1) can be refined in some neighborhood of the real line by using properties of analytic functions and the behavior of the function $\Lambda(\cdot)$ for the real values of the argument. On the basis of this, we single out the following classes of systems (1.3).

(1) System (1.3) is said to be *well-posed in the sense of Petrovskii* if there exists a constant $C > 0$ such that

$$\Lambda(\sigma) \le C, \qquad \sigma \in \mathbb{R}^q; \tag{3.2}$$

important subclasses of systems well-posed in the sense of Petrovskii are as follows:

(1a) *parabolic systems*, i.e., systems for which there exist constants $C > 0$, $h > 0$, and $C_1 > 0$ such that

$$\Lambda(\sigma) \le -C_1|\sigma|^h + C, \qquad \sigma \in \mathbb{R}^q;$$

(1b) *hyperbolic systems*, i.e., systems whose sharp power growth order p_0 is at most 1:

$$\Lambda(s) \le C_1|s| + C_2, \qquad s \in \mathbb{C}^q,$$

and the well-posedness condition in the sense of Petrovskii (3.2) holds.

(2) System (1.3) is said to be *conditionally well-posed* if there exist constants $C > 0$, $0 < h < 1$, and $C_1 > 0$ such that

$$\Lambda(\sigma) \le C|\sigma|^h + C_1, \qquad \sigma \in \mathbb{R}^q;$$

(3) System (1.3) is said to be *ill-posed* if the function $\Lambda(\cdot)$ increases for real $s = \sigma$ just as in the case of complex values:

$$\Lambda(\sigma) \le C|\sigma|^{p_0} + C_1, \qquad \sigma \in \mathbb{R}^q.$$

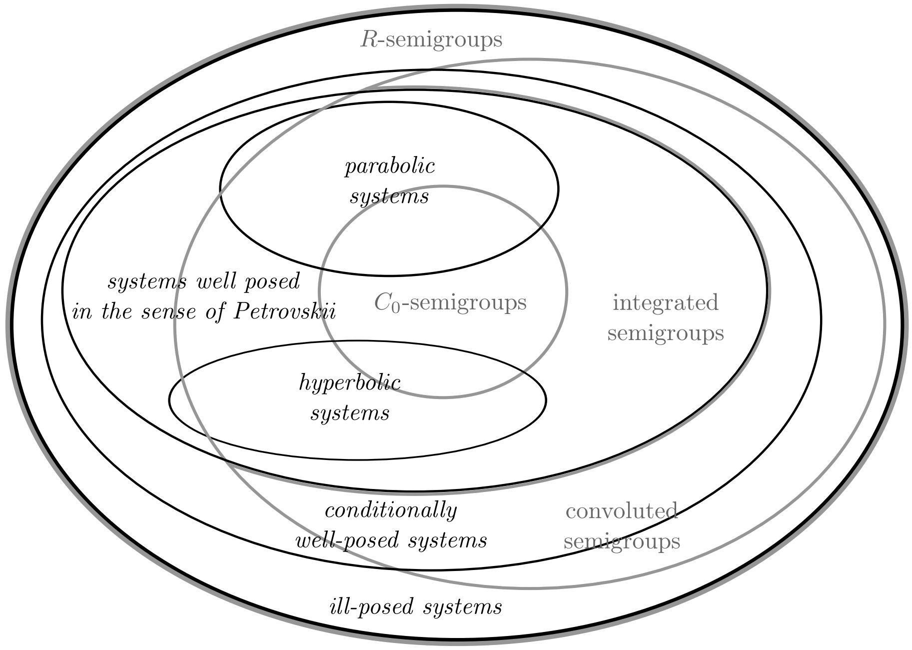

Figure.

4. MAIN RESULTS

We present the following relationship between the "semigroup" and "differential" classifications, whose proof consists of three theorems and two propositions and is illustrated by examples given below.

In the figure, the terms, such as "integrated semigroup," refer to the class of problems with operators A generating integrated semigroups.

Theorem 1. *If $A = A(i\,\partial/\partial x)$ generates a C_0-semigroup or an integrated semigroup (exponentially bounded or local) on the space $X = L_2^m(\mathbb{R}^q)$, then system (1.3) is well posed in the sense of Petrovskii.*

Proof. To begin with, we note that the operator $A = A(i\,\partial/\partial x)$ and the operator of multiplication by the matrix $A(\sigma)$ have a common resolvent set in the space $L_2^m(\mathbb{R}^q)$ because of the equality of the norms of their resolvents; therefore, they have a common spectrum.

