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Vector Lyapunov Functions and Ultimate Poisson Boundedness of Solutions of Systems of Differential Equations

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Abstract—Various forms of uniform-ultimate Poisson boundedness of solutions and of ultimate Poisson equiboundedness of solutions are introduced. Sufficient conditions for various forms of uniform-ultimate Poisson boundedness and of ultimate Poisson equiboundedness of solutions are obtained by using the method of vector Lyapunov functions.

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The theory of boundedness of solutions of systems of differential equations based on the method of Lyapunov functions was developed in [1]. The theory of boundedness of solutions of systems in part of the variables (or, for short, of partial boundedness of solutions) was developed on the basis of the method of Lyapunov functions in the monograph [2]. The basic forms of partial boundedness of solutions with partially controlled initial conditions were introduced and studied in [3]–[5] by using the method of Lyapunov functions. In the monograph [6], the author created the method of vector Lyapunov functions, also called the principle of comparison with vector Lyapunov functions, which offers much more possibilities for studying the boundedness of solutions than the method of Lyapunov functions. In [7] and [8], the method of vector Lyapunov functions was applied to study the partial boundedness of solutions and the partial boundedness of solutions with partially controlled initial conditions. Note that the methods of the theory of boundedness and of the theory of Lyapunov stability are very similar. This is caused by the fact that the notions of boundedness of solutions and of Lyapunov stability of an equilibrium position are dual in the sense that the quantifiers $\forall$ and $\exists$ preceding ε and δ are interchanged in the corresponding $\varepsilon\delta$-definitions. This duality ensures a large formal similarity between the theories of boundedness and stability in the sense of Lyapunov. The same considerations apply to the theories of partial boundedness and partial stability in the sense of Lyapunov. The theory of partial boundedness of solutions with partially controlled initial conditions and the theory of partial Lyapunov stability of a partial equilibrium position [9]–[11] are very similar formally as well. This similarity between the theory of boundedness and the theory of Lyapunov stability provides a natural motivation to consider the form of boundedness of solutions that dually corresponds to the notion of positive stability in the sense of Poisson [12] of the trajectory of motion of a dynamical system. A solution possessing this form of boundedness, called *Poisson boundedness* in this paper, is not necessarily contained entirely in a ball in the phase space, but returns to this ball countably many times. Obviously, the problem of finding an infinite system of time intervals on which the dynamical system functions normally, i.e., the parameters of motion of this dynamical system do not take arbitrarilyy large values, is very important. This implies the need for studying various forms of Poisson boundedness of solutions on the basis of the method of vector Lyapunov functions.

In the present paper, we introduce the notions of uniform-ultimate Poisson boundedness, partial uniform-ultimate Poisson boundedness, and partial uniform-ultimate Poisson boundedness of solutions with partially controlled initial conditions; these notions generalize the notions of the corresponding forms of uniform-ultimate boundedness of solutions. Using the method of vector Lyapunov functions, we obtain sufficient conditions for uniform-ultimate Poisson boundedness, partial uniform-ultimate

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Poisson boundedness, and the partial uniform-ultimate Poisson boundedness of solutions with partially controlled initial conditions. Further, we introduce the notions of ultimate Poisson equiboundedness, partial ultimate Poisson equiboundedness, and partial ultimate Poisson equiboundedness of solutions with partially controlled initial conditions; these notions generalize the notions of the corresponding forms of the ultimate equiboundedness of solutions. Using the method of vector Lyapunov functions, we obtain sufficient conditions for ultimate Poisson equiboundedness, partial ultimate Poisson equiboundedness, and ultimate Poisson partial equiboundedness of solutions with partially controlled initial conditions. We also exemplify the application of the obtained results to particular systems of differential equations. Now we pass to precise definitions and statements.

Suppose given an arbitrary system of differential equations of n variables

$$\frac{dx}{dt} = F(t,x), \qquad F(t,x) = (F_1(t,x), \dots, F_n(t,x))^T, \tag{1}$$

whose right-hand side is defined and continuous on $\mathbb{R}^+ \times \mathbb{R}^n$, where $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$. It is assumed that all solutions of of system (1) can be extended to the whole semiaxis $\mathbb{R}^+$. The uniqueness of a solution of the Cauchy problem for system (1) is not required.

