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Compactness of Some Operators of Convolution Type in Generalized Morrey Spaces

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Abstract—Sufficient conditions for the compactness in generalized Morrey spaces of the composition of a convolution operator and the operator of multiplication by an essentially bounded function are obtained. Very weak conditions on the function are also obtained under which the commutator of the operator of multiplication by such a function and a convolution operator is compact. The compactness of convolution operators in domains of cone type is investigated.

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Dedicated to the blessed memory of Nikolai Karapetovich Karapetyants

1. INTRODUCTION

In the past three decades, many authors have investigated Morrey-type spaces and operators acting on these spaces (see, e.g., the survey papers [1] and [2] by Burenkov and the references cited therein). The study of these spaces goes back to the work of Morrey [3] and is extensively continued at present. Along with the theory of Morrey-type spaces themselves, much attention has been paid to classical operators of analysis on these spaces, including the maximum operator, the Riesz potential, and a certain singular integral operator (see [2]). However, the important class of integral operators formed by convolution operators on Morrey spaces has been little studied. In this connection, first of all, it is worth to mention the papers [4] and [5], in which convolution operators on general Morrey-type spaces were considered and an analog of Young's inequality for convolutions in these spaces was obtained. In [6], the compactness of some operators of convolution type on the Morrey space was studied.

This paper is a continuation and generalization of [6]. In the paper, we consider products of the convolution operator with integrable kernel and the operator of multiplication by a function $a \in L_\infty(\mathbb{R}^n)$, on generalized Morrey spaces. Using results of [7], we show that, if the function a vanishes at infinity, then the product is a compact operator. We alo obtain weak conditions on a function a ensuring the compactness of the commutator of a convolution operator and the operator of multiplication by this function. In the concluding part of the paper, convolution operators on cone-type domains are considered.

We use the following notation:

- $\mathbb{R}^n$ is n-dimensional Euclidean space;
- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$;
- $|x| = \sqrt{x_1^2 + \dots + x_n^2}$;

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- $\mathbb{R}_+ = (0, \infty)$;
- $\mathbb{B}(x, r)$ is the open ball in $\mathbb{R}^n$ of radius r centered at x;
- $C\mathbb{B}(x, r) = \mathbb{R}^n \setminus \mathbb{B}(x, r)$;
- χ_D is the characteristic function of a measurable set $D \subset \mathbb{R}^n$;
- P_D is the operator of multiplication by the characteristic function χ_D;
- $C_0^\infty(\mathbb{R}^n)$ is the class of compactly supported infinitely differentiable functions.

2. PRELIMINARIES

Let $1 \le p \le \infty$, and let $D \subseteq \mathbb{R}^n$ be a measurable set. Then $L_p(D)$ is the space of (classes of) measurable complex-valued functions with norm

$$\|f\|_{L_p(D)} = \left(\int_D |f(x)|^p \, dx\right)^{1/p}, \qquad 1 \le p < \infty,$$
$$\|f\|_{L_\infty(D)} = \operatorname*{ess\,sup}_{x \in D} |f(x)|.$$

In the case $D = \mathbb{R}^n$, we use the notation $\|\cdot\|_p$ instead of $\|\cdot\|_{L_p(D)}$. We say that $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ if $f \in L_p(K)$ for any compact set $K \subset \mathbb{R}^n$.

Definition 1. Let $1 \le p \le \infty$, and let w be a nonnegative Lebesgue measurable function on $\mathbb{R}_+$ not equivalent to zero. The *generalized Morrey space* $L_{p,w}(\mathbb{R}^n)$ is the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L_{p,w}(\mathbb{R}^n)} \equiv \|f\|_{p,w} = \sup_{x \in \mathbb{R}^n} \|w(r)\|f\|_{L_p(\mathbb{B}(x,r))}\|_{L_\infty(\mathbb{R}_+)} < \infty. \tag{2.1}$$

If $w(r) = r^{-\lambda}$, where $0 \le \lambda \le n/p$, then the space $L_{p,w}(\mathbb{R}^n)$ coincides with the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$. For $w(r) \equiv 1$, the space $L_{p,w}(\mathbb{R}^n)$ coincides with the ordinary space $L_p(\mathbb{R}^n)$.

