ISSN 0001-4346, Mathematical Notes, 2018, Vol. 103, No. 4, pp. 626–634. © Pleiades Publishing, Ltd., 2018.
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On the Irrationality Measures of Certain Numbers. II

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Received June 19, 2012; in final form, June 13, 2017

Abstract—For the irrationality measures of the numbers $\sqrt{2k-1}\ \arctan(\sqrt{2k-1}/(k-1))$, where k is an even positive integer, upper bounds are presented.

DOI: 10.1134/S0001434618030306

Keywords: *irrationality measure.*

1. INTRODUCTION

By the *irrationality measure* $\mu(\alpha)$ of a number $\alpha \notin \mathbb{Q}$ we mean the least upper bound for the set of numbers $\varkappa$ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < q^{-\varkappa}$$

has an infinite number of solutions in rational p/q.

The present paper is devoted to the proof of upper bounds for the irrationality measures of the numbers

$$\alpha_k = \sqrt{2k-1}\ \arctan\frac{\sqrt{2k-1}}{k-1}, \qquad \text{where} \quad k = 2m, \quad m \in \mathbb{N}.$$

Some of the numerical results of this paper are summarized in Table 1 (see Theorem 1).

Table 1.

k	$\mu(\alpha_k) \le$
2	$4.60105\ldots$
4	$3.94704\ldots$
6	$3.76069\ldots$
8	$3.66666\ldots$
10	$3.60809\ldots$
12	$3.56730\ldots$

In particular, the bound for the irrationality measure of the number $\pi/\sqrt{3}$ is improved. In Hata's paper [1], it was proved that $\mu(\pi/\sqrt{3}) \le 4.60158\ldots$. For $k = 2$, we have $\mu(\pi/\sqrt{3}) \le 4.60105\ldots$ (see

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the table). The same estimate was also obtained by Androsenko and Salikhov in [2] by using the Marcovecchio integral (see [3]). Quite recently, in [4], Androsenko proved that $\mu(\pi/\sqrt{3}) \le 4.230464\dots$; the proof uses a modified version of the Marcovecchio integral. For the numbers α_k, where $k = 2m > 2$ and m is an integer, upper bounds for the irrationality measures are obtained for the first time.

The present paper is a continuation of the papers [5]–[9], in which a general integral construction was considered and for the principal parameter a real positive number was taken. In this paper, we use a complex-valued parameter.

The paper consists of five sections. The introduction briefly presents the history of the problem. In Sec. 2, auxiliary functions are introduced. In Sec. 3, arithmetic properties of these functions at certain points are proved. In Sec. 4, asymptotic properties of the values of these functions at cerain points are determined. In the last section, we conclude the proof of bounds for the irrationality exponents of the numbers α_k by using the information obtained Secs. 3 and 4 and Hata's lemma.

2. AUXILIARY INTEGRALS

Consider the polynomial

$$A_n(x) := \binom{x+5n}{3n}\binom{x+6n}{5n}\binom{x+7n}{7n} = \frac{(x+2n+1)\cdots(x+5n)}{(3n)!}\,\frac{(x+n+1)\cdots(x+6n)}{(5n)!}\,\frac{(x+1)\cdots(x+7n)}{(7n)!},$$

where n is an odd natural number. We indroduce the functions

$$\Psi_{1,n}(\zeta,t) := A_n(\zeta)\left(\frac{\pi}{\sin\pi\zeta}\right)(-t)^{-\zeta}, \qquad \Psi_{2,n}(\zeta,t) := A_n(\zeta)\left(\frac{\pi}{\sin\pi\zeta}\right)^2 t^{-\zeta},$$

where t is a fixed number. We assume that

$$(-t)^{-\zeta} = e^{-\zeta\ln(-t)} \qquad \text{and} \qquad t^{-\zeta} = e^{-\zeta\ln t},$$

where the branches of the logarithms are chosen as follows:

$$\ln(-t) = \ln|t| + i\arg t + i\pi, \quad \ln t = \ln|t| + i\arg t, \qquad \text{where} \quad -2\pi < \arg t < 0.$$

Now we introduce the integrals

$$J_{i,n}(t) := t^{-(7n+1)/2} I_{i,n}(t) := \frac{t^{-(7n+1)/2}}{2\pi i}\int_L \Psi_{i,n}(\zeta,t)\,d\zeta \qquad \text{for} \quad i = 1,2,$$

where $t \ne 0$ and the vertical line L is given by $\operatorname{Re}\zeta = -7n/2$ and is passed from below upward.