Before introducing notation, we recall, following [2], some basic facts concerning vector Lyapunov functions. Suppose given a continuously differentiable vector function

$$V(t,x) = (V_1(t,x), \dots, V_l(t,x))^T, \qquad l \geq 1, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

The derivative of this vector function subject to system (1) is defined by

$$\dot V(t,x) = (\dot V_1(t,x), \dots, \dot V_l(t,x))^T,$$

where $\dot V_i(t,x)$ is the derivative (subject to (1)) of the function $V_i(t,x)$, $1 \leq i \leq l$. Given vectors $\xi = (\xi_1, \dots, \xi_l)^T, \eta = (\eta_1, \dots, \eta_l)^T \in \mathbb{R}^l$, we write $\xi \leq \eta$ if

$$\xi_i \leq \eta_i \qquad \text{for any} \quad 1 \leq i \leq l.$$

Now, suppose given a continuous vector function

$$f(t,\xi) = (f_1(t,\xi), \dots, f_l(t,\xi))^T$$

on $\mathbb{R}^+ \times \mathbb{R}^l$. We write $f(t,\xi) \in W$ if $f(t,\xi)$ satisfies the Ważewski condition, namely, for each $1 \leq s \leq l$, the function $f_s(t,\xi)$ is nondecreasing in the variables $\xi_1, \dots, \xi_{s-1}, \xi_{s+1}, \dots, \xi_l$, i.e.,

$$\text{if} \quad \xi_i \leq \eta_i, \quad 1 \leq i \leq l, \quad i \neq s, \quad \xi_s = \eta_s \qquad \text{then} \quad f_s(t,\xi) \leq f_s(t,\eta).$$

It is easy to see that, for $l = 1$, the condition $f(t,\xi) \in W$ always holds. A continuously differentiable vector function $V(t,x)$ and a system

$$\frac{d\xi}{dt} = f(t,\xi), \qquad f(t,\xi) \in W, \tag{2}$$

are called, respectively, the *vector Lyapunov function* and the *comparison system* for system (1) if the following condition holds:

$$\dot V(t,x) \leq f(t, V(t,x)). \tag{3}$$

Since the right-hand side of system (2) is continuous, a solution of the Cauchy problem for this system may be nonunique. However, using the condition $f(t,\xi) \in W$, among all solutions of system (2) passing through an arbitrary point (t_0, ξ_0) we can choose the upper solution $\overline{\xi}(t, t_0, \xi_0,)$, i.e., the solution for which

$$\xi(t, t_0, \xi_0,) \leq \overline{\xi}(t, t_0, \xi_0,) \qquad \text{for all} \quad t \geq t_0,$$

where $\xi(t, t_0, \xi_0)$ is an arbitrary solution of system (2). It follows from Ważewski's theorem (see, e.g., [2]) that the solutions $x(t, t_0, x_0)$ of system (1), the vector Lyapunov function $V(t,x)$, and the upper solution $\overline{\xi}(t, t_0, V(t_0, x_0))$ of the comparison system (2) for system (1) are related for all $t \geq t_0$ by

$$V(t, x(t, t_0, t_0) \leq \overline{\xi}(t, t_0, V(t_0, x_0)). \tag{4}$$

In what follows, we denote the usual Euclidean norm by $\|\cdot\|$. For a solution $x = x(t)$ of system (1) passing through the point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, we use the notation $x = x(t, t_0, x_0)$. For any $t_0 \in \mathbb{R}^+$, by $\mathbb{R}^+(t_0)$ we denote the set $\{t \in \mathbb{R} \mid t \geq t_0\}$. By a *$\mathcal{P}$-sequence* we mean a nonnegative increasing number sequence $\tau = \{\tau_i\}_{i\geq 1}$ with $\lim_{i\to\infty} \tau_i = +\infty$. For each $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geq 1}$, we set

$$M(\tau) = \bigcup_{i=1}^{\infty} [\tau_{2i-1}; \tau_{2i}].$$

Recall [1] that a solution $x = x(t, t_0, x_0)$ of system (1) is said to be *bounded* if there exists a number $\beta > 0$ such that

$$\|x(t, t_0, x_0)\| < \beta \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0).$$

Definition 1. We say that a solution $x = x(t, t_0, x_0)$ of system (1) is *Poisson bounded* if there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geq 1}$, where $t_0 \in M(\tau)$, and a number $\beta > 0$ such that

$$\|x(t, t_0, x_0)\| < \beta \qquad \text{for all} \quad t \in R^+(t_0) \cap M(\tau).$$

Geometrically, Definition 1 means that the solution starting at some instant of time from the ball of radius $\beta > 0$ centered at the origin will return to this ball countably many times. Obviously, if a solution of system (1) is bounded, then this solution is also Poisson bounded.

