ISSN 0001-4346, Mathematical Notes, 2018, Vol. 103, No. 4, pp. 649–655. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © A. R. Chekhlov, 2018, published in Matematicheskie Zametki, 2018, Vol. 103, No. 4, pp. 609–616.

Homomorphically Stable Abelian Groups

A. R. Chekhlov*

Tomsk State University, Tomsk, Russia

Received February 14, 2017; in final form, September 4, 2017

Abstract—A group is said to be homomorphically stable with respect to another group if the union of the homomorphic images of the first group in the second group is a subgroup of the second group. A group is said to be homomorphically stable if it is homomorphically stable with respect to every group. It is shown that a group is homomorphically stable if it is homomorphically stable with respect to its double direct sum. In particular, given any group, the direct sum and the direct product of infinitely many copies of this group are homomorphically stable; all endocyclic groups are homomorphically stable as well. Necessary and sufficient conditions for the homomorphic stability of a fully transitive torsion-free group are found. It is proved that a group is homomorphically stable if and only if so is its reduced part, and a split group is homomorphically stable if and only if so is its torsion-free part. It is shown that every group is homomorphically stable with respect to every periodic group.

DOI: 10.1134/S0001434618030331

Keywords: *homomorphic stability, homomorphic image, group of homomorphisms, homomorphic association, direct sum of groups.*

All groups are assumed to be Abelian. Recall that a torsion-free group G is said to be *rigid* if its endomorphism ring $\mathrm{E}(G)$ is a subring of the field $\mathbb{Q}$ of rational numbers; a subset H of a group A is said to be *fully invariant* if $f(H) \subseteq H$ for every $f \in \mathrm{E}(A)$. Let $\mathbb{N}$ denote the set of positive integers. Given a cardinal number α and a group A, by $A^{(\alpha)}$ we denote the direct sum of α copies of A; the corresponding direct product is denoted by A^{α}. In particular, if $n \in \mathbb{N}$, then $A^{(n)} = A^n$.

A group A is said to be *homomorphically stable with respect to a group B* if the fully invariant subset

$$H = \bigcup_{\alpha \in \mathrm{Hom}(A,B)} \mathrm{Im}\,\alpha(A)$$

is a subgroup of the group B [1]–[3], i.e.,

$$H = \sum_{\alpha \in \mathrm{Hom}(A,B)} \mathrm{Im}\,\alpha(A).$$

If a group A is homomorphically stable with respect to a group B, then A and B are also said to form a *homomorphically stable pair* (A, B). Clearly, a group A is homomorphically stable with respect to a group B if and only if, for every $a \in \alpha(A)$ and $b \in \beta(A)$, where $\alpha, \beta \in \mathrm{Hom}(A, B)$, there are a $c \in A$ and a $\gamma \in \mathrm{Hom}(A, B)$ with the property $\gamma(c) = a + b$. A group homomorphically stable with respect to every group is said to be simply *homomorphically stable*. The homomorphic stability of periodic groups was mentioned in [4, Chap. VIII, Sec. 43, Exercise 11 (a)]; ibidem (Exercise 11 (b)), an example of groups which are not homomorphically stable is given, and, in [5] and [6], noncommutative homomorphically stable groups were studied.

In [1]–[3], the problem of the homomorphic stability of direct sums of groups was considered (see Theorem 1), the homomorphic stability of groups with respect to direct products was investigated, and the homomorphic stability of divisible, periodic, separable, etc. groups was shown.

We give the following simple example of homomorphically stable groups.

*E-mail: cheklov@math.tsu.ru

Example 1. (1) Every group containing a direct summand isomorphic to the group $\mathbb{Z}$ of integers is homomorphically stable.

(2) Every group A with unbounded reduced p-component is homomorphically stable with respect to every p-group.

(3) Every group is homomorphically stable with respect to the direct product (direct sum) of arbitrarily many copies of the group $\mathbb{Z}$.

Proof. (2) For every $n \in \mathbb{N}$, a group A satisfying the conditions in (2) contains a cyclic direct summand of order $\geq p^n$. Therefore, every element of a p-group B is the image of some homomorphism $A \to B$. Clearly, A is homomorphically stable with respect to B.