Let λ_k denote the following points on the unit circle:

$$\lambda_k := e^{-i\arctan(\sqrt{2k-1}/(k-1))} = \frac{k-1-i\sqrt{2k-1}}{k}, \qquad \text{where} \quad k = 2m, \quad m \in \mathbb{N}.$$

Choosing the parameter value $t = \lambda_k$, we obtain bounds for the irrationality measures of the numbers α_k.

Proposition 1. *For any $t \in \mathbb{C}$ satisfying the conditions $0 < |t| \le 1$ and $-2\pi < \arg t < 0$,*

$$J_{1,n}(t) = -U_n(t), \qquad J_{2,n}(t) = U_n(t)\ln t - V_n(t), \tag{2.1}$$

where the functions $U_n(t), V_n(t) \in \mathbb{Q}(t)$, $0 < |t| \le 1$, and $-2\pi < \arg t < 0$ are defined by

$$U_n(t) := t^{-(7n+1)/2}\left(\frac{t}{t-1}\right)^{7n+1}\sum_{j=7n}^{15n} a_j\left(\frac{t}{t-1}\right)^{j-7n},$$

where

$$a_j := \sum_{k=7n+1}^{j+1} (-1)^{k-1} A_n(-k) \begin{pmatrix} j \\ k-1 \end{pmatrix}, \qquad V_n(t) := t^{-(5n+1)/2}\left(\frac{t}{t-1}\right)^{5n+1} \sum_{j=5n}^{15n-1} b_j \left(\frac{t}{t-1}\right)^{j-5n};$$

here

$$b_j := \sum_{k=5n+1}^{j+1} (-1)^{k-1} A_n'(-k-n) \begin{pmatrix} j \\ k-1 \end{pmatrix}.$$

Proof. In [5] (see the second formula from above on p. 552), it was proved that, for $|t| < 1$,

$$J_{2,n}(t) = t^{-(7n+1)/2}\Bigg(-\Bigg(\sum_{k=7n+1}^{+\infty} A_n(-k)t^k\Bigg)\ln t + \sum_{k=6n+1}^{+\infty} A_n'(-k)t^k\Bigg).$$

It can also be shown in a similar way that, for $|t| < 1$,

$$J_{1,n}(t) = t^{-(7n+1)/2} \sum_{k=7n+1}^{+\infty} A_n(-k)t^k.$$

Therefore, it suffices to show that

$$U_n(t) = -t^{-(7n+1)/2} \sum_{k=7n+1}^{+\infty} A_n(-k)t^k, \qquad V_n(t) = -t^{-(7n+1)/2} \sum_{k=5n+1}^{+\infty} A_n'(-k)t^k. \qquad (2.2)$$

To this end, we use Lemma 1 from [5] cited below.

Lemma 1. *Let $P(x) \in \mathbb{C}[x]$ be a polynomial of degree m. Then, for each $t \in \mathbb{C}$, $|t| < 1$,*

$$-\sum_{k=1}^{+\infty} P(-k)t^k = \sum_{j=0}^{m} h_j \left(\frac{t}{t-1}\right)^{j+1},$$

where

$$h_j := \sum_{k=1}^{j+1} (-1)^{k-1} P(-k) \begin{pmatrix} j \\ k-1 \end{pmatrix}, \qquad j \geq 0.$$