Further, the spectrum of the operator of multiplication by the matrix $A(\sigma)$ contains all characteristic curves

$$\lambda = \lambda_j(\sigma), \qquad j = 1, \dots, m,$$

which are power functions of the parameter $\sigma \in \mathbb{R}^q$. In this case, the real and imaginary parts of these curves satisfy power relations.

Let us assume that system (1.3) is not well-posed in the sense of Petrovskii, i.e., there exists a characteristic curve $\lambda = \lambda_k(\sigma)$ for which the estimate $\operatorname{Re}\lambda_k(\sigma) \le C$ does not hold and, hence,

$$\forall\, C > 0 \quad \exists\, \sigma \in \mathbb{R}^q: \qquad \operatorname{Re}\lambda_k(\sigma) > C. \tag{4.1}$$

If the operator $A = A(i\,\partial/\partial x)$ generates a C_0-semigroup or an exponentially bounded n-times integrated semigroup, then its resolvent set contains the right half-plane $\operatorname{Re}\lambda > \omega$. In this case, condition (4.1) means that the set of regular points of the operator A contains points of the spectrum.

If A is the generator of a local n-times integrated semigroup, then its resolvent set contains the domain

$$\Lambda^{\ln}_{n,\gamma,\omega} = \{\lambda = \mu + i\eta \in \mathbb{C} : \mu > n\gamma \ln|\lambda| + \omega\}, \qquad n \in \mathbb{N}, \quad \gamma > 0, \quad \omega \in \mathbb{R},$$

whose boundary is given by the equality

$$\mu^2 + \eta^2 = e^{2(\mu-\omega)/(n\gamma)} \quad \Longleftrightarrow \quad \eta = \pm\sqrt{e^{2(\mu-\omega)/(n\gamma)} - \mu^2},$$

i.e., the real and imaginary parts of this curve satisfy the exponential relation $\eta = \mathscr{O}(e^{\mu})$ as $\mu \to +\infty$. In this case, in view of the power relations between μ_k and η_k, the characteristic curve

$$\lambda = \lambda_k(\sigma) = \mu_k(\sigma) + i\eta_k(\sigma)$$

intersects the boundary of the domain $\Lambda^{\ln}_{n,\gamma,\omega}$ as $\mu_k \to +\infty$ and thus is in the set of regular points of the operator A.

The resulting contradictions prove that system (1.3) is well posed in the sense of Petrovskii. □

Example 1. *An equation well-posed in the sense of Petrovskii with operator $A = A(i\,\partial/\partial x)$ generating a C_0-semigroup.*

Consider the Schrödinger equation

$$\frac{\partial u(t,x)}{\partial t} = i\,\frac{\partial^2 u(t,x)}{\partial x^2}, \qquad x \in \mathbb{R}, \quad t \ge 0. \tag{4.2}$$

Here $A = A(i\,\partial/\partial x) = -i(i\,\partial/\partial x)^2$ and the characteristic root is $\lambda(s) = -is^2$. We have the function

$$\Lambda(s) = 2\sigma\tau, \qquad \Lambda(\sigma) = 0;$$

hence (4.2) belongs to the class of systems well-posed in the sense of Petrovskii. The equation is neither parabolic nor hyperbolic.

Let us find what semigroup is generated by the operator A on the space $X = L_2(\mathbb{R})$ from the behavior of its resolvent. We shall use the fact that the Fourier operator on $L_2(\mathbb{R})$ is isometric and estimate the norm of the resolvent of the operator of multiplication by the function $A(\sigma) = -i\sigma^2$, the operator dual to A with respect to the Fourier transform:

$$(\lambda - A(\sigma))\widetilde{f}(\sigma) = \widetilde{g}(\sigma) \quad \Longleftrightarrow \quad \widetilde{f}(\sigma) = \frac{\widetilde{g}(\sigma)}{\lambda - A(\sigma)} = \frac{\widetilde{g}(\sigma)}{\lambda + i\sigma^2}, \qquad \sigma \in \mathbb{R}.$$

This implies

$$\|f\|^2 = \|\widetilde{f}\|^2 = \int_{\mathbb{R}} \frac{|\widetilde{g}(\sigma)|^2}{|\lambda + i\sigma^2|^2}\,d\sigma \le \max_{\sigma\in\mathbb{R}} \frac{1}{|\lambda + i\sigma^2|^2}\,\|\widetilde{g}\|^2 = \frac{\|g\|^2}{(\operatorname{Re}\lambda)^2};$$

therefore,

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} \le \frac{1}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 0,$$

i.e., condition (2.2) holds, and hence the operator A is the generator of a C_0-semigroup.