Recall [1] that the solutions of system (1) are said to be uniform-ultimately bounded (uniform-ultimately bounded in the limit) if there exists a number $B > 0$ such that, given any $\alpha \geq 0$, there is a $T \geq 0$ for which any solution $x = x(t, t_0, x_0)$ of system (1) with $t_0 \geq 0$ and $\|x_0\| \leq \alpha$ satisfies the condition

$$\|x(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T).$$

Now we introduce the notion of uniform-ultimate Poisson boundedness of solutions, which generalizes the notion of uniform-ultimate boundedness.

Definition 2. We say that the solutions of system (1) are *uniform-ultimately Poisson bounded* if there exist a number $B > 0$ and a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geq 1}$ such that, given any $\alpha \geq 0$, there is a number $T \geq 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $t_0 \in M(\tau)$ and $\|x_0\| \leq \alpha$ satisfies the condition

$$\|x(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau).$$

When it is required to specify the corresponding $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geq 1}$, we say that the solutions of system (1) are *uniform-ultimately Poisson bounded with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geq 1}$*.

It is easy to see that if the solutions of system (1) are uniform-ultimately bounded, then they are also uniform-ultimately Poisson bounded. Moreover, it is easy to see that we can always assume without loss of generality that the function $T = T(\alpha)$ in Definition 2 is nondecreasing.

Further, let $a(r)$, $a(t, r)$, and $b(r)$ denote arbitrary functions, where $r \geq 0$ and $t \geq 0$, with the following properties:

(1) $a(r) > 0$ is an increasing function;

(2) $a(t, r) > 0$ is an increasing function in r for each fixed $t \geq 0$;

(3) $b(r) \geq 0$ is a nondecreasing function and $b(r) \to \infty$ as $r \to \infty$.

Below we state and prove a sufficient condition for the uniform-ultimate Poisson boundedness of the solutions of system (1), which is based on the method of vector Lyapunov functions.

Theorem 1. *Suppose that, for system* (1)*, there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geq 1}$ and a vector Lyapunov function $V(t, x)$ such that the following conditions hold*:

(1) $V_1(t, x) \geq 0, \dots, V_l(t, x) \geq 0$;

(2) $b(\|x\|) \leq \sum_{i=1}^{l} V_i(t, x) \leq a(\|x\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$.

Suppose also that the solutions of the comparison system (2) *for system* (1) *are uniform-ultimately Poisson bounded with respect to the* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\geq 1}$. *Then the solutions of system* (1) *are uniform-ultimately Poisson bounded.*

Proof. Given system (1), we must find a number $B > 0$ and numbers $T = T(\alpha) \geq 0$ for all $\alpha \geq 0$ such that any solution $x = x(t, t_0, x_0)$ of (1) with $t_0 \in M(\tau)$ and $\|x_0\| \leq \alpha$ satisfies the condition

$$\|x(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in R^+(t_0 + T) \cap M(\tau).$$

Using condition (2) and inequality (4),

we obtain the inequalities

$$b(\|x(t, t_0, t_0)\|) \leq \sum_{i=1}^{l} V_i(t, x(t, t_0, x_0)) \leq \sum_{i=1}^{l} \overline{\xi}_i(t, t_0, V(t_0, x_0))$$

for a solution of $x(t, t_0, x_0)$ and the upper solution $\overline{\xi}(t, t_0, V(t_0, x_0))$ of the comparison system (2); these inequalities hold for all $t \in M(\tau)$. In addition, for any $t \geq 0$, we have the obvious inequalities

$$\sum_{i=1}^{l} \overline{\xi}_i(t, t_0, V(t_0, x_0)) \leq \sum_{i=1}^{l} |\,\overline{\xi}_i(t, t_0, V(t_0, x_0))| \leq l \cdot \|\,\overline{\xi}(t, t_0, V(t_0, x_0))\|.$$

Under the assumptions of the theorem, the solutions of the comparison system (2) are uniform-ultimately Poisson bounded with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geq 1}$; therefore, for the upper solution $\overline{\xi}(t, t_0, V(t_0, x_0))$ and the numbers $\nu = \|V(t_0, x_0)\|$, there exist numbers $C > 0$ and $\Lambda(\nu) \geq 0$ such that

$$\|\overline{\xi}(t, t_0, V(t_0, x_0))\| < C \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + \Lambda) \cap M(\tau).$$

