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One Approach to the Computation of Asymptotics of Integrals of Rapidly Varying Functions

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Abstract—We consider integrals of the form

$$I(x,h)=\frac{1}{(2\pi h)^{k/2}}\int_{\mathbb{R}^k} f\left(\frac{S(x,\theta)}{h},x,\theta\right)d\theta,$$

where h is a small positive parameter and $S(x,\theta)$ and $f(\tau,x,\theta)$ are smooth functions of variables $\tau\in\mathbb{R}$, $x\in\mathbb{R}^n$, and $\theta\in\mathbb{R}^k$; moreover, $S(x,\theta)$ is real-valued and $f(\tau,x,\theta)$ rapidly decays as $|\tau|\to\infty$. We suggest an approach to the computation of the asymptotics of such integrals as $h\to 0$ with the use of the abstract stationary phase method.

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1. INTRODUCTION

Rapidly oscillating integrals of the form

$$I(x,h)=\frac{1}{(2\pi h)^{k/2}}\int_{\mathbb{R}^k} e^{(i/h)S(x,\theta)}A(x,\theta)\,d\theta, \tag{1}$$

where the phase $S(x,\theta)$ and the amplitude $A(x,\theta)$ are smooth functions of variables $x\in\mathbb{R}^n$ and $\theta\in\mathbb{R}^k$ and a small positive parameter h, the function $S(x,\theta)$ is real-valued, and $A(x,\theta)$ is either compactly supported or decaying sufficiently rapidly as $|x|+|\theta|\to\infty$, so that the integral is convergent, occur in numerous problems of theoretical and mathematical physics. In particular, they are used in the construction of the Maslov canonical operator [1], [2], which gives semiclassical asymptotics of solutions for a broad class of such problems with rapidly oscillating data. Accordingly, integrals of the form (1) are well studied. Their asymptotics as $h\to 0$ can be computed by the stationary phase method (see, e.g., [3]). The simplest formulas arise if the phase has only nondegenerate stationary points. In more complicated generic cases, the asymptotics can be expressed via standard integrals, and its computation involves studying the set of stationary points and normal forms in a neighborhood of these points [4], [5].

It turns out that, along with the integral (1), many problems contain a more general integral

$$I(x,h)=\frac{1}{(2\pi h)^{k/2}}\int_{\mathbb{R}^k} f\left(\frac{S(x,\theta)}{h},x,\theta\right)d\theta, \tag{2}$$

for which the integral (1) is the special case with $f(\tau,x,\theta)=e^{i\tau}A(x,\theta)$. Integral (2) is of special interest in the following two cases:

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(a) $f(\tau, x, \theta)$ is a smooth function decaying sufficiently rapidly as $\tau \to \pm\infty$; such integrals occur, in particular, in the construction of asymptotic solutions of hyperbolic equations and systems with localized initial conditions (see, e.g., [6] and the references cited therein);

(b) $f(\tau, x, \theta)$ is a smooth function 2π-periodic in the variable τ; this case is useful when studying cell problems in homogenization theory (see [7] and the bibliography there).

We mainly deal with case (a) in what follows. The asymptotics of such integrals have been studied much less than those of integrals (1). First, note that, although the integral (2) can formally be reduced to (1) as

$$I(x,h) = \frac{1}{\sqrt{2\pi}\,(2\pi h)^{k/2}} \int_{\mathbb{R}^{k+1}} e^{(i/h)\xi S(x,\theta)} \widetilde{f}(\xi, x, \theta)\, d\theta\, d\xi, \tag{3}$$

where

$$\widetilde{f}(\xi, x, \theta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi\tau} f(\tau, x, \theta)\, d\tau \tag{4}$$

is the Fourier transform of the function $f(\tau, x, \theta)$ with respect to the variable τ, the reduction is of little use for the asymptotic computation of the integral. Indeed, the set $\Omega(x)$ of stationary points of the phase function $\Phi(x, \theta, \xi) = \xi S(x, \theta)$ of the integral (3) is the union

$$\Omega(x) = \{(\theta, \xi) : S(x,\theta) = 0,\ \xi = 0\} \cup \{(\theta, \xi) : S(x,\theta) = 0,\ S_\theta(x,\theta) = 0\}.$$

Assume that $S(x, \theta)$ is a generic function. Then the first component of the union may consist of isolated points only if the number k of variables θ is 1. The second component is empty for some x, while for other x (for which the system of equations $S(x,\theta) = 0$, $S_\theta(x,\theta) = 0$ is solvable with respect to θ), it is the union of several straight lines on each of which θ is constant and ξ is arbitrary. Thus, if the integral (2) in question is either at least double or depends on at least one parameter x, then the phase function $\Phi(x, \theta, \xi)$ of the corresponding the integral (3) is not generic, which renders the methods in [3]–[5] useless.

A representation of localized functions via rapidly oscillating ones in the form of a parametric integral of the Maslov canonical operator on a Lagrangian manifold of a special form was proposed in [8]. This representation, which has later been used in [6] and many other papers, allows one to avoid the direct study of integrals of the form (2) when constructing asymptotic solutions of hyperbolic equations and systems with localized initial data in some cases. Nevertheless, the problem of the asymptotics of these integrals as $h \to 0$ remains topical, and the present paper discusses one of the possible approaches to the computation of the asymptotics of such integrals.