Definition 2 (see [8]). Let $1 \le p \le \infty$. The *set* $\Omega_{p\infty}$ is the family of all functions w that are nonnegative, Lebesgue measurable on $\mathbb{R}_+$, and not equivalent to zero which satisfy the conditions

$$\|w(r) r^{n/p}\|_{L_\infty(0,t)} < \infty, \qquad \|w(r)\|_{L_\infty(t,\infty)} < \infty$$

for some $t > 0$.

As is known (see [9] and [10]), the generalized Morrey space $L_{p,w}(\mathbb{R}^n)$ is nontrivial, i.e., contains functions not equivalent to zero on $\mathbb{R}^n$, if and only if $w \in \Omega_{p\infty}$.

Below we present conditions for the precompactness of a set contained in a generalized Morrey space.

Proposition 1 ([7]). *Given $1 \le p \le \infty$ and $w \in \Omega_{p\infty}$, let Ψ be a set of functions in $L_{p,w}(\mathbb{R}^n)$ satisfying the following conditions*:

i) $\sup_{\psi \in \Psi} \|\psi\|_{p,w} < \infty$;

ii) $\lim_{\delta \to 0} \sup_{\psi \in \Psi} \|\psi(\,\cdot\, + \delta) - \psi(\,\cdot\,)\|_{p,w} = 0$;

iii) $\lim_{R \to \infty} \sup_{\psi \in \Psi} \|\psi \chi_{C\mathbb{B}(0,R)}\|_{p,w} = 0$.

Then the set Ψ is precompact in the space $L_{p,w}(\mathbb{R}^n)$.

Consider the following convolution operator on $L_{p,w}(\mathbb{R}^n)$:

$$(H\varphi)(x) = \int_{\mathbb{R}^n} h(x-y)\varphi(y)\,dy, \qquad x \in \mathbb{R}^n, \tag{2.2}$$

where $h \in L_1(\mathbb{R}^n)$. The papers [4] and [5] devoted to convolutions in Morrey-type spaces contain, in particular, the following assertion.

Proposition 2 ([4]). *If $1 \le p \le \infty$, $w \in \Omega_{p\infty}$, and $h \in L_1(\mathbb{R}^n)$, then the operator H is bounded on the space $L_{p,w}(\mathbb{R}^n)$ and*

$$\|H\varphi\|_{p,w} \le \|h\|_1 \|\varphi\|_{p,w} \tag{2.3}$$

for every function $\varphi \in L_{p,w}(\mathbb{R}^n)$.

3. MAIN RESULTS

3.1. In this section, we study the compactness of the product of a convolution operator and the operator of multiplication by a bounded function.

Let M_a denote the operator of multiplication by a function $a \in L_\infty(\mathbb{R}^n)$. It can readily be seen that this operator is bounded on the space $L_{p,w}(\mathbb{R}^n)$; moreover, for any function $\varphi \in L_{p,w}(\mathbb{R}^n)$, we have

$$\|M_a\varphi\|_{p,w} \le \|a\|_\infty \|\varphi\|_{p,w}.$$

Let $C_0(\mathbb{R}^n)$ be the family of all functions a that are continuous on $\mathbb{R}^n$ and satisfy the condition $\lim_{x\to\infty} a(x) = 0$. The following assertion is an analog of Lemma 1 in [6]. For completeness, we present a detailed proof of this assertion.

Lemma 1. *If $1 \le p \le \infty$, $w \in \Omega_{p\infty}$, $a \in C_0(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$, then the operator $H_a := M_a H$ is compact on the space $L_{p,w}(\mathbb{R}^n)$.*

Proof. Let $\Phi = \{\varphi\}$ be an arbitrary bounded set in $L_{p,w}(\mathbb{R}^n)$, i.e., such that $\|\varphi\|_{p,w} \le C$ for every $\varphi \in \Phi$. We claim that the set $\{H_a\varphi\}$, where $\varphi \in \Phi$, is precompact in $L_{p,w}(\mathbb{R}^n)$. Let us verify conditions (i)–(iii) of Proposition 1.