Proposition 1. *The classes of hyperbolic and parabolic systems do not intersect.*

Proof. The reduced order of the parabolic system is greater than 1 [3], while, for a hyperbolic system, $p_0 \le 1$. □

Proposition 2. *The operator of a parabolic system can be the generator of a C_0-semigroup, of an integrated semigroup, and of an R-semigroup.*

The proof of this assertion is given by Example 2 and also by the following result on the generation of an R-semigroup by the operator $A = A(i\,\partial/\partial x)$ for each of the classes of differential systems; this result was proved in [10].

Theorem 2. *Let $X = L_2^m(\mathbb{R}^q)$, and let $\tau > 0$. Let $e^{tA(\sigma)}$ be the solution operator of problem* (1.5), (1.6). *Let $\mathscr{K}(\sigma)$, $\sigma \in \mathbb{R}^q$, be a function with the following properties: for any $T < \tau$,*

a) *the matrix function* $\mathscr{K}(\sigma)e^{tA(\sigma)}$ *is uniformly bounded on* $[0;T]\times\mathbb{R}^q$;

b) $\mathscr{K}(\,\cdot\,)e^{TA(\,\cdot\,)}\in X\times X$.

Then the family of convolution operators with kernel

$$G_{\mathscr{K}}(t,x):=\frac{1}{(2\pi)^q}\int_{\mathbb{R}^q}e^{i\langle\sigma,x\rangle}\mathscr{K}(\sigma)e^{tA(\sigma)}\,d\sigma$$

constitutes a local R*-semigroup of operators on the space* X *with operator* R:

$$Rf(x)=\frac{1}{(2\pi)^q}\int_{\mathbb{R}^q}e^{i\langle\sigma,x\rangle}\mathscr{K}(\sigma)\widetilde{f}(\sigma)\,d\sigma.$$

Here the function $\mathscr{K}(\sigma)$ *satisfying the conditions stated above is chosen as follows*:

1) *for systems well-posed in the sense of Petrovskii,*

$$\mathscr{K}(\sigma)=(1+|\sigma|^2)^{-p(m-1)/2-1};$$

2) *for conditionally well-posed systems,*

$$\mathscr{K}(\sigma)=e^{-a|\sigma|^h},\qquad \textit{where}\quad a>\mathrm{const}\cdot T;$$

3) *for ill-posed systems,*

$$\mathscr{K}(\sigma)=e^{-a|\sigma|^{p_0}},\qquad \textit{where}\quad a>\mathrm{const}\cdot T.$$

Example 2. *A parabolic system with operator* $A=A(i\,\partial/\partial x)$ *generating, depending on the parameter, a* C_0*-semigroup, an integrated exponentially bounded, or an* R*-semigroup.*

Consider the system of differential equations

$$\begin{cases}\dfrac{\partial u_1(t,x)}{\partial t}&=\dfrac{\partial^2 u_1(t,x)}{\partial x^2},\\[2mm] \dfrac{\partial u_2(t,x)}{\partial t}&=i^\nu\dfrac{\partial^\nu u_1(t,x)}{\partial x^\nu}+\dfrac{\partial^2 u_2(t,x)}{\partial x^2},\end{cases}\qquad x\in\mathbb{R},\quad t\ge 0,\quad \nu\in\mathbb{N}_0. \tag{4.3}$$

The operator $A=A(i\,\partial/\partial x)$ of this system and the matrix $A(s)$ are of the form

$$A=\begin{pmatrix}\dfrac{\partial^2}{\partial x^2} & 0\\[2mm] i^\nu\dfrac{\partial^\nu}{\partial x^\nu} & \dfrac{\partial^2}{\partial x^2}\end{pmatrix},\qquad A(s)=\begin{pmatrix}-s^2 & 0\\ s^\nu & -s^2\end{pmatrix}.$$

The characteristic root of system (4.3) is $\lambda(s)=-s^2$ and, therefore, $\Lambda(\sigma)=-\sigma^2$, and system (4.3) is parabolic for any ν.