From this inequality we see that $b(\|x(t, t_0, x_0)\|) \leq l \cdot C$ for any $t \in \mathbb{R}^+(t_0 + \Lambda) \cap M(\tau)$. Conditions (1) and (2) of the theorem imply

$$\|V(t_0, x_0)\| \leq \sum_{i=1}^{l} |\,V_i(t_0, x_0)| = \sum_{i=1}^{l} V_i(t_0, x_0) \leq a(\|x_0\|) \leq a(\alpha)$$

for each $t \in M(\tau)$. Since the function $\Lambda(r)$ is nondecreasing, we have $\Lambda(\|V(t_0, x_0)\|) \leq \Lambda(a(\alpha))$. Defining the required number $T = T(\alpha)$ by $T(\alpha) = \Lambda(a(\alpha))$, we obtain

$$b(\|x(t, t_0, x_0)\|) \leq l \cdot C \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau).$$

Now, since $b(r) \to \infty$ as $r \to \infty$ and the numbers l and C are fixed, we can choose a number $B > 0$ such that $l \cdot C < b(B)$. This gives

$$b(\|x(t, t_0, x_0)\|) < b(B) \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau),$$

where B is independent of the solution $x = x(t, t_0, x_0)$. Since the function $b(r)$ is nondecreasing, it follows from the last inequality that

$$\|x(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau).$$

Thus, we have shown that the solutions of system (1) are uniform-ultimately Poisson bounded. □

Taking $l = 1$ for the vector Lyapunov function in Theorem 1, i.e., considering conventional functions as vector ones, we obtain the following statement.

Corollary 1. *Suppose that, for system* (1)*, there exists a* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\geq 1}$*, a continuously differentiable function* $V(t, x) \geq 0$ *defined on* $\mathbb{R}^+ \times \mathbb{R}^n$*, and a continuous function* $f(t, \xi)$ *defined on* $\mathbb{R}^+ \times \mathbb{R}$ *for which the following conditions hold*:

(1) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$;

(2) $\dot{V}(t,x) \le f(t, V(t,x))$ *for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$;

(3) *the solutions of the equation* $\dot{\xi} = f(t,\xi)$ *are uniform-ultimately Poisson bounded with respect to the* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\ge 1}$.

Then the solutions of system (1) *are uniform-ultimately Poisson bounded.*

In what follows, for any $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ and any fixed number $1 \le k \le n$, we use the notation $y = (x_1, \dots, x_k)^T \in \mathbb{R}^k$.

Recall [2] that the solutions of system (1) are said to be uniform-ultimately y-bounded if there exists a number $B > 0$ such that, given any $\alpha \ge 0$, there is a number $T \ge 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $t_0 \ge 0$ and $\|x_0\| \le \alpha$ satisfies the condition

$$\|y(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T).$$

Below we introduce the notion of uniform-ultimate Poisson y-boundedness of solutions, which generalizes the notion of uniform-ultimate y-boundedness.

Definition 3. We say that the solutions of system (1) are *uniform-ultimately Poisson* y*-bounded* if there exists a number $B > 0$ and a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge 1}$ such that, given any $\alpha \ge 0$, there is a number $T \ge 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $t_0 \in M(\tau)$ and $\|x_0\| \le \alpha$, satisfies the condition

$$\|y(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau).$$

When it is required to specify the corresponding $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge 1}$, we say that the solutions of system (1) are *uniform-ultimately Poisson* y*-bounded with respect to the* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\ge 1}$.

Obviously, if the solutions of system (1) are uniform-ultimately y-bounded, then they are uniform-ultimately Poisson y-bounded.

In what follows, for any $\xi = (\xi_1, \dots, \xi_l)^T \in \mathbb{R}^l$ and any fixed number $1 \le p \le l$, we use the notation $\mu = (\xi_1, \dots, \xi_p)^T \in \mathbb{R}^p$.

The following statement, whose proof is similar to that of Theorem 1, provides a sufficient condition for the uniform-ultimate Poisson y-boundedness of solutions; it is based on the method of vector Lyapunov functions.