The proof of assertion (1) is similar.

Assertion (3) follows from the observation that if there is a nonzero homomorphism of a group A to the group $\mathbb{Z}^\alpha$ or to the group $\mathbb{Z}^{(\alpha)}$ for a cardinal $\alpha \geq 1$, then A has a direct summand isomorphic to the group $\mathbb{Z}$. □

The class of homomorphically stable groups is closed with respect to taking direct sums.

Theorem 1 ([1, Theorem 1]). *If $\{A_i\}_{i\in I}$ is a family of groups homomorphically stable with respect to B, then the pair of groups $(\bigoplus_{i\in I} A_i, B)$ is homomorphically stable.*

The following theorem gives a criterion for homomorphic stability.

Theorem 2. *For a group A, the following conditions are equivalent*:

1) *A is homomorphically stable with respect to A^2*;

2) *A is homomorphically stable*;

3) *for any $a, b \in A$, there exists a $c \in A$ and $\alpha, \beta \in \mathrm{E}(A)$ such that $\alpha(c) = a$ and $\beta(c) = b$.*

Proof. (1) ⇒ (3). By assumption, there is a $\gamma \in \mathrm{Hom}(A, A^2)$ and a $c \in A$ such that $\gamma(c) = (a, b)$. If π and θ are the projections of the group A^2 to the first and the second summand, respectively, then

$$\pi\gamma, \theta\gamma \in \mathrm{E}(A), \qquad \pi\gamma(c) = a, \quad \theta\gamma(c) = b.$$

(3) ⇒ (1). Suppose that $u = (x, y) \in \mathrm{Im}\,\alpha$ and $v = (z, g) \in \mathrm{Im}\,\beta$ for some $\alpha, \beta \in \mathrm{Hom}(A, A^2)$. Then $u + v = (a, b)$, where $a = x + z$ and $b = y + g$. By assumption, we have $\varphi(c) = a$ and $\psi(c) = b$. Therefore,

$$(\varphi, \psi) \in \mathrm{Hom}(A, A^2), \qquad (\varphi, \psi)(c) = u + v.$$

(3) ⇒ (2). Suppose that $a = \alpha(x)$ and $b = \beta(y)$, where $\alpha, \beta \in \mathrm{Hom}(A, B)$ and $x, y \in A$. By assumption, we have $x = \gamma(z)$ and $y = \delta(z)$ for some $z \in A$ and $\gamma, \delta \in \mathrm{E}(A)$. Hence

$$(\alpha\gamma + \beta\delta)(z) = a + b, \qquad \alpha\gamma + \beta\delta \in \mathrm{Hom}(A, B). \quad \square$$

Corollary 1. (1) *Every group A with the property $A^2 \cong A$ is homomorphically stable.*

(2) *For every group G and any cardinal number $\alpha \geq \aleph_0$, the groups $G^{(\alpha)}$ and G^α are homomorphically stable.*

Note that, in [1, Theorem 7], the equivalence of assertions (1) and (2) of Theorem 2 was proved for rigid groups.

Proposition 1. *If $A = C \oplus G$ is a homomorphically stable group, where the subgroup G is fully invariant in A, then the group C is homomorphically stable as well.*

Proof. Take $a \in \alpha(C)$ and $b \in \beta(C)$, where $\alpha, \beta \in \mathrm{Hom}(C, C^2)$. Since $A^2 = C^2 \oplus G^2$, we may assume that $\alpha, \beta \in \mathrm{Hom}(A, A^2)$. By assumption, we have

$$a + b = \gamma(c_1) + \gamma(c_2) \qquad \text{for some} \quad \gamma \in \mathrm{Hom}(A, A^2), \quad c_1 \in C, \quad c_2 \in G.$$

Hence if $\pi\colon A^2 \to C^2$ is the projection, then $a + b = \pi\gamma(c_1)$, where $\pi(\gamma \mid C) \in \mathrm{Hom}(C, C^2)$. □

All divisible [3, Theorem 4] and periodic [2, Corollary 2] groups are homomorphically stable; thus, the corollary given below follows from Theorem 1 and Proposition 1.