Let us find what semigroup is generated by the operator $A=A(i\,\partial/\partial x)$ on the space $X=L_2(\mathbb{R})$. To do this, just as in the previous example, we estimate the norm of the resolvent of the operator of multiplication by the matrix $A(\sigma)$:

$$(\lambda I-A(\sigma))^{-1}=\frac{1}{(\lambda+\sigma^2)^2}\begin{pmatrix}\lambda+\sigma^2 & 0\\ \sigma^\nu & \lambda+\sigma^2\end{pmatrix},$$

$$\|(\lambda I-A(\sigma))^{-1}\|_{\mathscr{L}(X)}=\max\biggl\{\sup_{\sigma\in\mathbb{R}}\frac{1}{|\lambda+\sigma^2|},\,\sup_{\sigma\in\mathbb{R}}\frac{|\sigma|^\nu}{|\lambda+\sigma^2|^2}\biggr\}.$$

For the first function, in the case $\operatorname{Re}\lambda > 0$, we obtain

$$\sup_{\sigma\in\mathbb{R}} \frac{1}{|\lambda+\sigma^2|} \le \sup_{\sigma\in\mathbb{R}} \frac{1}{\operatorname{Re}\lambda+\sigma^2} = \frac{1}{\operatorname{Re}\lambda}.$$

For the second function, in the case $\operatorname{Re}\lambda > 0$,

$$\sup_{\sigma\in\mathbb{R}} \frac{|\sigma|^\nu}{|\lambda+\sigma^2|^2} \le \sup_{\sigma\in\mathbb{R}} \frac{|\sigma|^\nu}{(\operatorname{Re}\lambda+\sigma^2)^2} = \begin{cases} \dfrac{\nu^{\nu/2}(4-\nu)^{2-\nu/2}}{16}\cdot\dfrac{1}{(\operatorname{Re}\lambda)^{2-\nu/2}} & \text{for } \nu<4,\\ 1 & \text{for } \nu=4,\\ \infty & \text{for } \nu>4. \end{cases}$$

Thus, for $\nu = 0, 1, 2$,

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} = \|(\lambda I - A(\sigma))^{-1}\|_{\mathscr{L}(X)} \le \frac{1}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 1.$$

We have obtained estimate (2.2); therefore, the operator A is the generator of a C_0-semigroup.

For $\nu = 3$,

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} = \|(\lambda I - A(\sigma))^{-1}\|_{\mathscr{L}(X)} \le \frac{3^{3/2}}{16}\cdot\frac{1}{\sqrt{\operatorname{Re}\lambda}}, \qquad \operatorname{Re}\lambda > 1; \tag{4.4}$$

this estimate is attainable, and hence the operator A is not the generator of a C_0-semigroup. On the basis of the inequalities

$$\left\|\frac{\mathscr{R}(\lambda)}{\lambda}\right\|_{\mathscr{L}(X)} \le \frac{1}{(\operatorname{Re}\lambda)^{3/2}} < \frac{1}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 1, \tag{4.5}$$

we can assume that the operator A generates a 1-time integrated semigroup. However, in contrast to C_0-semigroups, inequalities (4.5) do not imply the series of estimates (2.3) with common constant $C > 0$. So, in the present example, the derivative of order k satisfies the following estimate:

$$\left\|\frac{d^k}{d\lambda^k}\left(\frac{\mathscr{R}(\lambda)}{\lambda}\right)\right\|_{\mathscr{L}(X)} \le \frac{k!}{(\operatorname{Re}\lambda)^{k+3/2}}\left(\frac{3}{2}\right)^{3/2}\left[\frac{1}{4\sqrt{2}} + \sum_{m=1}^{k}\frac{m+1}{m^{3/2}}\cdot\frac{(1+1/(2m))^{m+1/2}}{(1+2/m)^{m+2}}\right]$$

for $\operatorname{Re}\lambda > \omega_k$, where $\omega_k \to +\infty$ as $k \to \infty$; this estimate is attained on the real axis, and the sum in brackets increases without limit. It follows that A is not a generator of once integrated semigroup of operators on X.