Theorem 2. *Suppose that, for system* (1)*, there exists a* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\ge 1}$*, a vector Lyapunov function* $V(t,x)$*, and a number* $1 \le p \le l$ *for which the following conditions hold*:

(1) $V_1(t,x) \ge 0, \dots, V_l(t,x) \ge 0$;

(2) $b(\|y\|) \le \sum_{i=1}^{p} V_i(t,x)$ *for all* $(t,x) \in M(\tau) \times \mathbb{R}^n$;

(3) $\sum_{i=1}^{l} V_i(t,x) \le a(\|x\|)$ *for all* $(t,x) \in M(\tau) \times \mathbb{R}^n$.

Suppose also that the solutions of the comparison system (2) *for system* (1) *are uniform-ultimately Poisson* μ*-bounded with respect to the* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\ge 1}$*. Then the solutions of system* (1) *are uniform-ultimately Poisson* y*-bounded.*

Corollary 2. *Suppose that, for system* (1)*, there exists a* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\ge 1}$*, a continuously differentiable function* $V(t,x) \ge 0$ *defined on* $\mathbb{R}^+ \times \mathbb{R}^n$*, and a continuous function* $f(t,\xi)$ *defined on* $\mathbb{R}^+ \times \mathbb{R}$ *for which the following conditions hold*:

(1) $b(\|y\|) \le V(t,x) \le a(\|x\|)$ *for all* $(t,x) \in M(\tau) \times \mathbb{R}^n$;

(2) $\dot{V}(t,x) \le f(t, V(t,x))$ *for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$;

(3) *the solutions of the equation $\dot{\xi} = f(t, \xi)$ are uniform-ultimately Poisson bounded with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$.*

Then the solutions of system (1) *are uniform-ultimately Poisson y-bounded.*

In what follows, for any element $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and any fixed numbers $1 \leq k \leq m \leq n$, we use the notations $y = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $z = (x_1, \dots, x_m) \in \mathbb{R}^m$.

Recall [3] that the solutions of system (1) are said to be uniform-ultimately *y-bounded with z_0-control* if there exists a number $B > 0$ such that, given any $\alpha \geq 0$, there is a number $T \geq 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $t_0 \geq 0$ and $\|z_0\| \leq \alpha$ satisfies the condition

$$\|y(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T).$$

Below we introduce the notion of uniform-ultimate y-boundedness with Poisson z_0-control of solutions, which generalizes the notion of uniform-ultimate y-boundedness with z_0-control.

Definition 4. We say that the solutions of system (1) are *uniform-ultimately y-bounded with Poisson z_0-control* if there exists a number $B > 0$ and a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$ such that, for any $\alpha \geq 0$, there is a number $T \geq 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $t_0 \in M(\tau)$ and $\|z_0\| \leq \alpha$ satisfies the condition

$$\|y(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau).$$

When it is required to specify the corresponding $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, we say that the solutions of system (1) are *uniform-ultimately y-bounded with Poisson z_0-control with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$.*

It is easy to see that if the solutions of system (1) are uniform-ultimately y-bounded with z_0-control, then they are also uniform-ultimately y-bounded with Poisson z_0-control.

In what follows, for any element $\xi = (\xi_1, \dots, \xi_l)^T \in \mathbb{R}^l$ and arbitrary fixed numbers $1 \leq p \leq q \leq l$, we use the notations $\mu = (\xi_1, \dots, \xi_p)^T$ and $\gamma = (\xi_1, \dots, \xi_q)^T$.

The following statement, whose proof is similar to that of Theorems 1 and 2, provides a sufficient condition for the uniform-ultimate y-boundedness with Poisson z_0-control of solutions; it is based on the method of vector Lyapunov functions.

Theorem 3. *Suppose that, for system* (1)*, there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, a vector Lyapunov function $V(t, x)$, and numbers $1 \leq p \leq q \leq l$ for which the following conditions hold*:

(1) $V_1(t, x) \geq 0, \dots, V_q(t, x) \geq 0$;

(2) $b(\|y\|) \leq \sum_{i=1}^{p} V_i(t, x)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$;

(3) $\sum_{i=1}^{q} V_i(t, x) \leq a(\|z\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$.

Suppose also that the solutions of the comparison system (2) *for system* (1) *are uniform-ultimately μ-bounded with Poisson γ_0-control with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$. Then the solutions of system* (1) *are uniform-ultimately y-bounded with Poisson z_0-control.*

Corollary 3. *Suppose that, for system* (1)*, there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, a continuously differentiable function $V(t, x) \geq 0$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, and a continuous function $f(t, \xi)$, defined on $\mathbb{R}^+ \times \mathbb{R}$ for which the following conditions hold*:

(1) $b(\|y\|) \leq V(t, x) \leq a(\|z\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$;

(2) $\dot{V}(t, x) \leq f(t, V(t, x))$ *for all* $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$;

(3) *the solutions of the equation* $\dot{\xi} = f(t, \xi)$ *are uniform-ultimately Poisson bounded with respect to the* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i \geq 1}$.