On the one hand, the idea of our approach is suggested by the observation in [9] that "*…the qualitative behavior of the solution* [of a strictly hyperbolic system with localized initial data] *must resemble the propagation of discontinuities in the problem with the delta function at the initial time…*"; in other words, there is an analogy between the asymptotics with respect to the parameter of solutions with localized initial data and the asymptotics with respect to smoothness of solutions with initial data like the delta function. On the other hand, the paper [10] provides the asymptotics with respect to smoothness in terms of the $\partial/\partial\tau$-canonical operator based on the *abstract stationary phase method* introduced in the book [1, pp. 347–393], i.e., on the stationary phase method for functions ranging in a Banach space with an unbounded operator A playing the role of a large parameter. (In [10], A is the operator $-i\partial/\partial\tau$ on the Sobolev spaces on $\mathbb{R}^1$.) These considerations naturally suggest that the asymptotics of integrals of the form (2) of localized functions can be computed with the use of the abstract stationary phase method. It is this idea which is developed in the present paper.

2. ASYMPTOTICS OF INTEGRAL (2)

Let us describe the classes of functions that will be used in the integral (2).

Definition 1. For any real number $m > -1$, let $\mathcal{P}_m$ be the space of functions $f(\tau)$, $\tau \in \mathbb{R}$, such that the Fourier transform $\widetilde{f}(\xi)$ belongs to the space $\widetilde{\mathcal{P}}_m$ of functions that lie in $L^1(\mathbb{R})$, are infinitely differentiable outside the point $\xi = 0$, and satisfy the estimates

$$\left|\left(\frac{d}{d\xi}\right)^s [|\xi|^{-m}\widetilde{f}(\xi)]\right| \le C_{sN}(1+|\xi|)^{-N}, \qquad s, N = 0, 1, 2, \dots \quad \xi \ne 0, \tag{5}$$

with some constants C_{sN} outside that point.

Lemma 1. (a) *The space $\mathcal{P}_m$ is a Fréchet space with respect to the system of seminorms given by the best possible values of the constants C_{sN} in* (5).

(b) *If $f \in \mathcal{P}_m$, then the function f is infinitely differentiable and satisfies the estimates*

$$|f^{(l)}(\tau)| \le C_l(1+|\tau|)^{-m-1-l}, \qquad l = 0, 1, 2, \dots, \tag{6}$$

with some constants C_l, while the function $|\xi|^{-m}\widetilde{f}(\xi)$ and all of its derivatives has finite right and left one-sided limits at the point $\xi = 0$.

The proof of the lemma is given in the Supplement at the end of the paper.

Now we can state conditions to be imposed on the functions S and f.

Condition 1. The function $f(\tau, x, \theta)$ lies in the space $C_0^\infty(\mathbb{R}^{n+k}, \mathcal{P}_m)$ of smooth compactly supported $\mathcal{P}_m$-valued functions of variables $(x, \theta) \in \mathbb{R}^{n+k}$.

Condition 2. The function $S(x, \theta)$ is smooth and real-valued, and all of its stationary points with respect to θ on the support $\operatorname{supp} f$ of the function f are nondegenerate.

Theorem 1. *Assume that Condition* 1 *with some $m > k/2 - 1$ and Condition* 2 *are satisfied. Then the integral* (2) *has the following asymptotics as $h \to 0$:*

$$I(x, h) = \sum_{\theta \in \Omega(x)} \frac{[e^{i(\pi/4)\operatorname{sign} S_{\theta\theta}(x,\theta)\operatorname{sign}\widehat{\xi}}|\widehat{\xi}|^{-k/2} f](S(x,\theta)/h, x, \theta)}{\sqrt{|\det S_{\theta\theta}(x,\theta)|}} + R(x, h), \tag{7}$$

where the sum is over the set $\Omega(x) = \{\theta : S_\theta(x, \theta) = 0\}$ of stationary points of the phase function $S(x, \theta)$ for each x,

$$\operatorname{sign} S_{\theta\theta}(x, \theta) = \sigma_+(S_{\theta\theta}(x, \theta)) - \sigma_-(S_{\theta\theta}(x, \theta))$$

is the signature (*the difference of the numbers of positive and negative eigenvalues*) *of the nonsingular symmetric matrix $S_{\theta\theta}(x, \theta)$, $\widehat{\xi} = -i\partial/\partial\tau$, the operator $A(\hat{\xi})$, which is a function of the operator $\hat{\xi}$, is defined by the formula*

$$F(\tau) = A(\widehat{\xi})f(\tau) : \widetilde{F}(\xi) = A(\xi)\widetilde{f}(\xi),$$

and the remainder $R(x, h)$ satisfies the following estimates as $h \to 0$: if $m > k/2$, then

$$\left|\left(-ih\frac{\partial}{\partial x}\right)^\alpha R(x, h)\right| \le C_{|\alpha|} h \tag{8}$$

with some constants C_j, $j = 0, 1, 2, \dots$; if $k/2 - 1 < m \le k/2$, then estimates (8) *remain valid for $|\alpha| > 0$, while the estimate for the function $R(x, h)$ itself has the form*

$$|R(x, h)| \le C_0 \cdot \begin{cases} h|\ln h|, & m = k/2, \\ h^{m-k/2+1}, & k/2 - 1 < m < k/2. \end{cases} \tag{9}$$