Instead of studying estimates of the norms of operators of the form $\mathscr{R}(\lambda)/\lambda^n$ in search for a concrete n for which conditions (2.3) hold, we shall argue as follows: it suffices to use one estimate (4.4) for constructing a generalized solution of the Cauchy problem for system (4.3) in the space of distributions $S'_\omega(X)$. By the structural theorem in the space $S'_\omega(X)$, this solution is the generalized derivative of order n_0 of a continuous X-valued function. At the same time, the Laplace transform of the generalized solution operator coincides with the resolvent of the operator A and satisfies the series of estimates (2.3) with $n = n_0$ (for the proof of these facts, see [8] and [4]). This means that A generates an n_0-times integrated exponentially bounded semigroup.

For $\nu = 4$, the situation is similar: the resolvent of the operator A exists in the right half-plane and satisfies the estimates

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} = \|(\lambda I - A(\sigma))^{-1}\|_{\mathscr{L}(X)} \le 1,$$

$$\left\|\frac{\mathscr{R}(\lambda)}{\lambda}\right\|_{\mathscr{L}(X)} = \left\|\frac{1}{\lambda}(\lambda I - A(\sigma))^{-1}\right\|_{\mathscr{L}(X)} \le \frac{1}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 0;$$

for $n = 1$, the series of estimates (2.3) do not hold with one constant $C > 0$, and hence the operator A does not generate a once integrated semigroup, but, just as in the similar case $\nu = 3$, A is the generator of an n-times ($n \ge n_0$) integrated exponentially bounded semigroup.

For $\nu > 4$, the operator $(\lambda I - A(\sigma))^{-1}$ is unbounded for $\lambda > 0$; therefore, the operator A generates neither integrated nor convolutional C_0-semigroup. By Theorem 2, A is the generator of an R-semigroup on the space X.

The operators of hyperbolic systems can also be the generators of C_0-semigroups, of integrated semigroups, and of R-semigroups (see Examples 3–5). If the operator of a hyperbolic system has a regular point, we can prove the following refining result.

Theorem 3. *Let $A = A(i\,\partial/\partial x)$ be the operator of the hyperbolic system* (1.3)*, and let $\rho(A) \neq \varnothing$. Then A generates either a C_0-semigroup or an integrated semigroup on the space $X = L_2^m(\mathbb{R}^q)$.*

Proof. For any $T > 0$ and $t \in [0;T]$, the matrix function $e^{tA(s)}$ of the hyperbolic system (1.3) satisfies the following estimates:

$$\|e^{tA(s)}\|_m \le C(1+|s|)^{p(m-1)}e^{bT\cdot|s|}, \qquad s \in \mathbb{C}^q,$$
$$\|e^{tA(\sigma)}\|_m \le C_T(1+|\sigma|)^{p(m-1)}, \qquad \sigma \in \mathbb{R}^q.$$

By the Paley–Wiener–Schwarz theorem, it follows that, for each $t \in [0;T]$,

$$G(t,\,\cdot\,) = \mathscr{F}^{-1}[e^{tA(\sigma)}]$$

is an element of the space $\mathscr{D}'$ with support $\operatorname{supp} G(t,\,\cdot\,) = \{x \in \mathbb{R}^q : |x| \le bT\}$ for any $\varepsilon > 0$ expressible as

$$G(t,x) = P\biggl(\frac{\partial}{\partial x}\biggr)g(t,x), \qquad x \in \mathbb{R}^q, \quad t \in [0;T],$$

where $g(t,x) = \{g_{l,j}(t,x)\}_{l,j=1}^m$ is a continuous matrix function of the variable x with support in the domain $|x| < bT + \varepsilon$ and $P(\partial/\partial x)$ is a differential operator of order at most $p(m-1)+q$.