Then the solutions of system (1) *are uniform-ultimately* y*-bounded with Poisson* z_0*-control.*

Recall [1] that the solutions of system (1) are said to be *ultimately equibounded* if there exists a number $B > 0$ such that, given any $t_0 \geq 0$ and $\alpha \geq 0$, there is a number $T \geq 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $\|x_0\| \leq \alpha$ satisfies the condition

$$\|x(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T).$$

Below we introduce the notion of ultimate Poisson equiboundedness of solutions, which generalizes the notion of ultimate equiboundedness.

Definition 5. We say that the solutions of system (1) are *ultimately Poisson equibounded* if there exists a number $B > 0$ and a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$ such that, given any $t_0 \in M(\tau)$ and $\alpha \geq 0$, there is a number $T \geq 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $\|x_0\| \leq \alpha$ satisfies the condition

$$\|x(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau).$$

When it is required to specify the corresponding $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, we say that the solutions of system (1) are *ultimately Poisson equibounded with respect to the* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i \geq 1}$.

It is easy to see that if the solutions of system (1) are ultimately equibounded, then they are ultimately Poisson equibounded.

The following statement, whose proof is similar to that of Theorem 1, gives a sufficient condition for ultimate Poisson equiboundedness of solutions; it is based on the method of vector Lyapunov functions.

Theorem 4. *Suppose that, for system* (1)*, there exists a* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i \geq 1}$ *and a vector Lyapunov function* $V(t, x)$ *such that the following conditions hold*:

(1) $V_1(t, x) \geq 0, \dots, V_l(t, x) \geq 0$;

(2) $b(\|x\|) \leq \sum_{i=1}^{l} V_i(t, x) \leq a(t, \|x\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$.

Suppose also that the solutions of the comparison system (2) *for system* (1) *are ultimately Poisson equibounded with respect to the* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i \geq 1}$. *Then the solutions of system* (1) *are ultimately Poisson equibounded.*

Corollary 4. *Suppose that, for system* (1)*, there exists a* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i \geq 1}$*, a continuously differentiable function* $V(t, x) \geq 0$ *defined on* $\mathbb{R}^+ \times \mathbb{R}^n$*, and a continuous function* $f(t, \xi)$ *defined on* $\mathbb{R}^+ \times \mathbb{R}$ *for which the following conditions hold*:

(1) $b(\|x\|) \leq V(t, x) \leq a(t, \|x\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$;

(2) $\dot{V}(t, x) \leq f(t, V(t, x))$ *for all* $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$;

(3) *the solutions of the equation* $\dot{\xi} = f(t, \xi)$ *are ultimately Poisson equibounded with respect to the* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i \geq 1}$.

Then the solutions of system (1) *are ultimately Poisson equibounded.*

Recall [2] that the solutions of system (1) are said to be *ultimately* y*-equibounded* if there exists a number $B > 0$ such that, given any $t_0 \geq 0$ and $\alpha \geq 0$, there is a number $T \geq 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $\|x_0\| \leq \alpha$ satisfies the condition

$$\|y(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T).$$

Below we introduce the notion of ultimate Poisson y-equiboundedness of solutions, which generalizes the notion of ultimate y-equiboundedness of solutions.

Definition 6. We say that the solutions of system (1) are *ultimate Poisson y-equibounded* if there exists a number $B > 0$ and a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$ such that, given any $t_0 \in M(\tau)$ and any $\alpha \ge 0$, there is a number $T \ge 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $\|x_0\| \le \alpha$ satisfies the condition

$$\|y(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau).$$

When it is required to specify the corresponding $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$, we say that the solutions of system (1) are *ultimate Poisson y-equibounded with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$*.

It is easy to see that if the solutions of system (1) are ultimately y-equibounded, then they are ultimate Poisson y-equibounded.

The following statement, whose proof is similar to that of Theorem 2, provides a sufficient condition for ultimate Poisson y-equiboundedness of solutions; it is based on the method of vector Lyapunov functions.