Proof. In view of Condition 2, for any $x = x_0$, the set $\Omega(x_0)$ contains an at most finite subset of points θ_0 such that $(x_0, \theta_0) \in \operatorname{supp} f$; moreover, each of these points is determined by a smooth function $\theta(x)$ such that $\theta_0 = \theta(x_0)$ and $\theta(x) \in \Omega(x)$ whenever x is sufficiently close to x_0. Hence one can use a partition of unity and reduce the proof to the case in which the support $\operatorname{supp} f$ is small enough, so that either (a) the function $S(x, \theta)$ has no stationary points with respect to θ on $\operatorname{supp} f$ at all or (b) $\operatorname{supp} f \subset B_1 \times B_2$, where $B_1 \subset \mathbb{R}^n_x$ and $B_2 \subset \mathbb{R}^n_\theta$ are some balls, and, for each $x \in B_1$, the equation $S_\theta(x, \theta) = 0$ of stationary points of $S(x, \theta)$ has a unique solution $\theta = \theta(x) \in B_2$, where $\theta(x)$ is a smooth function.

In case (a), the sum on the right-hand side in (7) is lacking, and the assertion of the theorem is a consequence of the following lemma, whose proof is given at the end of the paper.

Lemma 2. *In case* (a),

$$\left|\frac{\partial^\alpha I}{\partial x^\alpha}(x, h)\right| \le C_\alpha h^{m-k/2+1}, \qquad |\alpha| = 0, 1, 2, \dots, \qquad \textit{as} \quad h \to 0.$$

In case (b), the sum consists of a single term, and formula (7) becomes

$$I(x, h) = \frac{[e^{i(\pi\sigma/4)\operatorname{sign}\widehat{\xi}}|\widehat{\xi}|^{-k/2} f](S_*/h, x, \theta(x))}{\sqrt{|\det S_{\theta\theta*}|}} + R(x, h). \tag{10}$$

Here and throughout the proof of the theorem, we use the following shorthand notation:

$$S_* \equiv S_*(x) := S(x, \theta(x))$$

is the value of the phase function at a stationary point;

$$S_{\theta\theta*} \equiv S_{\theta\theta*}(x) := S_{\theta\theta}(x, \theta(x))$$

is the Hessian matrix of the phase function at a stationary point;

$$\sigma := \operatorname{sign} S_{\theta\theta*} = \sigma_+ - \sigma_- \qquad \text{and} \qquad \sigma_\pm := \sigma_\pm(S_{\theta\theta*})$$

are the signature and the positive and negative indices of inertia, respectively, of this matrix. (Note that σ and $\sigma_\pm$ are independent of $x \in B_1$).

Let us prove the theorem in case (b). First, we show that the first term on the right-hand side in (10) is well defined. The denominator is nonzero by the definition of a nondegenerate stationary point. It remains to show that the function

$$F = e^{i(\pi\sigma/4)\operatorname{sign}\widehat{\xi}}|\widehat{\xi}|^{-k/2} f$$

is well defined. To this end, we use the definition of functions of the operator $\widehat{\xi} = -i\partial/\partial\tau$ via the Fourier transform and find that the Fourier transform $\widetilde{F}$ of the function F with respect to τ must be given by the formula

$$\widetilde{F}(\xi, x, \theta) = e^{i(\pi\sigma/4)\operatorname{sign}\xi}|\xi|^{-k/2}\widetilde{f}(\xi, x, \theta). \tag{11}$$

Since $\widetilde{f} \in C_0^\infty(\mathbb{R}^{n+k}, \widetilde{\mathcal{P}}_m)$, it follows from estimates (5) that $\widetilde{F} \in C_0^\infty(\mathbb{R}^{n+k}, \widetilde{\mathcal{P}}_{m-k/2})$, and since $m - k/2 > -1$, we see that F is well defined; further, $F \in C_0^\infty(\mathbb{R}^{n+k}, \mathcal{P}_{m-k/2})$ (and, accordingly, $F(\tau, x, \theta(x)) \in C_0^\infty(\mathbb{R}^n, \mathcal{P}_{m-k/2})$).

Thus, we need to prove formula (10). First, let us informally explain why this formula is true; a mathematically rigorous argument being postponed until later on in this paper.

Note that the integral (2) can be represented in the form

$$I(x, h) = \frac{1}{(2\pi h)^{k/2}}\left[\int_{\mathbb{R}^k} e^{iAS(x,\theta)} f(\tau, x, \theta)\, d\theta\right]\bigg|_{\tau=0}, \tag{12}$$

where

$$A = -\frac{i}{h}\frac{\partial}{\partial\tau} = \frac{1}{h}\cdot\widehat{\xi}. \tag{13}$$

Indeed, the one-parameter group e^{itA} of linear operators generated by A is given by the formula

$$e^{itA}u(\tau,h)=u\biggl(\tau+\frac{t}{h}\,,h\biggr); \tag{14}$$

i.e., it is the *family of operators of shift by t/h with respect to the variable τ*. Thus, the integrand in (2) has the form

$$f\biggl(\frac{S(x,\theta)}{h}\,,x,\theta\biggr)=[e^{iAS(x,\theta)}f](\tau,x,\theta)|_{\tau=0}.$$