The validity of condition (i) follows from the boundedness of the operator H_a. Let us prove (ii). For every function $\varphi \in \Phi$, we have

$$\begin{aligned}
&\|(H_a\varphi)(\,\cdot + \delta) - (H_a\varphi)(\,\cdot\,)\|_{p,w} \\
&\quad \le \|(a(\,\cdot + \delta) - a(\,\cdot\,))(H\varphi)(\,\cdot + \delta)\|_{p,w} + \|a(\,\cdot\,)((H\varphi)(\,\cdot + \delta) - (H\varphi)(\,\cdot\,))\|_{p,w} \\
&\quad \le \|a(\,\cdot + \delta) - a(\,\cdot\,)\|_\infty \|H\varphi\|_{p,w} + \|a\|_\infty \|(H\varphi)(\,\cdot + \delta) - (H\varphi)(\,\cdot\,)\|_{p,w}.
\end{aligned}$$

Applying inequality (2.3) and taking into account the assumption $\|\varphi\|_{p,w} \le C$, we obtain

$$\begin{aligned}
&\|(H_a\varphi)(\,\cdot + \delta) - (H_a\varphi)(\,\cdot\,)\|_{p,w} \\
&\quad \le C(\|a(\,\cdot + \delta) - a(\,\cdot\,)\|_\infty \|h\|_1 + \|a\|_\infty \|h(\,\cdot + \delta) - h(\,\cdot\,)\|_1).
\end{aligned}$$

The first summand on the right-hand side of this inequality tends to zero as $\delta \to 0$ because $a \in C_0(\mathbb{R}^n)$, and the other summands tend to zero because the function $h \in L_1(\mathbb{R}^n)$ is continuous with respect to the L_1 norm. Hence

$$\lim_{\delta\to 0} \sup_{\varphi\in\Phi} \|(H_a\varphi)(\,\cdot + \delta) - (H_a\varphi)(\,\cdot\,)\|_{p,w} = 0.$$

We proceed to condition (iii). Again using inequality (2.3), we obtain

$$\|\chi_{C\mathbb{B}(0,R)} H_a\varphi\|_{p,w} \le \|\chi_{C\mathbb{B}(0,R)} a\|_\infty \|H\varphi\|_{p,w} \le C\|h\|_1 \sup_{|x|\ge R} |a(x)|.$$

Since $\lim_{x\to\infty} a(x) = 0$, it follows that

$$\lim_{R\to\infty} \sup_{\varphi\in\Phi} \|\chi_{C\mathbb{B}(0,R)} H_a\varphi\|_{p,w} = 0.$$

This completes the proof of the lemma. □

Lemma 2. *If $1 \le p \le \infty$, $w \in \Omega_{p\infty}$, $h \in L_1(\mathbb{R}^n)$, and D is a bounded measurable set in $\mathbb{R}^n$, then the operator $P_D H$ is compact on the space $L_{p,w}(\mathbb{R}^n)$.*

Proof. Let $a \in C_0(\mathbb{R}^n)$ be a function such that $a(x) \equiv 1$ for all $x \in D$. Then we have

$$P_D H = P_D M_a H.$$

By Lemma 1, the operator $M_a H$ is compact on $L_{p,w}(\mathbb{R}^n)$; thus, so is the operator $P_D M_a H$. □

Let us extend the class of coefficients under consideration.

Definition 3 ([11, p. 37]). We say that a function $a \in L_\infty(\mathbb{R}^n)$ *belongs to the class* $B_0^{\sup}(\mathbb{R}^n)$ if

$$\lim_{N\to\infty} \operatorname*{ess\,sup}_{|x|>N} |a(x)| = 0.$$

Note that the class $B_0^{\sup}(\mathbb{R}^n)$ is the closure with respect to the L_∞ norm of the set of all compactly supported functions in $L_\infty(\mathbb{R}^n)$.

Theorem 1. *If $1 \le p \le \infty$, $w \in \Omega_{p\infty}$, $a \in B_0^{\sup}(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$, then the operator $M_a H$ is compact on the space $L_{p,w}(\mathbb{R}^n)$.*

Proof. Consider the function $a_N(x) = a(x)\chi_{\mathbb{B}(0,N)}(x)$. We have

$$M_{a_N} H = M_{a_N} P_{\mathbb{B}(0,N)} H,$$

and the operator $M_{a_N} H$ is compact by Lemma 2. Since $a \in B_0^{\sup}(\mathbb{R}^n)$, it follows that

$$\|M_a H - M_{a_N} H\|_{L_{p,w}\to L_{p,w}} \le \operatorname*{ess\,sup}_{|x|>N} |a(x)|\, \|H\|_{L_{p,w}\to L_{p,w}} \to 0$$

as $N \to \infty$. Hence the operator $M_a H$ is compact on $L_{p,w}(\mathbb{R}^n)$. □