Applying Lemma 1 to the polynomial $A_n(x)$, we obtain

$$-\sum_{k=7n+1}^{+\infty} A_n(-k)t^k = -\sum_{k=1}^{\infty} A_n(-k)t^k = \sum_{j=0}^{15n} a_j \left(\frac{t}{t-1}\right)^{j+1} = \left(\frac{t}{t-1}\right)^{7n+1} \sum_{j=7n+1}^{15n} a_j \left(\frac{t}{t-1}\right)^{j+1},$$

because $a_j = 0$ for $1 \leq j \leq 7n$. Similarly, applying the same lemma to $A_n'(x+n)$, we obtain

$$-\sum_{k=5n+1}^{+\infty} A_n'(-k)t^k = -\sum_{k=n+1}^{\infty} A_n'(-k)t^k = -t^n \sum_{k=1}^{\infty} A_n'(-k-n)t^k$$

$$= t^n \sum_{j=0}^{15n} b_j \left(\frac{t}{t-1}\right)^{j+1} = t^n \left(\frac{t}{t-1}\right)^{5n+1} \sum_{j=5n+1}^{15n-1} b_j \left(\frac{t}{t-1}\right)^{j+1},$$

because $b_j = 0$ for $1 \leq j \leq 5n$. Substituting these expressions for the sums of series into the right-hand side of (2.2), we obtain the required identity.

To complete the proof of the proposition, it remains to consider the case where $|t| = 1$ and $-2\pi < \arg t < 0$. For $|t| \leq 1$, the right- and left-hand sides of relations in (2.1) are defined, and we have proved the required relations for $|t| < 1$; by continuity, they hold also for $|t| = 1$. $\square$

3. ARITHMETIC PROPERTIES

In [5, Lemma 1], the following lemma was proved.

Lemma 2. *The set Ω of numbers y, $0 \le y < 1$, such that*

$$[x-2y]+[x-y]+[x]-[x-5y]-[x-6y]-[x-6y]-[3y]-[5y]-[7y] \ge 1$$

for each $x \in \mathbb{R}$, has the form

$$\Omega = \left[\frac{1}{6}, \frac{3}{7}\right) \cup \left[\frac{1}{2}, \frac{5}{7}\right) \cup \left[\frac{3}{4}, \frac{6}{7}\right).$$

Let $\Delta = \prod p$, where the multiplication is over all primes $p > \sqrt{7n}$ for which $\{n/p\} \in \Omega$, and let d_n denote the least common multiple of the numbers $1, 2, \dots, n$. We set

$$D_{1,n,k} = m^{-(7n+1)/2}, \quad D_{2,n,k} = m^{-(5n+1)/2}\frac{d_{7n}}{\Delta}, \qquad \text{if} \quad k = 2m.$$

Lemma 3. *The numbers $D_{1,n,k}U_n(\lambda_k)$, $D_{2,n,k}(i\sqrt{2k-1}\,V_n(\lambda_k))$ are integers.*

Proof. Let us prove that $X := D_{2,n,k}(i\sqrt{2k-1}\,V_n(\lambda_k))$ is integer (the proof for $D_{1,n,k}U_n(\lambda_k)$ is completely similar). Note that it suffices to prove that X^2 is an algebraic integer (and hence so is X) and X is rational.

Let us first prove that X is rational. Bashmakova proved the relations (see Statement 2 and Lemma 1 in [6])

$$U_n(t) = U_n\left(\frac{1}{t}\right) = \widehat{U}\left(t+\frac{1}{t}\right), \qquad V_n(t) = -V_n\left(\frac{1}{t}\right) = \left(t-\frac{1}{t}\right)\widehat{V}\left(t+\frac{1}{t}\right), \tag{3.1}$$

where $\widehat{U}(t), \widehat{V}(t) \in \mathbb{Q}(t)$, i.e.,

$$U_n(t), \frac{V_n(t)}{t-1/t} \in \mathbb{Q}$$

provided that $t + 1/t \in \mathbb{Q}$. Note that $\lambda_k + 1/\lambda_k = 2(k-1)/k \in \mathbb{Q}$; hence X is a rational number.