Let us use the abstract stationary phase method [1, pp. 347–393] in the form given in [2, Theorem 1.19, pp. 54–55]. By that theorem (in view of the fact that the operator A is *not positive definite*), the bracketed expression in (12) has the following "asymptotics" for each given $\varepsilon>0$:

$$\begin{aligned}&\int_{\mathbb{R}^k}e^{iAS(x,\theta)}f(\tau,x,\theta)\,d\theta\\&\qquad=\frac{(2\pi)^{k/2}}{\sqrt{|\det S_{\theta\theta*}|}}\bigl[e^{i\pi\sigma/4+iAS_*}(A+i\varepsilon E)^{-\sigma_+/2}(A-i\varepsilon E)^{-\sigma_-/2}f\bigr](\tau.x,\theta(x))+g_1,\end{aligned}$$

where the definition of the operators $(A\pm i\varepsilon E)^{-1/2}$ uses the branch of the square root defined in the plane with a cut along the negative real line and positive for positive values of the argument and the element $g_1=g_1(\varepsilon)$ lies in the domain of the operator $A^{[k/2]+1}$ and hence can be represented in the form $g_1(\varepsilon)=A^{-[k/2]-1}w(\varepsilon)$ for some element $w(\varepsilon)$. We use the explicit form (13) of the operator A to rewrite the last relation in the form

$$\begin{aligned}&\int_{\mathbb{R}^k}e^{iAS(x,\theta)}f(\tau,x,\theta)\,d\theta\\&\qquad=\frac{(2\pi h)^{k/2}[e^{i\pi\sigma/4}(\widehat{\xi}+ih\varepsilon E)^{-\sigma_+/2}(\widehat{\xi}-ih\varepsilon E)^{-\sigma_-/2}f]}{\sqrt{|\det S_{\theta\theta*}|}}\biggl(\tau+\frac{S_*}{h}\,,x,\theta(x)\biggr)\\&\qquad\quad+h^{[k/2]+1}(\widehat{\xi}^{-[k/2]-1}w(\varepsilon)).\end{aligned} \tag{15}$$

Since one can readily see that

$$\lim_{\varepsilon\to+0}e^{i\pi\sigma/4}(\xi+ih\varepsilon)^{-\sigma_+/2}(\xi-ih\varepsilon)^{-\sigma_-/2}=e^{i(\pi\sigma/4)\operatorname{sign}\xi}|\xi|^{-k/2}$$

for the above-indicated choice of a branch of the square root, we obtain formula (10) (with a somewhat weaker remainder estimate than in the theorem), provided that we can appropriately choose the function space on which the operator A acts and prove that the function $\widehat{\xi}^{-[k/2]-1}w(\varepsilon)$ is bounded uniformly with respect to h and ε and that (since f lies in $C_0^\infty(\mathbb{R}^{n+k},\mathcal{P}_m)$ for sufficiently large m) one can pass to the limit as $\varepsilon\to 0$ in (15).

Let us proceed to a rigorous argument. A straightforward implementation of the above scheme of proof encounters some technical difficulties; to avoid these, we make the Fourier transform with respect to τ and carry out all estimates in terms of Fourier transforms. By the assumption of the theorem, the Fourier transform of $f(\tau,x,\theta)$ with respect to τ can be written in the form

$$\widetilde{f}(\xi,x,\theta)=|\xi|^m g(\xi,x,\theta),$$

where the function $g(\xi,x,\theta)$ is compactly supported in (x,θ), is continuous and infinitely differentiable for $\xi\neq0$, and satisfies the estimates

$$\biggl|\frac{\partial^{s+|\alpha|+|\beta|}g}{\partial\xi^s\,\partial x^\alpha\,\partial\theta^\beta}(\xi,x,\theta)\biggr|\le C_{s\alpha\beta N}(1+|\xi|)^{-N},\qquad s,\alpha,\beta,N=0,1,2,\dots,\quad \xi\neq0.$$

Accordingly, the integral (2) can be written in the form

$$I(x,h)=\frac{1}{\sqrt{2\pi}\,(2\pi h)^{k/2}}\int_{-\infty}^{\infty}|\xi|^m\biggl\{\int_{\mathbb{R}^k}e^{i\lambda S(x,\theta)}g(\xi,x,\theta)\,d\theta\biggr\}\bigg|_{\lambda=\xi/h}d\xi. \tag{16}$$

The integral in braces on the right-hand side in (16) is given by the formula

$$\int_{\mathbb{R}^k} e^{i\lambda S(x,\theta)} g(\xi,x,\theta)\,d\theta = \left(\frac{2\pi}{|\lambda|}\right)^{k/2} \frac{e^{i(\pi\sigma/4)\operatorname{sign}\lambda + i\lambda S_*}}{\sqrt{|\det S_{\theta\theta *}|}}[g(\xi,x,\theta(x)) + r(\xi,x,\lambda)], \tag{17}$$

where the remainder $r(\xi,x,\lambda)$ satisfies the estimates

$$\left|\frac{\partial^\alpha r}{\partial x^\alpha}(\xi,x,\lambda)\right| \le C_{N\alpha}(1+|\lambda|)^{-1}(1+|\xi|)^{-N}, \qquad |\alpha|, N = 0,1,2,\dots\,. \tag{18}$$

Indeed, for $|\lambda| \ge 1$, formulas (17) and (18) are none other than the standard stationary phase method [3]. (The factor $(1+|\xi|)^{-N}$ is due to the fact that each constant in the remainder estimate in the stationary phase method linearly depends on finitely many constants in the estimates for the derivatives of g.) For $|\lambda| \le 1$, estimate (18) readily follows from the fact that, for these λ, the left-hand side of (17), together with all of its derivatives, is $O(1)$, and hence $g + r = O(\lambda^{k/2})$ (together with all derivatives as well).