Corollary 2. (1) *A split group $A = T \oplus G$ is homomorphically stable if and only if its torsion-free part G is homomorphically stable.*

(2) *An arbitrary group is homomorphically stable if and only if its reduced part is homomorphically stable.*

Note that the *if* part in assertion (2) of Corollary 2 follows also from the fact that if the reduced part of the group A is homomorphically stable with respect to a group B, then so is the entire group A [3, Theorem 7].

Proposition 2. *If $G \neq 0$ is a homomorphically stable group with commutative ring $E = \mathrm{E}(G)$ all of whose nonzero elements are monomorphisms of G, then the rank of the module ${}_EG$ is equal to* 1.

Proof. Let $0 \neq x, y \in G$ be elements independent over E. For $\alpha, \beta \in \mathrm{Hom}(G, G^2)$, there exists a $c \in G$ and a $\gamma \in \mathrm{Hom}(G, G^2)$ such that $\gamma(c) = \alpha(x) + \beta(y)$. Passing to coordinates, we obtain

$$\delta(c) = \zeta(x) + \eta(y), \qquad \vartheta(c) = \xi(x) + \lambda(y),$$

where $\delta, \zeta, \eta, \vartheta, \xi, \lambda \in \mathrm{E}(G)$, $\alpha = (\zeta, \xi)$, $\beta = (\eta, \lambda)$, and $\gamma = (\delta, \vartheta)$. Hence

$$(\vartheta\zeta - \delta\xi)x + (\vartheta\eta - \delta\lambda)y = 0,$$

and therefore $\vartheta\zeta = \delta\xi$ and $\vartheta\eta = \delta\lambda$, i.e., $\xi\eta = \zeta\lambda$, which contradicts the arbitrariness of the homomorphisms α and β. □

In particular, any homomorphically stable rigid group has rank 1 [1, Theorem 7].

Recall that a group A is said to be *n-endogenerated* if A is an n-generated $\mathrm{E}(A)$-module. A 1-endogenerated group is also said to be *endocyclic*.

Example 2. If A is an n-endogenerated group for some positive integer n, then the group A^n is homomorphically stable. In particular, every endocyclic group is homomorphically stable.

Proof. Let

$$a = \alpha(x), \qquad b = \beta(y),$$

where $\alpha, \beta \in \mathrm{Hom}(A^n, B)$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in A^n$, $c_1, \dots, c_n$ are generators of the $\mathrm{E}(A)$-module A, and

$$x_i = \varphi_{i1}(c_1) + \cdots + \varphi_{in}(c_n), \qquad y_i = \psi_{i1}(c_1) + \cdots + \psi_{in}(c_n),$$

where $\varphi_{ij}, \psi_{ij} \in \mathrm{E}(A)$ and $i, j = 1, \dots, n$. Then

$$\varphi = (\varphi_{ij}), \quad \psi = (\psi_{ij}) \in \mathrm{E}(A^n), \qquad x = \varphi(c), \quad y = \psi(c),$$

where $c = (c_1, \dots, c_n) \in A^n$ and $a + b = (\alpha\varphi + \beta\psi)c$. □

Remark 1. A direct summand of a group homomorphically stable with respect to a group B need not be homomorphically stable with respect to B. Indeed, every group is homomorphically stable with respect to itself; however, a rigid group G of rank > 1 is not homomorphically stable with respect to $G \oplus G$. Further, according to Example 1, the group $\mathbb{Z} \oplus G$ is homomorphically stable for every (not necessarily homomorphically stable) direct summand G. As far as the direct summands of the group B are concerned, the following fact holds: if a group A is homomorphically stable with respect to B, then A is homomorphically stable with respect to every direct summand of B [1, Theorem 2].

Example 3. If a group A has a direct summand isomorphic to a group B, then A is homomorphically stable with respect to B.

Proof. Indeed, in this case, for every $b \in B$, there is an $a \in A$ and a homomorphism $A \to B$ taking a to b. $\square$

Recall that a torsion-free group A is said to be *fully transitive* if, for any elements $0 \neq a, b \in A$, the condition $\chi(a) \leq \chi(b)$ on their characteristics implies that $\alpha(a) = b$ for some $\alpha \in E(A)$. A generalization of this class of groups was studied in [7].