Let us now prove that X^2 is an algebraic integer. To do this, we use Lemma 4 from Nesterenko's paper [5], which claims that

$$\frac{d_{7n}}{\Delta} A_n'(-k) \in \mathbb{Z}, \qquad \text{where} \quad k \in \mathbb{Z},$$

i.e., $b_j \in \mathbb{Z}$ (see the definition of b_j in Proposition 1). Note that

$$\frac{\lambda_k}{\lambda_k - 1} = \frac{1+i\sqrt{2k-1}}{2} =: t_1, \qquad \frac{1}{1-\lambda_k} = \frac{1-i\sqrt{2k-1}}{2} =: t_2$$

are the roots of the equation $t^2 - t + k/2 = 0$. Obviously, t_1 and t_2 are algebraic integers.

Applying (3.1) and the expression for $V_n(z)$ in Proposition 1, we obtain

$$\begin{aligned}
-(V_n(\lambda_k))^2 = V(\lambda_k)V_n\left(\frac{1}{\lambda_k}\right) &= (t_1t_2)^{5n+1}\sum_{j=5n}^{15n-1} b_j t_1^{j-5n} \sum_{j=5n}^{15n-1} b_j t_2^{j-5n} \\
&= \left(\frac{k}{2}\right)^{5n+1}\sum_{j=5n}^{15n-1} b_j t_1^{j-5n} \sum_{j=5n}^{15n-1} b_j t_2^{j-5n}.
\end{aligned}$$

Since the numbers t_1, t_2, and b_j are algebraic integers, it follows that the number

$$X^2 = -\sum_{j=5n}^{15n-1} b_j t_1^{j-5n} \sum_{j=5n}^{15n-1} b_j t_2^{j-5n}$$

is an algebraic integer. This completes the proof of the lemma. ☐

4. ASYMPTOTIC PROPERTIES

We shall repeatedly use Stirling's formula (see [10, Sec. 12.31])

$$\begin{gathered}\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln 2\pi+r(z),\qquad\text{where}\quad |r(z)|<\frac{c}{\operatorname{Re}z},\\ c>0 \text{ is a constant},\qquad \ln\Gamma(1)=0,\qquad \ln z=\ln|z|+i\arg z,\quad |\arg z|<\frac{\pi}{2}.\end{gathered}\tag{4.1}$$

In this section, we obtain asymptotic expressions for $J_{1,n}(\gamma)$, $J_{2,n}(\gamma)$, where γ is any point such that $|\gamma|=1$ and $\operatorname{Im}\gamma<0$.

Consider the equation

$$\frac{(z+7)(z+6)(z+5)}{(-z)(-z-1)(-z-2)(-\gamma)}=1\tag{4.2}$$

and the function

$$\begin{aligned}g_\gamma(y)&=\arg(3.5+iy)+\arg(2.5+iy)+\arg(1.5+iy)\\&\quad-\arg(3.5-iy)-\arg(2.5-iy)-\arg(1.5-iy)-\pi-\arg\gamma,\end{aligned}$$

where $-\pi<\arg\gamma<0$ and $-\pi/2\le\arg(c+iy)\le\pi/2$ for $c=1.5, 2.5, 3.5$.

Equation (4.2) has exactly three roots, and they all lie on the line $\operatorname{Re}z=-3.5$. Indeed, in the domain $\operatorname{Re}z<-3.5$, the following inequalities hold:

$$|z+7|<|-z|,\qquad |z+6|<|-(z+1)|,\qquad |z+5|<|-(z+2)|.$$

Therefore, since $|\gamma|=1$, it follows that the absolute value of the left-hand side of Eq. (4.2) in the half-plane $\operatorname{Re}z<-3.5$ is less than 1. In the same way, we can prove that the absolute value of the left-hand side of Eq. (4.2) in the half-plane $\operatorname{Re}z>-3.5$ is greater than 1. Hence all the three roots of Eq. (4.2) lie on the line $\operatorname{Re}z=-3.5$.