Let us substitute expansion (17) into (16). The first term gives the leading term in (10) (cf. (11)), and we arrive at formula (10) with a remainder $R(x,h)$ of the form

$$R(x,h) = \frac{1}{\sqrt{2\pi|\det S_{\theta\theta *}|}} \int_{-\infty}^{\infty} |\xi|^{m-k/2} e^{i(\pi\sigma/4)\operatorname{sign}\xi + i(\xi/h)S_*} r\left(\xi, x, \frac{\xi}{h}\right) d\xi. \tag{19}$$

Formula (19), in view of the estimates (18) for $r(x,\xi,\lambda)$, implies that

$$\begin{aligned} \left|\frac{\partial^\alpha R}{\partial x^\alpha}(x,h)\right| &\le C \sum_{j=0}^{|\alpha|} \int_0^\infty \xi^{m-k/2}\left(\frac{\xi}{h}\right)^j \left(1+\frac{\xi}{h}\right)^{-1}(1+\xi)^{-N_j}\,d\xi \\ &= Ch^{1-|\alpha|} \sum_{j=0}^{|\alpha|} h^{|\alpha|-j} \int_0^\infty \frac{\xi^{m+j-k/2}\,d\xi}{(h+\xi)(1+\xi)^{N_j}}\,, \end{aligned} \tag{20}$$

where the N_j will be chosen below to ensure the convergences of the integrals at infinity. Here and in what follows, one letter C is used to denote various constants. Let us estimate individual terms in the sum multiplying $Ch^{1-|\alpha|}$. Consider all possible cases.

(a) Let $m + j > k/2$. Since $m > k/2 - 1$, we see that this is always the case for $j > 1$. Taking a sufficiently large N_j, we obtain the estimate

$$h^{|\alpha|-j}\int_0^\infty \frac{\xi^{m+j-k/2}\,d\xi}{(h+\xi)(1+\xi)^{N_j}} \le h^{|\alpha|-j}\int_0^\infty \frac{\xi^{m+j-1-k/2}\,d\xi}{(1+\xi)^{N_j}} = Ch^{|\alpha|-j} \le C$$

for the corresponding term.

For $j = 0$, there are two more possible cases.

(b) Let $m = k/2$ and $j = 0$. Then we take $N_0 = 1$ and obtain

$$h^{|\alpha|}\int_0^\infty \frac{d\xi}{(h+\xi)(1+\xi)} = \frac{h^{|\alpha|}|\ln h|}{1-h} \le \begin{cases} C, & |\alpha| > 0, \\ C|\ln h|, & |\alpha| = 0. \end{cases}$$

(c) Let $k/2 - 1 < m < k/2$ and $j = 0$. Then we take $N_0 = 0$ and obtain

$$h^{|\alpha|}\int_0^\infty \frac{\xi^{m-k/2}\,d\xi}{h+\xi} = h^{m-k/2+|\alpha|}\int_0^\infty \frac{\xi^{m-k/2}\,d\xi}{1+\xi} \le \begin{cases} C, & |\alpha| > 0, \\ Ch^{m-k/2}, & |\alpha| = 0. \end{cases}$$

We substitute these estimates for the summands into (20) and arrive at estimates (8) and (9). The proof of the theorem is complete. □

3. EXAMPLES

1. Consider the integral

$$J(x,h)=\iint_{\mathbb{R}^2}\frac{\varphi(x,\theta)\,d\theta_1\,d\theta_2}{(S(x,\theta)+ih)^2}\,, \tag{21}$$

where $S(x,\theta)$ is a smooth real-valued function of the variables $\theta=(\theta_1,\theta_2)\in\mathbb{R}^2$ and the parameters $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $\varphi(x,\theta)$ is a smooth compactly supported function, and h is a small positive parameter. This integral can be written in the form

$$J(x,h)=\frac{2\pi}{h}I(x,h),$$

where $I(x,h)$ is the following integral of the form (2):

$$I(x,h)=\frac{1}{2\pi h}\iint_{\mathbb{R}^2}f\biggl(\frac{S(x,\theta)}{h},x,\theta\biggr)\,d\theta_1\,d\theta_2,\qquad f(\tau,x,\theta)=\frac{\varphi(x,\theta)}{(\tau+i)^2}\,.$$

The Fourier transform of $f(\tau,x,\theta)$ can be written as

$$\widetilde{f}(\xi,x,\theta)=\begin{cases}-i\xi e^{-\xi}\varphi(x,\theta), & \xi>0,\\ 0, & \xi<0.\end{cases} \tag{22}$$

Thus, the function $f(\tau,x,\theta)$ lies in the space $C_0^\infty(\mathbb{R}^{n+2},\mathcal{P}_1)$; i.e., Condition 1 holds with $m=1=k/2$. Further, let the function $S(x,\theta)$ satisfy Condition 2; for simplicity, we assume that the stationary point $\theta=\theta(x)$ is unique.