Theorem 5. *Suppose that, for system* (1)*, there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$, a vector Lyapunov function $V(t, x)$, and a number $1 \le p \le l$ for which the following conditions hold*:

(1) $V_1(t, x) \ge 0, \dots, V_l(t, x) \ge 0$;

(2) $b(\|y\|) \le \sum_{i=1}^{p} V_i(t, x)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$;

(3) $\sum_{i=1}^{l} V_i(t, x) \le a(t, \|x\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$.

Suppose also that the solutions of the comparison system (2) *for system* (1) *are ultimate Poisson μ-equibounded with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$. Then the solutions of system* (1) *are ultimate Poisson y-equibounded.*

Corollary 5. *Suppose that, for system* (1)*, there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$, a continuously differentiable function $V(t, x) \ge 0$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, and a continuous function $f(t, \xi)$, defined on $\mathbb{R}^+ \times \mathbb{R}$ for which the following conditions hold*:

(1) $b(\|y\|) \le V(t, x) \le a(t, \|x\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$;

(2) $\dot{V}(t, x) \le f(t, V(t, x))$ *for all* $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$;

(3) *the solutions of the equation $\dot{\xi} = f(t, \xi)$ are ultimately Poisson equibounded with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$.*

Then the solutions of system (1) *are ultimate Poisson y-equibounded.*

Recall [3] that the solutions of system (1) are said to be *ultimately y-equibounded with z_0-control* if there exists a number $B > 0$ such that, given any $t_0 \ge 0$ and any $\alpha \ge 0$, there is a number $T \ge 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $\|z_0\| \le \alpha$, satisfies the condition

$$\|y(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T).$$

Below we introduce the notion of ultimate y-equiboundedness with Poisson z_0-control of solutions, which generalizes the notion of ultimate y-equiboundedness with z_0-control.

Definition 7. We say that the solutions of system (1) are *ultimately y-equibounded with Poisson z_0-control* if there exists a number $B > 0$ and a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$ such that, given any $t_0 \in M(\tau)$ and any $\alpha \ge 0$, there is a number $T \ge 0$ for which any solution $x = x(t, t_0, x_0)$ of (1) with $\|z_0\| \le \alpha$ satisfies the condition

$$\|y(t, t_0, x_0)\| < B \qquad \text{for all} \quad t \in \mathbb{R}^+(t_0 + T) \cap M(\tau).$$

When it is required to indicate the exact corresponding $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$, we say that the solutions of system (1) are *ultimately y-equibounded with Poisson z_0-control with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$*.

Obviously, if the solutions of system (1) are ultimately y-equibounded with z_0-control, then they are ultimately y-equibounded with Poisson z_0-control.

The following statement, whose proof is similar to that of Theorem 3, provides a sufficient condition for the ultimate y-equiboundedness with Poisson z_0-control of solutions; it is based on the method of vector Lyapunov functions.

Theorem 6. *Suppose that, for system* (1)*, there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$, a vector Lyapunov function $V(t, x)$, and numbers $1 \le p \le q \le l$ for which the following conditions hold*:

(1) $V_1(t, x) \ge 0, \dots, V_q(t, x) \ge 0$;

(2) $b(\|y\|) \le \sum_{i=1}^{p} V_i(t, x)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$;

(3) $\sum_{i=1}^{q} V_i(t, x) \le a(t, \|z\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$.

Suppose also that the solutions of the comparison system (2) *for system* (1) *are ultimately μ-equibounded with Poisson γ_0-control with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$. Then the solutions of system* (1) *are ultimately y-equibounded with Poisson z_0-control.*

Corollary 6. *Suppose that, for system* (1)*, there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$, a continuously differentiable function $V(t, x) \ge 0$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, and a continuous function $f(t, \xi)$ defined on $\mathbb{R}^+ \times \mathbb{R}$ for which the following conditions hold*:

(1) $b(\|y\|) \le V(t, x) \le a(t, \|z\|)$ *for all* $(t, x) \in M(\tau) \times \mathbb{R}^n$;

(2) $\dot V(t, x) \le f(t, V(t, x))$ *for all* $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$;

(3) *the solutions of the equation $\dot\xi = f(t, \xi)$ are ultimately Poisson equibounded with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\ge1}$.*

Then the solutions of system (1) *are ultimately y-equibounded with Poisson z_0-control.*