Theorem 3. *A fully transitive torsion-free group A is homomorphically stable if and only if, given any types t_1 and t_2 in the set $T(A)$ of types of all nonzero elements of the group A, there is a $\tau \in T(A)$ such that $\tau \leq t_1, t_2$.*

Proof. *Necessity.* By Theorem 2, for any $a, b \in A$, there exists a $c \in A$ and $\alpha, \beta \in \mathrm{E}(A)$ such that $\alpha(c) = a$ and $\beta(c) = b$. In particular, for the types of these elements, we have

$$t(c) \leq t(a), t(b).$$

Sufficiency. Let

$$a = \alpha(x), \qquad b = \beta(y),$$

where $\alpha, \beta \in \mathrm{Hom}(A, B)$, $x, y \in A$, and $t(x), t(y) \geq \tau \in T(A)$. There is an element z of type τ such that $\chi(z) \leq \chi(x)$. Let $n \in \mathbb{N}$ be the least number for which $\chi(z) \leq \chi(ny)$; we have $z \in nA$, because n is minimal. If $nz_0 = z$, then

$$\chi(z_0) \leq \chi(x), \qquad \chi(z_0) \leq \chi(y).$$

By assumption, we have $\varphi(z_0) = x$ and $\psi(z_0) = y$ for some $\varphi, \psi \in \mathrm{E}(A)$. Hence

$$(\alpha\varphi + \beta\psi)(z_0) = a + b. \quad \square$$

A group A is said to be *finitely homomorphically associated* with a family of groups $\{B_i\}_{i \in I}$ if, for every finite subset $J \subseteq I$ and any homomorphisms $\alpha_j \in \mathrm{Hom}(A, B_j)$ and elements $a_j \in A$, there exists an element $a \in A$ and a family of homomorphisms $\beta_j \in \mathrm{Hom}(A, B_j)$ such that $\alpha_j(a_j) = \beta_j(a)$ for every $j \in J$. Note that the notion of a *homomorphically associated* group (without the finiteness condition) was introduced in [1] to study homomorphic stability with respect to direct products of groups.

Theorem 4. *Let $B = \bigoplus_{i \in I} B_i$. A pair of groups (A, B) is homomorphically stable if and only if every pair (A, B_i) is homomorphically stable and A is finitely homomorphically associated with the family of groups $\{B_i\}_{i \in I}$.*

Proof. The *only if* part is obvious due to the aforementioned property of direct summands of the group B in Example 1.

Sufficiency. Let

$$a = \alpha(x), \qquad b = \beta(y),$$

where $\alpha, \beta \in \mathrm{Hom}(A, B)$ and $x, y \in A$. If $\pi_i \colon B \to B_i$ are the projections, then $\pi_i\alpha \in \mathrm{Hom}(A, B_i)$ and

$$a = \sum_{j \in J} \pi_j \alpha(x), \quad b = \sum_{j \in J} \pi_j \beta(y) \qquad \text{for a finite set} \quad J \subseteq I.$$

By assumption, we have $\pi_j\alpha(x) + \pi_j\beta(y) = \gamma_j(c_j)$ for some $\gamma_j \in \mathrm{Hom}(A, B_j)$ and $c_j \in A$. If $\delta_j(c) = \gamma_j(c_j)$, then

$$\delta = \sum_{j \in J} \delta_j \in \mathrm{Hom}(A, B), \qquad \delta(c) = a + b. \quad \square$$

The following technical assertion is valid.

Corollary 3. *Every group A is finitely homomorphically associated with any family $\{B_p\}_{p\in\Pi}$ of p-groups B_p, where Π is a set of primes.*

Proof. Let $p_1,\dots,p_n$ be a finite family of distinct primes, and let $b_i=\alpha_i(a_i)$, where $\alpha_i\in\operatorname{Hom}(A,B_{p_i})$. We set $s_i=\prod_{j\neq i}p_j^{m_{ji}}$, where each $p_j^{m_{ji}}$ is the order of $\alpha_j(a_i)$. Note that every number s_i is coprime to p_i. Therefore, $u_is_ib_i=b_i$ for some integer u_i. Since $\alpha_j(u_is_ia_i)=0$ for $j\neq i$, it follows that

$$\alpha_j(u_1s_1a_1+\dots+u_ns_na_n)=\alpha_j(u_js_ja_j)=b_j.$$

Note that the proof uses only the finiteness of the orders of the b_i and does not use the condition $\alpha_i\in\operatorname{Hom}(A,B_{p_i})$. □

As mentioned above, all periodic groups are homomorphically stable; the following assertion also holds.