We apply Theorem 1, take into account the explicit expression for the function $\widetilde{f}$ (in particular, the fact that its Fourier transform is zero for $\xi<0$, which permits one to omit sign $\widehat{\xi}$ in the formula and get rid of the absolute value sign in the expression $|\widehat{\xi}|^{-1}$), and obtain

$$J(x,h)=\frac{2\pi}{h}\,\frac{[e^{i(\pi/4)\operatorname{sign}S_{\theta\theta}(x,\theta(x))}(\widehat{\xi})^{-1}f](S(x,\theta)/h,x,\theta(x))}{\sqrt{|\det S_{\theta\theta}(x,\theta(x))|}}+\frac{2\pi}{h}R(x,h),$$

where $R(x,h)$ satisfies the estimates specified in the theorem for the case of $m=k/2$. We explicitly compute the leading term and obtain

$$J(x,h)\asymp\frac{2\pi e^{-(i\pi/2)\sigma_-(S_{\theta\theta}(x,\theta(x))}}{\sqrt{|\det S_{\theta\theta}(x,\theta(x))|}}\,\frac{\varphi(x,\theta(x))}{S(x,\theta(x))+ih}\,. \tag{23}$$

2. As a second example, consider the following integral, which was studied in Kuksin's paper[1] [11]:

$$I(\nu)=\int_{\mathbb{R}^{2d}}\frac{F(x,y)\,dx\,dy}{(x\cdot y)^2+\nu^2\Gamma^2(x,y)}\,, \tag{24}$$

where $d\ge 2$, $x\cdot y=x_1y_1+\dots+x_dy_d$, ν is a small positive parameter, $F(x,y)$ and $\Gamma(x,y)$ are real-valued functions satisfying certain conditions (which we omit for brevity) concerning smoothness and behavior at infinity; in particular, $\Gamma(x,y)$ is strictly positive. Consider the asymptotics of this integral as $\nu\to 0$. Integral (24) does not contain free parameters (which are denoted by x in the other parts of the present paper). If, following [11], we denote the integration variables (x,y) by z (earlier in the present paper, the integration variables were denoted by θ), then we can write the integral (24) as the integral (2) multiplied by ν^{-2} with the small parameter $h=\nu$ and the functions

$$f(\tau,z)=\frac{F(z)}{\tau^2+\Gamma^2(z)}\,,\qquad S(z)=x\cdot y. \tag{25}$$

[1] In this section, we use the notation of [11], which is different from ours.

The phase function $S(z)$ has the unique stationary point $x = y = 0$, which is easily seen to be nondegenerate. Nevertheless, according to the main theorem in [11], that the integral (24) has the asymptotics

$$I(\nu) = \frac{\pi}{\nu}\int_{\Sigma_*} \frac{F(z)\,d_{\Sigma^*} z}{|z|\Gamma(z)} + \text{lower-order remainder as } \nu \to +0, \tag{26}$$

where $\Sigma_* = \{z \neq 0 : x \cdot y = 0\}$ and $d_{\Sigma^*} z$ is the Riemannian volume form on Σ^* corresponding to the restriction to Σ^* of the standard Riemannian metric on $\mathbb{R}^{2d}$. Formula (26) is clearly inconsistent with the formula produced by the abstract stationary phase method, because the leading asymptotic term in the latter is independent of the values of $F(z)$ and $\Gamma(z)$ at any points other than the stationary point $z = 0$. This is because the function $f(\tau, z)$ given by (25) does not satisfy Condition 1. Indeed,

$$\widetilde{f}(0, z)|_{\xi=0} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau, z)\,d\tau > 0.$$

Thus, this example shows that Condition 1 in Theorem 1 cannot be dropped.

The compete rigorous proof of formula (26) is rather cumbersome (it can be found in [11]), and rather than reproduce it, we present a simple heuristic argument explaining why the formula is true, at least in the special case where the support of F does not contain the singular point $(0, 0)$. Using a partition of unity and permuting coordinates where necessary, we see that it suffices to prove the assertion for the case in which the support of F is contained in a sufficiently small neighborhood U of the part of Σ^* covered by the coordinate neighborhood with coordinates $(x_2, \dots, x_d, y_1, \dots, y_d)$, where $y_1 \neq 0$ and the form $d_{\Sigma^*} z$ can be expressed as

$$d_{\Sigma^*} z = y_1^{-1}|z|\,dx_2 \wedge \dots \wedge dx_d \wedge dy_1 \wedge \dots \wedge dy_d.$$

Thus, we can equip U with the coordinates $(s, x_2, \dots, x_d, y_1, \dots, y_d)$, where $s = x \cdot y$; in these coordinates, Σ^* is given by the equation $s = 0$, and

$$dx_1 \wedge dx_2 \wedge \dots \wedge dy_d = y^{-1} ds \wedge dx_2 \wedge \dots \wedge dy_d = |z|^{-1} ds \wedge d_{\Sigma^*} z \quad \text{in } U.$$