3.2. We proceed to study the commutator $[M_a, H]$ of operators M_a and H. As is well known, the commutator is defined by the formula

$$[M_a, H] = M_a H - H M_a.$$

By virtue of (2.2), we have

$$\begin{aligned}([M_a, H]\varphi)(x) &= \int_{\mathbb{R}^n} (a(x) - a(y)) h(x-y)\varphi(y)\, dy \\ &= \int_{\mathbb{R}^n} (a(x) - a(x-t)) h(t)\varphi(x-t)\, dt, \qquad x \in \mathbb{R}^n.\end{aligned}$$

In [6], to describe compactness conditions for the commutator $[M_a, H]$ in classical Morrey spaces, the author introduced the class $\boldsymbol{\Omega}_\infty(\mathbb{R}^n)$ which consists of all functions $a \in L_\infty(\mathbb{R}^n)$ such that the following equation holds for every compact set $K \subset \mathbb{R}^n$:

$$\lim_{N\to\infty} \operatorname*{ess\,sup}_{|x|>N} \|a(x) - a(x - \,\cdot\,)\|_{L_\infty(K)} = 0.$$

Here we introduce and use a wider class of functions.

Definition 4. We say that a function $a \in L_\infty(\mathbb{R}^n)$ *belongs to the class* $\mathcal{A}_0(\mathbb{R}^n)$ if the following condition holds for every compact set $K \subset \mathbb{R}^n$:

$$\lim_{N\to\infty} \operatorname*{ess\,sup}_{|x|>N} \int_K |a(x) - a(x-t)|\, dt = 0. \tag{3.1}$$

It can readily be seen that $\boldsymbol{\Omega}_\infty(\mathbb{R}^n) \subset \mathcal{A}_0(\mathbb{R}^n)$. As shown below, the condition $a \in \mathcal{A}_0(\mathbb{R}^n)$ is sufficient for the compactness of the operator $[M_a, H]$.

Lemma 3. *If $a \in \mathcal{A}_0(\mathbb{R}^n)$, then, for every function $f \in L_1(\mathbb{R}^n)$,*

$$\lim_{N\to\infty} \operatorname*{ess\,sup}_{|x|>N} \int_{\mathbb{R}^n} |a(x) - a(x-t)||f(t)|\,dt = 0. \tag{3.2}$$

Proof. Take an arbitrary $\varepsilon > 0$. Using the density of the class $C_0^\infty(\mathbb{R}^n)$ in the space $L_1(\mathbb{R}^n)$, we choose $g \in C_0^\infty(\mathbb{R}^n)$ so that

$$\|f - g\|_1 < \frac{\varepsilon}{4\|a\|_\infty}$$

and fix this function g. We have

$$\begin{aligned}\int_{\mathbb{R}^n} |a(x) - a(x-t)||f(t)|\,dt &\le \int_{\mathbb{R}^n} |a(x) - a(x-t)||f(t) - g(t)|\,dt \\ &\qquad + \int_{\operatorname{supp} g} |a(x) - a(x-t)||g(t)|\,dt \\ &\le 2\|a\|_\infty \|f - g\|_1 + \sup_{t\in\mathbb{R}^n} |g(t)| \int_{\operatorname{supp} g} |a(x) - a(x-t)|\,dt \\ &< \frac{\varepsilon}{2} + \|g\|_\infty \int_{\operatorname{supp} g} |a(x) - a(x-t)|\,dt,\end{aligned}$$

where $\operatorname{supp} g$ stands for the support of the function $g \in C_0^\infty(\mathbb{R}^n)$. By virtue of (3.1), there is an $N_0 > 0$ such that, for all $N > N_0$,

$$\operatorname*{ess\,sup}_{|x|>N} \int_{\operatorname{supp} g} |a(x) - a(x-t)|\,dt < \frac{\varepsilon}{2\|g\|_\infty}\,.$$

Thus, the inequality

$$\operatorname*{ess\,sup}_{|x|>N} \int_{\mathbb{R}^n} |a(x) - a(x-t)||f(t)|\,dt < \varepsilon$$

holds for all $N > N_0$, which proves (3.2). □

Theorem 2. *If $1 < p < \infty$, $w \in \Omega_{p\infty}$, $a \in \mathcal{A}_0(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$, then the commutator $[M_a, H]$ is compact on the space $L_{p,w}(\mathbb{R}^n)$.*