Proposition 3. *Every group A is homomorphically stable with respect to every periodic group B.*

Proof. By virtue of Corollary 3 and Theorem 5, we may assume that B is a p-group and the group A is reduced. Take $a\in\alpha(A)$ and $b\in\beta(A)$, where $\alpha,\beta\in\operatorname{Hom}(A,B)$, and let p^k be the order of the element $a+b$. If the group A contains a cyclic direct summand of order $\geq p^k$ with a generator c, then $\gamma(c)=a+b$ for some $\gamma\in\operatorname{Hom}(A,B)$. If the group A has no such summands, then its p-component is a bounded group for which the orders of all elements are $<p^k$ and, therefore, this p-component is a direct summand of A: $A=A_p\oplus G$. The periodic part of the group G is contained in the kernel of every homomorphism $A\to B$; thus, we may assume that G is a torsion-free group. Since at least one of the elements a and b is of order $\geq p^k$, it follows that this element is an image of an element of A of infinite order, and since $p^{k-1}A_p=0$, it follows that $pG\neq G$. Hence the element $a+b$ is a homomorphic image of some element of the quotient group G/p^kG and, therefore, a homomorphic image of some element of the group A itself. □

Corollary 4. *Every group A is homomorphically stable with respect to any direct sum B of cyclic groups.*

Proof. If there are homomorphic images of A which are not contained in the periodic part of B, then, since the torsion-free part of B is a free group, it follows that the group A has a direct summand isomorphic to the group $\mathbb{Z}$. In this case, in view of Example 1, the group A is homomorphically stable. If B is a periodic group, then the assertion follows from Proposition 3. □

Proposition 4. *Every group A is homomorphically stable with respect to any group B of rank* 1.

Proof. A group of rank 1 is either a periodic group isomorphic to a subgroup of a quasicyclic group for some prime p or a torsion-free group isomorphic to a subgroup of the group of rational numbers. Every group is homomorphically stable with respect to any divisible group [3, Theorem 4] and with respect to any cyclic group (Corollary 4). Therefore, suppose that B is a torsion-free group of rank 1 and

$$0\neq\alpha\in\operatorname{Hom}(A,B),\qquad b=\alpha(x)\neq 0.$$

We may assume that A is a torsion-free group of rank 1. If $0\neq g\in B$, then $nb=mg$ for some coprime integers n and m. Let q be a prime divisor of m. If $x=qy$ for some $y\in A$, then $\alpha(y)=z$, where $qz=b$. If $x\notin qA$, then $\gamma(x)=z$ for some $\gamma\in\operatorname{Hom}(A,B)$. Arguing by induction, we see that there is an $a\in A$ and a $\delta\in\operatorname{Hom}(A,B)$ for which $\delta(a)=b/m$. Hence

$$\delta(ma+na)=b+g.\quad\square$$

Proposition 5. *A pair of groups (A,B) is homomorphically stable if and only if the group A is homomorphically stable with respect to the reduced part of B.*

Proof. The *only if* part holds because every group homomorphically stable with respect to B is homomorphically stable with respect to every direct summand of the group B [1, Theorem 2]; in turn, the reduced part is a direct summand of the group.

Sufficiency. All periodic and divisible groups are homomorphically stable [2, Corollary 2] and [3, Theorem 4]. Further, if the reduced part of A is homomorphically stable with respect to B, then A is homomorphically stable with respect to B [3, Theorem 7]. Therefore, we suppose that A is a reduced mixed group, $\alpha, \beta \in \mathrm{Hom}(A, B)$, and $x, y \in A$. We have $B = D \oplus G$, where D is the divisible part and G is the reduced part of B. If $\pi\colon B \to D$ and $\theta\colon B \to G$ are the projections, then, by assumption, we have

$$\gamma(z) = a \qquad \text{for some} \quad \gamma \in \mathrm{Hom}(A, G), \quad z \in A,$$

where $a = \theta\alpha(x) + \theta\beta(y)$. If the element z is of infinite order, then

$$\delta(z) = \pi\alpha(x) + \pi\beta(y) \qquad \text{for some} \quad \delta \in \mathrm{Hom}(\langle z\rangle, D).$$