(Here we have extended the form $d_{\Sigma^*} z$ to the neighborhood U by using the same formula as on the submanifold Σ_*.) We rewrite the integral in the new variables, set $z' = (x_2, \dots, x_d, y_1, \dots, y_d)$ for brevity (which is quite natural in this coordinate neighborhood), and denote the functions F and Γ expressed via the new variables by the same letters; next, we transform the multiple integral into a repeated one and obtain

$$\int_{\mathbb{R}^{2d}} \frac{F(x, y)\,dx\,dy}{(x \cdot y)^2 + \nu^2\Gamma^2(x, y)} = \frac{1}{\nu}\int\left(\int \frac{\nu F(s, z')\,ds}{s^2 + \nu^2\Gamma^2(s, z')}\right)\frac{d_{\Sigma^*} z}{|z|}.$$

The change of variables $s = \nu t$ in the inner integral gives

$$\int \frac{\nu F(s, z')\,ds}{s^2 + \nu^2\Gamma^2(s, z')} = \int \frac{F(\nu t, z')\,dt}{t^2 + \Gamma^2(\nu t, z')} \xrightarrow{\nu \to 0} \int \frac{F(0, z')\,dt}{t^2 + \Gamma^2(0, z')} = \frac{\pi F(0, z')}{\Gamma(0, z')},$$

and we arrive at the assertion of Kuksin's theorem.

4. CONCLUSIONS

We have constructed the asymptotics of integrals of the form

$$\int f\left(\frac{S(x, \theta)}{h}, x, \theta\right) d\theta \qquad \text{as} \quad h \to 0$$

under certain assumptions about the functions $f(\tau, x, \theta)$ and $S(x, \theta)$. Let us make some remarks about the result.

Remark 1. The assertion of Theorem 1 is informative only if the support $\operatorname{supp} f$ contains a stationary point (x_0, θ_0) of $S(x, \theta)$ such that $S(x_0, \theta_0) = 0$. Otherwise, the leading term itself is small (rather than $O(1)$).

Remark 2. One can extend the classes $\mathcal{P}_m$ by requiring the elements of the space $\widetilde{\mathcal{P}}_m$ to be functions smooth outside zero, rapidly decaying at infinity, and satisfying estimates of the form $|f^{(l)}(\xi)| \le C_l|\xi|^{m-l}$ for $\xi \to 0$. Then the estimates in Lemma 1 remain valid for noninteger m and acquire the additional factor $\ln|\tau|$ as $|\tau| \to \infty$ for integer m. The proof is the same, except that now the computation for $m = 0$ must be carried out in the same way as for $-1 < m < 0$. Further, one must have in mind that sometimes the point $\xi = 0$ can produce a substantial contribution to the asymptotics; in that case, the integrals to be studied must be split into integrals over half-spaces, and the contribution of boundary stationary points must be taken into account. In the present paper, we restrict ourselves to the class introduced in Definition 1.

Remark 3. If the stationary points of $S(x,\theta)$ are degenerate but still generic, the methods in [3]–[5] are likely to produce the answer. However, the exponential in this answer will be replaced by more complicated functions (for example, the Airy function in the case of a fold) of the operator A, and the analysis of the corresponding expression becomes more difficult (even apart from the fact that the corresponding theorem has not been proved in [1], so that one will have to prove it separately). We do not dwell on this problem in the present paper. We also point out that, in contrast to the classical stationary phase method (so much the more in the case of degenerate stationary points), the asymptotics of the original integral turns out to be expressed in the form of some integrals with respect to the variable τ, and it is only rarely that these integrals can be expressed via special functions or even via elementary functions. In any case, these are one-dimensional integrals with nonoscillating integrands, which can readily be calculated on a computer with the use of software such as Wolfram Mathematica or MatLab.

Remark 4. The method used in the second example for the heuristic computation of the integral in [11] can be used to construct a general procedure for the asymptotic computation of integrals of the form (2) for the case in which $\widetilde{f}(0,x,\theta) \ne 0$ (i.e., Condition 1 fails). This problem will be considered elsewhere.

Remark 5. Consider the integral (2) in case (b) assuming that $f(\tau,x,\theta)$ is a smooth function 2π-periodic in the variable τ. Our approach also applies to this case; the difference is that the continuous Fourier transform is replaced by the discrete Fourier transform. The parameter ξ is accordingly replaced by an integer k, and so one avoids any "trouble" related to a neighborhood of $\xi = 0$. Namely, let $f(\tau,x,\theta)$ be a 2π-periodic function of τ with Fourier series

$$f(\tau,x,\theta) = \sum_{j=-\infty}^{\infty} f^j(x,\theta)e^{ij\tau}, \qquad \overline{f}(x,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\tau,x,\theta)\,d\tau.$$

The above argument also works *mutatis mutandis* in this case (see also [3]). For example, on can apply (7), at least formally, and obtain

$$I(x,h) = \frac{1}{(2\pi h)^{k/2}}\int_{\mathbb{R}^k} \overline{f}(x,\theta)\,d\theta + \mathcal{I}(x,h) + R(x,h),$$
$$\mathcal{I}(x,h) = \sum_{\theta\in\Omega(x)} \frac{[e^{i(\pi/4)\operatorname{sign} S_{\theta\theta}(x,\theta)\operatorname{sign}\widehat{\xi}}|\widehat{\xi}|^{-k/2}(f - \overline{f}(x,\theta))](S(x,\theta)/h, x, \theta)}{\sqrt{|\det S_{\theta\theta}(x,\theta)|}}. \tag{27}$$

The term $\mathcal{I}(x,h)$ can be rewritten with the use of the Fourier series in the form

$$\mathcal{I}(x,h) = \sum_{\theta\in\Omega(x)}\sum_{j\ne 0} \frac{[e^{i(\pi/4)\operatorname{sign} S_{\theta\theta}(x,\theta)\operatorname{sign} j}|j|^{-k/2} f^j(x,\theta)]e^{(i/h)jS(x,\theta)}}{\sqrt{|\det S_{\theta\theta}(x,\theta)|}}.$$

Just as in the case of integrands decaying with respect to the variable τ, the series in the last formula can rarely be expressed via special functions and usually have to be studied with the use of a computer. Sometimes, the Poisson formula is convenient to use in the analysis (see [3], [12, Introduction, Sec. 8]).