Example. Consider the system of differential equations

$$\begin{cases} \dot x_1 = \dfrac{\sin(t) + \cos(t) - f(t, x_1, x_2)}{1 + \sin(t) + e^{-t}} x_1 + \dfrac{\sin(t) - \cos(t)}{1 + \sin(t) + e^{-t}} x_2, \\ \dot x_2 = \dfrac{\sin(t) - \cos(t)}{1 + \sin(t) + e^{-t}} x_1 + \dfrac{\sin(t) + \cos(t) - f(t, x_1, x_2)}{1 + \sin(t) + e^{-t}} x_2, \end{cases} \tag{5}$$

where $t \in \mathbb{R}^+$, $(x_1, x_2) \in \mathbb{R}^2$ and $f(t, x_1, x_2)$ is any continuous function satisfying the condition $f(t, x_1, x_2) \ge 4$ for which all solutions of system (5) can be continued to the whole semiaxis $\mathbb{R}^+$. The uniqueness of a solution of the Cauchy problem for system (5) is not required. Let us show by using Theorem 1 that the solutions of (5) are uniform-ultimately Poisson bounded. Consider the increasing sequence $\tau = \{\tau_i\}_{i\ge1}$, where $\tau_1 = 0$ and $\tau_2 < \tau_3 < \dots < \tau_i < \dots$ is the increasing sequence of the roots of the equation $\sin(t) + e^{-t} = 0$. Obviously, we have $\lim_{i\to\infty} \tau_i = +\infty$; therefore, $\tau = \{\tau_i\}_{i\geq 1}$ is a $\mathcal{P}$-sequence. Now consider the vector function $V\colon \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by as follows for each $(t, x) = (t, x_1, x_2) \in \mathbb{R}^+ \times \mathbb{R}^2$:

$$V_1(t,x) = \frac{1}{2}(1 + \sin(t) + e^{-t})(x_1 + x_2)^2 \geq 0,$$
$$V_2(t,x) = \frac{1}{2}(1 + \sin(t) + e^{-t})(x_1 - x_2)^2 \geq 0.$$

Obviously, for all $t \in \mathbb{R}^+$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we have

$$V_1(t,x) + V_2(t,x) \leq a(\|x\|), \qquad \text{where} \quad a(r) = 3r^2.$$

Since $1 \leq 1 + \sin(t) + e^{-t}$ on each closed interval $[\tau_{2i-1}; \tau_{2i}]$, $i \geq 1$, if follows that we have, for all pairs $(t, x) \in M(\tau) \times \mathbb{R}^2$,

$$b(\|x\|) \leq V_1(t,x) + V_2(t,x), \qquad \text{where} \quad b(r) = r^2.$$

This inequality implies that, for all $(t, x) \in M(\tau) \times \mathbb{R}^2$, the double inequality

$$b(\|x\|) \leq V_1(t,x) + V_2(t,x) \leq a(\|x\|)$$

holds. Thus, conditions (1) and (2) of Theorem 1 are satisfied. Direct calculations show that, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^2$, we have

$$\dot{V}_1(t,x) = \frac{(\cos(t) - e^{-t}) + 4\sin(t) - 2f(t, x_1, x_2)}{1 + \sin(t) + e^{-t}} V_1(t,x),$$
$$\dot{V}_2(t,x) = \frac{(\cos(t) - e^{-t}) + 4\cos(t) - 2f(t, x_1, x_2)}{1 + \sin(t) + e^{-t}} V_2(t,x),$$

from which we obtain the inequalities

$$\dot{V}_1(t,x) \leq -V_1(t,x), \quad \dot{V}_2(t,x) \leq -V_2(t,x) \qquad \text{for all} \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2,$$

because

$$1 + \sin(t) + e^{-t} \leq 3, \qquad -2f(t, x_1, x_2) \leq -8,$$
$$4\sin(t) \leq 4, \qquad 4\cos(t) \leq 4, \qquad \cos(t) - e^{-t} \leq 1.$$

Obviously, the vector function $f(t, \xi_1, \xi_2) = (-\xi_1, -\xi_2)$ satisfies the condition $f(t, \xi_1, \xi_2) \in W$. Therefore, the comparison system for system (5) is

$$\dot{\xi}_1 = -\xi_1, \qquad \dot{\xi}_2 = -\xi_2.$$

It is easy to see that the solutions of this system are uniform-ultimately bounded and, therefore, uniform-ultimately Poisson bounded with respect to the $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geq 1}$. Thus, all assumptions of Theorem 1 hold and, therefore, the solutions of system (5) are uniform-ultimately Poisson bounded.

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