Proof. Let us show that the operator $[M_a, H]$ can be approximated with respect to the operator norm by compact operators. Take arbitrary $\varepsilon > 0$. Using Lemma 3, we choose an $N > 0$ so that

$$A_N := \operatorname*{ess\,sup}_{|x|\ge N} \int_{\mathbb{R}^n} |a(x) - a(x-t)||h(t)|\,dt < \frac{\varepsilon^{p'}}{(2\|a\|_\infty \|h\|_1)^{p'/p}}\,,$$

and fix this N. Let

$$P_N = P_{\mathbb{B}(0,N)}, \qquad Q_N = I - P_N,$$

where I is the identity operator. Applying Hölder's inequality, we obtain

$$|(Q_N[M_a, H]\varphi)(x)| \le \int_{\mathbb{R}^n} |a(x) - a(x-t)||h(t)||\varphi(x-t)|\,dt$$

$$
\begin{aligned}
&\le \left(\int_{\mathbb{R}^n} |a(x)-a(x-t)||h(t)|\,dt\right)^{1/p'} \left(\int_{\mathbb{R}^n} |a(x)-a(x-t)||h(t)||\varphi(x-t)|^p\,dt\right)^{1/p} \\
&\le A_N^{1/p'} \left(2\|a\|_\infty \int_{\mathbb{R}^n} |h(t)||\varphi(x-t)|^p\,dt\right)^{1/p} \\
&< \varepsilon \|h\|_1^{-1/p} \left(\int_{\mathbb{R}^n} |h(t)||\varphi(x-t)|^p\,dt\right)^{1/p}
\end{aligned}
$$

for almost all $x \in C\mathbb{B}(0,N)$. Thus, for arbitrary $x \in \mathbb{R}^n$ and $r>0$, we have

$$
\begin{aligned}
\|Q_N[M_a,H]\varphi\|_{L_p(\mathbb{B}(x,r))} &< \varepsilon \|h\|_1^{-1/p} \left(\int_{\mathbb{B}(x,r)} dy \int_{\mathbb{R}^n} |h(t)||\varphi(y-t)|^p\,dt\right)^{1/p} \\
&= \varepsilon \|h\|_1^{-1/p} \left(\int_{\mathbb{R}^n} |h(t)|\,dt \int_{\mathbb{B}(x,r)} |\varphi(y-t)|^p\,dy\right)^{1/p} \\
&= \varepsilon \|h\|_1^{-1/p} \left(\int_{\mathbb{R}^n} |h(t)|\|\varphi\|^p_{L_p(\mathbb{B}(x-t,r))}\,dt\right)^{1/p}.
\end{aligned}
$$

Taking into account (2.1), we obtain the following relations for the norm of the function $Q_N[M_a,H]\varphi$ in the space $L_{p,w}(\mathbb{R}^n)$:

$$
\begin{aligned}
\|Q_N[M_a,H]\varphi\|_{p,w} &= \sup_{x\in\mathbb{R}^n} \|w(r)\|Q_N[M_a,H]\varphi\|_{L_p(\mathbb{B}(x,r))}\|_{L_\infty(\mathbb{R}_+)} \\
&< \varepsilon \|h\|_1^{-1/p} \sup_{x\in\mathbb{R}^n} \left\| w(r) \left(\int_{\mathbb{R}^n} |h(t)|\,\|\varphi\|^p_{L_p(\mathbb{B}(x-t,r))}\,dt\right)^{1/p} \right\|_{L_\infty(\mathbb{R}_+)} \\
&= \varepsilon \|h\|_1^{-1/p} \sup_{x\in\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} |h(t)|[w(r)\|\varphi\|_{L_p(\mathbb{B}(x-t,r))}]^p\,dt \right\|^{1/p}_{L_\infty(\mathbb{R}_+)} \\
&= \varepsilon \|h\|_1^{-1/p} \sup_{x\in\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |h(t)|\,\|w(r)\|\varphi\|_{L_p(\mathbb{B}(x-t,r))}\|^p_{L_\infty(\mathbb{R}_+)}\,dt\right)^{1/p} \\
&\le \varepsilon \|h\|_1^{-1/p} \|\varphi\|_{p,w} \left(\int_{\mathbb{R}^n} |h(t)|\,dt\right)^{1/p} = \varepsilon \|\varphi\|_{p,w}.
\end{aligned}
$$