SUPPLEMENT. PROOFS OF TECHNICAL ASSERTIONS

Proof of Lemma 1. The infinite differentiability of the function F follows from the fact that its Fourier transform rapidly decays. Let us prove estimates (6). Note that if $F \in \mathcal{P}_m$, then $F^{(l)} \in \mathcal{P}_{m+l}$, so that it suffices to consider the case of $l = 0$. The function $\widetilde{F}$ or its derivatives may have discontinuities at the point $\xi = 0$, and so we split the Fourier integral representation of F into two integrals (from 0 to ∞ and from $-\infty$ to 0) and estimate, say, the first of them. (The second integral can be estimated in a similar way.) Thus, let us prove (assuming without loss in generality that $\tau > 0$) that

$$\left|\int_0^\infty p^m f(p) e^{ip\tau}\,dp\right| \le C\tau^{-m-1} \qquad \text{as} \quad \tau\to\infty \quad \text{if} \quad f \in \widetilde{\mathcal{P}}_0. \tag{28}$$

For $m > 0$, one can integrate by parts in (28) by using the formulas

$$e^{ip\tau} = -\frac{i}{\tau}\frac{\partial}{\partial p}e^{ip\tau}, \qquad \frac{\partial}{\partial p}(p^m f(p)) = p^{m-1}(mf(p)+pf'(p)) \equiv p^{m-1}f_1(p), \qquad f_1 \in \widetilde{\mathcal{P}}_0,$$

thus reducing the proof of this inequality to the proof of the same inequality in which m is less by 1. By induction, we conclude that it suffices to prove inequality (28) for $-1 < m \le 0$. First, let $m = 0$. Then we obtain the desired inequality by once more integrating by parts. (The integrated term does not affect the estimate.) Now let $-1 < m < 0$. Then

$$\int_0^\infty p^m f(p)e^{ip\tau}\,dp = \int_0^{1/\tau} p^m f(p)e^{ip\tau}\,dp + \int_{1/\tau}^\infty p^m f(p)e^{ip\tau}\,dp \equiv I_1(\tau)+I_2(\tau),$$

$$|I_1(\tau)| \le C\int_0^{1/\tau} p^m\,dp = \frac{Cp^{m+1}}{m+1}\bigg|_0^{1/\tau} = C_1\tau^{-m-1},$$

$$I_2(\tau) = -\frac{i}{\tau}\left(\frac{1}{\tau}\right)^m f\left(\frac{1}{\tau}\right)e^i + \frac{i}{\tau}\int_{1/\tau}^\infty p^{m-1}f_1(p)e^{ip\tau}\,dp,$$

$$|I_2(\tau)| \le C_2\left(\tau^{-m-1} + \tau^{-1}\int_{1/\tau}^\infty p^{m-1}\,dp\right) = C_3\tau^{-m-1}.$$

The proof of the lemma is complete. □

Proof of Lemma 2. We rewrite (16) in the form

$$\begin{aligned} I(x,h) &= \frac{1}{\sqrt{2\pi}\,(2\pi h)^{k/2}}\int_{|\xi|\le h}|\xi|^m\left\{\int_{\mathbb{R}^k} e^{i\lambda S(x,\theta)}g(\xi,x,\theta)\,d\theta\right\}\bigg|_{\lambda=\xi/h}d\xi \\ &\quad + \frac{1}{\sqrt{2\pi}(2\pi h)^{k/2}}\int_{|\xi|\ge h}|\xi|^m\left\{\int_{\mathbb{R}^k} e^{i\lambda S(x,\theta)}g(\xi,x,\theta)\,d\theta\right\}\bigg|_{\lambda=\xi/h}d\xi \\ &\equiv I_1 + I_2. \end{aligned}$$

For I_1, we obtain the estimate

$$\left|\frac{\partial^\alpha I_1}{\partial x^\alpha}(x,h)\right| \le C\int_0^h \xi^m\,d\xi = \widetilde{C}h^{m-k/2+1}.$$

For I_2, we obtain the estimate

$$\left|\frac{\partial^\alpha I_2}{\partial x^\alpha}(x,h)\right| \le C\sum_{j=0}^{|\alpha|}\int_h^\infty \xi^m\left(\frac{h}{\xi}\right)^s\left(\frac{\xi}{h}\right)^j d\xi$$

(where the factors $(\xi/h)^j$ arise when differentiating with respect to x in the integrand, and the factor $(h/\xi)^s$ results from s-fold integration by parts). We take s large enough for the integral to converge, compute the integral, and obtain

$$\left|\frac{\partial^\alpha I_2}{\partial x^\alpha}(x,h)\right| \le Ch^{m-k/2+1}.$$

The proof of the lemma is complete. □

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