Since the group D is injective, we may assume that $\delta \in \mathrm{Hom}(A, D)$. Then

$$\gamma + \delta \in \mathrm{Hom}(A, B) \qquad \text{and} \qquad (\gamma + \delta)(z) = \alpha(x) + \beta(y).$$

Therefore, it remains to show that the element z in the equation $\gamma(z) = a$ can be chosen to be of infinite order. This is the case if at least one of the elements $\theta\alpha(x)$ and $\theta\beta(y)$ has infinite order.

Now, suppose that $\theta\alpha(x)$ and $\theta\beta(y)$ are contained in the periodic part of the group G. Considerations similar to those in the proof of Corollary 3 show that it sufiices to consider the case in which these elements belong to some p-component. Let p^k denote the order of the element a. The order of the element z is $\geq p^k$. Since the group A is reduced, it follows that the p-component of A contains a cyclic direct summand $\langle g\rangle$ of order $\geq p^k$. For some $\eta \in \mathrm{Hom}(\langle g\rangle, G)$, we have $\eta(g) = a$. Let us extend η to a homomorphism $A \to G$ by setting the action of the homomorphism on the complementary direct summand $\langle g\rangle$ to be equal to 0. If c is an element of infinite order in the complementary direct summand, then $\lambda(c) = 0$ and, therefore, $\lambda(c + g) = a$. □

Proposition 5 was proved in [3, Theorem 6] for torsion-free groups. Before passing to Corollary 5, we state Theorem 5, which was stated in [3] for the case in which A is a torsion-free group; however, its proof remains valid for an arbitrary group A. Corollary 5 was also proved in [3, Corollary 10] in the case of torsion-free groups.

Theorem 5 ([3, Theorem 8])**.** *A pair of groups (A, B) is homomorphically stable if and only if the reduced part of A is homomorphically stable with respect to the group B.*

Proposition 5 and Theorem 5 imply the following corollary.

Corollary 5. *A pair of groups (A, B) is homomorphically stable if and only if the reduced part of A is homomorphically stable with respect to the reduced part of B.*

In conclusion, we pose two open problems.

Problem 1. *Study the homomorphic stability of pairs of groups (A^n, A^m), where $n, m \in \mathbb{N}$ and $n < m < 2n$.*

Problem 2. *Study those homomorphically stable groups A for which there are groups not homomorphically stable with respect to A.*

REFERENCES

1. S. Ya. Grinshpon and T. A. El'tsova, "Homomorphic images of Abelian groups," Fundam. Prikl. Mat. **13** (3), 17–24 (2007) [J. Math. Sci. (New York) **154** (3), 290–294 (2008)].
2. S. Ya. Grinshpon and T. A. El'tsova, "Homomorphic stability of Abelian groups," Fundam. Prikl. Mat. **14** (5), 67–76 (2008) [J. Math. Sci. (New York) **163** (6), 670–676 (2009)].
3. S. Ya. Grinshpon and T. A. El'tsova, "Connection of divisible and reduced groups with homomorphic stability," Vestn. Tomsk. Gos. Univ. Mat. Mekh., No. 2 (6), 14–19 (2009).
4. L. Fuchs, *Infinite Abelian Groups*, Vol. 1 (Academic Press, New York, 1970; Mir, Moscow, 1974).
5. I. A. Shilin, V. V. Kityukov and A. A. Aleksandrov, "Computation of homomorphism groups and the verification of homomorphic stability of pairs of finite groups," Prikl. Inform. **37** (1), 111–115 (2012).
6. M. I. Kabenyuk, "Homomorphic stability of finite groups," Prikladnaya Diskretnaya Matematika, No. 35, 5–13 (2017).
7. A. R. Chekhlov, "On fully quasitransitive Abelian groups," Sibirsk. Mat. Zh. **57** (5), 1184–1192 (2016) [Sib. Mat. J. **57** (5), 929–934 (2016)].