A solution of problem (1.3), (1.4) is the generalized function (distribution)

$$u(t,x) = \mathscr{F}^{-1}[e^{tA(\sigma)}\widetilde{f}(\sigma)] = G(t,x) * f(x) = P\biggl(\frac{\partial}{\partial x}\biggr)g(t,x) * f(x), \quad t \in [0;T].$$

If the initial function $f(\,\cdot\,)$ is differentiable a sufficient number of times, then, by the properties of convolution, the solution is the convolution of two continuous functions, one of which is compactly supported:

$$u(t,x) = g(t,x) * P\biggl(\frac{\partial}{\partial x}\biggr)f(x),$$

i.e., the solution is a continuous function of the variable x. The continuity and differentiability of the solution with respect to t are proved in the same way as in Theorem 2 from [10].

Further, using the inequality

$$\|g * f\|_{L_p(\mathbb{R}^q)} \le \|g\|_{L_1(\mathbb{R}^q)} \cdot \|f\|_{L_p(\mathbb{R}^q)}, \qquad p \ge 1,$$

we obtain

$$\|u(t,\,\cdot\,)\|_{L_2^m(\mathbb{R}^q)} \le C\max_{l,j}\|g_{l,j}(t,\,\cdot\,)\|_{L_1(\mathbb{R}^q)}\|P\biggl(\frac{\partial}{\partial x}\biggr)f(x)\|_{L_2^m(\mathbb{R}^q)}, \qquad t \in [0;T],$$

where the integral of the functions $g_{l,j}(t,\,\cdot\,)$ is taken over the bounded support of these functions. Hence, for $f \in \operatorname{dom} A^{n+1}$, under the condition $np \ge p(m-1)+q$, it follows that the solution is n-stable:

$$\sup_{t\in[0;T]}\|u(t,\,\cdot\,)\| \le C_T\|f\|_n,$$

which means that the Cauchy problem is n-well-posed for system (1.3) on any half-interval $[0;\tau)$. Since the resolvent set of the operator A is nonempty, it follows that A is the generator of a local n-times integrated semigroup of operators on the space X. $\square$

Let us give examples of hyperbolic systems whose operators generate a C_0-semigroup, an integrated semigroup, and an R-semigroup.

Example 3. *A hyperbolic system with operator $A = A(i\,\partial/\partial x)$ generating a C_0-semigroup.* Consider the system of differential equations

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = \dfrac{\partial u_2(t,x)}{\partial x}, \\ \dfrac{\partial u_2(t,x)}{\partial t} = \dfrac{\partial u_1(t,x)}{\partial x}, \end{cases} \qquad x \in \mathbb{R}, \quad t \ge 0. \tag{4.6}$$

The operator $A = A(i\,\partial/\partial x)$ of this system and the matrix $A(s)$ are of the form

$$A = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & 0 \end{pmatrix}, \qquad A(s) = \begin{pmatrix} 0 & -is \\ -is & 0 \end{pmatrix}.$$

Then

$$\lambda_{1,2}(s) = \pm is, \qquad \Lambda(s) \le |s|, \qquad \Lambda(\sigma) = 0,$$

i.e., system (4.6) is well-posed in the sense of Petrovskii and hyperbolic.

The resolvent of the operator of multiplication by the matrix $A(\sigma)$ is of the form

$$(\lambda I - A(\sigma))^{-1} = \frac{1}{\lambda^2 + \sigma^2} \begin{pmatrix} \lambda & -i\sigma \\ -i\sigma & \lambda, \end{pmatrix}$$

and if $\operatorname{Re}\lambda > 0$, then, on the space $X = L_2(\mathbb{R})$, we obtain

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} = \|(\lambda I - A(\sigma))^{-1}\|_{\mathscr{L}(X)} = \sup_{\sigma\in\mathbb{R}} \left\{ \frac{|\lambda|}{|\lambda^2 + \sigma^2|}, \frac{|\sigma|}{|\lambda^2 + \sigma^2|} \right\} \le \frac{1}{\operatorname{Re}\lambda};$$

therefore, the operator $A = A(i\,\partial/\partial x)$ generates a C_0-semigroup of operators on the space X.