Since the function $\varphi \in L_{p,w}(\mathbb{R}^n)$ is arbitrary, we have

$$
\|[M_a,H] - P_N[M_a,H]\|_{L_{p,w}\to L_{p,w}} = \|Q_N[M_a,H]\|_{L_{p,w}\to L_{p,w}} < \varepsilon.
$$

Since ε is an arbitrary positive number and $P_N[M_a,H]$ is a compact operator (by Lemma 2), it follows that the operator $[M_a,H]$ is compact as well. This completes the proof of the theorem. □

Corollary 1. *If $1<p<\infty$, $w \in \Omega_{p\infty}$, $a \in B_0^{\sup}(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$, then the operator HM_a is compact on the space $L_{p,w}(\mathbb{R}^n)$. In particular, the operator HP_D is compact for every measurable bounded set $D \subset \mathbb{R}^n$.*

Proof. Since $a \in B_0^{\sup}(\mathbb{R}^n)$, it follows that $a \in \mathcal{A}_0(\mathbb{R}^n)$. Therefore, the equation

$$
HM_a = M_aH - [M_a,H]
$$

and Theorems 1 and 2 imply the compactness of the operator HM_a. □

In conclusion of this section, we mention the papers [12]–[14], in which commutators of certain operators acting on Morrey spaces were considered, too.

3.3. As shown above, the operators $P_D H$ and HP_D are compact if the set D is bounded. In this subsection, we study the compactness of operators of the form $P_{D_1}HP_{D_2}$ in the case of unbounded sets D_1 and D_2. Given arbitrary $D \subset \mathbb{R}^n$, we set

$$D_N = D \cap \mathbb{B}(0, N), \qquad CD_N = D \setminus D_N.$$

We denote the distance between sets D_1 and D_2 by $\operatorname{dist}(D_1, D_2)$.

Theorem 3. *If $1 \le p \le \infty$, $w \in \Omega_{p\infty}$, $h \in L_1(\mathbb{R}^n)$, and D_1 and D_2 are measurable unbounded sets in $\mathbb{R}^n$ such that $\operatorname{dist}(CD_{1N}, CD_{2N}) \to \infty$ as $N \to \infty$, then the operator $P_{D_1}HP_{D_2}$ is compact on the space $L_{p,w}(\mathbb{R}^n)$.*

Proof. First, suppose that $h \in C_0^\infty(\mathbb{R}^n)$. Then there is a $\rho > 0$ for which $\operatorname{supp} h \subset \mathbb{B}(0, \rho)$. Choose an $N > 0$ so large that $|x - y| > \rho$ for all $x \in CD_{1N}$ and $y \in D_2$. We have $P_{CD_{1N}}HP_{D_2} = 0$. Hence

$$P_{D_1}HP_{D_2} = P_{D_{1N}}HP_{D_2} + P_{CD_{1N}}HP_{D_2} = P_{D_{1N}}HP_{D_2}$$

and the operator $P_{D_1}HP_{D_2}$ is compact by Lemma 2.

Now suppose that h is an arbitrary function in $L_1(\mathbb{R}^n)$. Then there is a sequence $\{h_k\} \subset C_0^\infty(\mathbb{R}^n)$ such that $\|h - h_k\|_1 \to 0$ as $k \to \infty$. By virtue of (2.3), we have

$$\|P_{D_1}HP_{D_2} - P_{D_1}H_kP_{D_2}\|_{L_{p,w}\to L_{p,w}} \le \|h - h_k\|_1 \to 0.$$

Since the operators $P_{D_1}H_kP_{D_2}$ are compact, it follows that the operator $P_{D_1}HP_{D_2}$ is compact on $L_{p,w}(\mathbb{R}^n)$. This completes the proof of the theorem. □

Corollary 2. *If $1 \le p \le \infty$, $w \in \Omega_{p\infty}$, $h \in L_1(\mathbb{R}^n)$, and Γ_1 and Γ_2 are cones in $\mathbb{R}^n$ whose closures have no common points except the origin. Then the operator $P_{\Gamma_1}HP_{\Gamma_2}$ is compact on the space $L_{p,w}(\mathbb{R}^n)$.*

In conclusion, we mention that the operators of convolution in cones on L_p-spaces were first considered in [15].

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