Example 4. *A hyperbolic system with operator $A = A(i\,\partial/\partial x)$ generating an integrated exponentially bounded semigroup.* Consider the hyperbolic equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2}, \qquad x \in \mathbb{R}, \quad t \ge 0.$$

We shall write it as a system of first-order differential equations in t:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = u_2(t,x), \\ \dfrac{\partial u_2(t,x)}{\partial t} = \dfrac{\partial^2 u_1(t,x)}{\partial x^2}, \end{cases} \qquad x \in \mathbb{R}, \quad t \ge 0. \tag{4.7}$$

For this system, the operator $A = A(i\,\partial/\partial x)$ and the matrix $A(s)$ are of the form

$$A = \begin{pmatrix} 0 & I \\ \dfrac{\partial^2}{\partial x^2} & 0 \end{pmatrix}, \qquad A(s) = \begin{pmatrix} 0 & 1 \\ -s^2 & 0 \end{pmatrix}.$$

Let us find the characteristic roots:

$$\det(A(s) - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -s^2 & -\lambda \end{vmatrix} = \lambda^2 + s^2 = 0, \qquad \lambda_{1,2}(s) = \pm is = \pm i(\sigma + i\tau).$$

Then

$$\Lambda(s) = \max\{\operatorname{Re}\lambda_1(s), \operatorname{Re}\lambda_2(s)\} = \max\{\pm\tau\} = |\tau|, \qquad \Lambda(\sigma) = 0,$$

and hence system (4.7) is well posed in the sense of Petrovskii. The growth order of the function $\Lambda(s)$ in the complex plane is equal to 1:

$$\Lambda(s) = |\tau| \le |s|, \qquad s = \sigma + i\tau \in \mathbb{C};$$

therefore, system (4.7) is hyperbolic.

The resolvent of the operator of multiplication by the matrix $A(\sigma)$ is of the form

$$(\lambda I - A(\sigma))^{-1} = \frac{1}{\lambda^2 + \sigma^2} \begin{pmatrix} \lambda & 1 \\ -\sigma^2 & \lambda, \end{pmatrix}$$

and if $\operatorname{Re}\lambda > 0$, then, on the space $X = L_2(\mathbb{R})$, we obtain

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} = \|(\lambda I - A(\sigma))^{-1}\|_{\mathscr{L}(X)} = \sup_{\sigma\in\mathbb{R}} \left\{ \frac{1}{|\lambda^2+\sigma^2|}, \frac{|\lambda|}{|\lambda^2+\sigma^2|}, \frac{|\sigma|^2}{|\lambda^2+\sigma^2|} \right\} = 1.$$

In this case, just as in Example 2, for $\nu = 3, 4$, the operator A is the generator of an integrated exponentially bounded semigroup of operators on X.

Example 5. *A hyperbolic system with operator $A = A(i\,\partial/\partial x)$ generating an R-semigroup.* Consider the system of differential equations

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = \dfrac{\partial u_1(t,x)}{\partial x}, \\ \dfrac{\partial u_2(t,x)}{\partial t} = i^\kappa \dfrac{\partial^\kappa u_1(t,x)}{\partial x^\kappa} - \dfrac{\partial u_2(t,x)}{\partial x}, \end{cases} \qquad x \in \mathbb{R}, \quad t \ge 0. \tag{4.8}$$

The operator $A = A(i\,\partial/\partial x)$ of this system and the matrix $A(s)$ are of the form

$$A = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ i^\kappa \dfrac{\partial^\kappa}{\partial x^\kappa} & -\dfrac{\partial}{\partial x} \end{pmatrix}, \qquad A(s) = \begin{pmatrix} -is & 0 \\ s^\kappa & is \end{pmatrix}.$$

Just as in the previous example, $\lambda_{1,2}(s) = \pm is$, and system (4.8) is hyperbolic.

The resolvent of the operator of multiplication by the matrix $A(\sigma)$ in the space $X = L_2(\mathbb{R})$ is

$$(\lambda I - A(\sigma))^{-1} = \frac{1}{\lambda^2 + \sigma^2} \begin{pmatrix} \lambda + i\sigma & 0 \\ \sigma^\kappa & \lambda - i\sigma \end{pmatrix},$$

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} = \|(\lambda I - A(\sigma))^{-1}\|_{\mathscr{L}(X)} = \sup_{\sigma\in\mathbb{R}} \left\{ \frac{|\lambda \pm i\sigma|}{|\lambda^2+\sigma^2|}, \frac{|\sigma^\kappa|}{|\lambda^2+\sigma^2|} \right\}.$$

Just as in Example 2 for $\nu > 4$, the operator $(\lambda I - A(\sigma))^{-1}$ is unbounded for $\kappa > 2$, and hence the operator A generates neither integrated nor convolutional C_0-semigroup; it is the generator of an R-semigroup in X.

The operators of conditionally well-posed and ill-posed systems that are not well-posed in the sense of Petrovskii can no longer generate C_0-semigroups or integrated semigroups (Theorem 1). We present examples showing the existence of convolutional and R-semigroups for such systems.

Example 6. *A conditionally well-posed differential equation with operator $A = A(i\,\partial/\partial x)$ generating an R-semigroup of operators.* The differential equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = ia\frac{\partial u(t,x)}{\partial x}$$

has the characteristic roots $\lambda_{1,2}(s) = \pm\sqrt{as}$. For $a > 0$,

$$\Lambda(\sigma) = \sqrt{a\sigma}, \qquad \sigma \ge 0;$$

therefore, the corresponding system is conditionally well posed. Here the spectrum of the operator $A = A(i\,\partial/\partial x)$ fills the positive real semiaxis, and hence A is not the generator of an integrated or a convolutional semigroup; by Theorem 2, it generates an R-semigroup.

Example 7. *An ill-posed differential equation with operator $A = A(i\,\partial/\partial x)$ generating a K-convolutional semigroup of operators on X.* Consider the equation

$$\frac{\partial u(t,x)}{\partial t} = i\,\frac{\partial^4 u(t,x)}{\partial x^4} - \frac{\partial^2 u(t,x)}{\partial x^2}, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{4.9}$$

Here

$$A = A\biggl(i\,\frac{\partial}{\partial x}\biggr) = i\,\frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial x^2}.$$

The characteristic root of the equation is

$$\lambda(s) = is^4 + s^2 = i(\sigma + i\tau)^4 + (\sigma + i\tau)^2.$$

The function $\Lambda(s)$ increases on the real axis with exponent greater than 1:

$$\Lambda(\sigma) = \operatorname{Re}\lambda(\sigma) = \sigma^2;$$

therefore, Eq. (4.9) belongs to the class of ill-posed equations.

The operator A is the operator $i\,\partial^4/\partial x^4$, perturbed by the unbounded operator $-\partial^2/\partial x^2$. The differential operator $i\,\partial^4/\partial x^4$, just as the Schrödinger operator (see Example 1), generates a C_0-semigroup on the space $X = L_2(\mathbb{R})$. In this case (see [11, Theorem 6.2]), the resolvent of the operator A on the space X satisfies the condition

$$\|\mathscr{R}(\lambda)\|_{\mathscr{L}(X)} = \mathscr{O}\biggl(\frac{|\lambda|}{\operatorname{Re}\lambda}\biggr) \qquad \text{for} \quad |\lambda| \to \infty, \quad \lambda \notin \{z \in \mathbb{C} : z = \mu^2 + i\mu^4\}.$$

Thus, we see that conditions (2.6) and (2.7) hold for $M(\xi) = \sqrt{\xi}$, $\alpha = 1$, $\gamma = 1$, $\omega = 0$, and any $\beta > 0$.

Further, the function $M(\xi) = \sqrt{\xi}$ is the Borel transform of the sequence $M_k = (k!)^2$. According to [12], there exists a differential operator $P(d/dt)$ of infinite order bounded in the space $\mathscr{D}_a^{\{M_k\}}$ of Roumier ultradifferentiable functions and such that the corresponding polynomial is of the form

$$P(s) = \prod_{k=1}^{\infty}\biggl(1 + \frac{l_k s}{k^2}\biggr),$$

where l_k is an infinitely small sequence. In addition, the polynomial $P(s)$ satisfies the following condition: for any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$|P(s)| \leq C_\varepsilon e^{\varepsilon\sqrt{|s|}}, \qquad s \in \mathbb{C}.$$

Hence the function $K(t)$, which is a fundamental solution of the ultradifferential equation

$$P\biggl(\frac{d}{dt}\biggr)K(t) = \delta,$$

can be constructed by means of the inverse Fourier transform

$$K(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{it\sigma}}{P(\sigma)}\,d\sigma, \qquad t \geq 0;$$

for this function, the following estimate of the form (2.5) holds:

$$|K(t)| \leq Ce^{\varepsilon\sqrt{t}}, \qquad t \geq 0.$$

Therefore, the operator A generates a K-convolutional semigroup on the space X with this function $K(t)$.

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