Hence, clearly, $g_\gamma(y')=2\pi m$ for $m\in\mathbb{Z}$ if and only if $z=-b/2+iy'$ is a root of Eq. (4.2). The function $g(y)$ increases continuously from $-4\pi-\arg\gamma$ (which is greater than -4π) to $2\pi-\arg\gamma$ (which is greater than 2π) as y varies from $-\infty$ to $+\infty$. Since $g(0)=-\pi-\arg\gamma<0$, it follows that two roots of Eq. (4.2) lie in the upper half-plane and one root lies in the lower half-plane. These are the points

$$z_{1,\gamma}=-3.5+iy_{1,\gamma},\qquad z_{2,\gamma}=-3.5+iy_{2,\gamma},\qquad z_{3,\gamma}=-3.5+iy_{3,\gamma},$$

at which

$$g_\gamma(y_{1,\gamma})=2\pi,\qquad g_\gamma(y_{2,\gamma})=0,\qquad g_\gamma(y_{3,\gamma})=-2\pi.$$

It is also easy to verify that $y_{1,\gamma}>|y_{3,\gamma}|>y_{2,\gamma}>0$.

Consider the function

$$h(z)=\ln\frac{(z+7)^7(z+6)^6(z+5)^5}{(-z-1)(-z-2)^2 7^7 5^5 3^3}.\tag{4.3}$$

The main statement of this section is as follows.

Proposition 2. *The following asymptotic relations hold*:

$$\begin{aligned}&\lim_{n\to+\infty}\frac{1}{n}\ln|J_{1,n}(\gamma)|=\lim_{n\to+\infty}\frac{1}{n}\ln|U_n(\gamma)|=\operatorname{Re}h(z_{1,\gamma})=:M_1(\gamma);\\&\lim_{n\to+\infty}\frac{1}{n}\ln|J_{2,n}(\gamma)|=\operatorname{Re}h(z_{3,\gamma})=:M_2(\gamma).\end{aligned}$$

Proof of Proposition 2. We give the proof for $J_{1,n}(\gamma)$. The case of $J_{2,n}(\gamma)$ is similar. It suffices to find an asymptotics for $I_{1,n}(\gamma)$, because $|\gamma| = 1$.

Thus, the function $\Psi_{1,n}(\zeta, \gamma)$ can be represented as

$$\Psi_{1,n}(\zeta,\gamma) = \frac{\Gamma(\zeta+5n+1)\Gamma(-\zeta-2n)}{\Gamma(3n+1)} \frac{\Gamma(\zeta+6n+1)\Gamma(-\zeta-n)}{\Gamma(5n+1)} \times \frac{\Gamma(\zeta+7n+1)\Gamma(-\zeta)}{\Gamma(7n+1)} \left(\frac{\sin\pi\zeta}{\pi}\right)^2 e^{-\zeta(\ln\gamma + i\pi)}.$$

Now, we represent the integral $I_{1,n}(\gamma)$ as the sum of the two integrals

$$I_{1,n}(\gamma) = \frac{1}{2\pi i}\int_{L'} \Psi_{1,n}(\zeta,\gamma)\,d\zeta + \frac{1}{2\pi i}\int_{L''} \Psi_{1,n}(\zeta,\gamma)\,d\zeta := I'_{1,n}(\gamma) + I''_{1,n}(\gamma);$$

here L' is the ray determined by $\operatorname{Re}\zeta = -3.5n$ and $\operatorname{Im}\zeta \ge 0$ and L'' is the ray determined by $\operatorname{Re}\zeta = -3.5n$ and $\operatorname{Im}\zeta \le 0$.

Consider $I'_{1,n}(\gamma)$. The case of $I''_{1,n}(\gamma)$ is similar. In view of (4.1), we have

$$\Psi_{1,n}(zn,\gamma) = e^{nf_\gamma(z)}\varphi_n(z)\left(1 + O\left(\frac{1}{n}\right)\right),$$

where

$$\begin{aligned} f_\gamma(z) := {} & (z+7)\ln(z+7) + (z+6)\ln(z+6) + (z+5)\ln(z+5) \\ & - z\ln(-z) - (z+1)\ln(-z-1) - (z+2)\ln(-z-2) \\ & - 3\ln 3 - 5\ln 5 - 7\ln 7 - z\ln\gamma - iz\pi - 2iz\pi, \end{aligned}$$

$$\varphi_n(z) := \left(\frac{e^{2iz\pi n} - 1}{2i}\right)^2 \sqrt{\left(\frac{2\pi}{n}\right)^3 \frac{(z+7)(z+6)(z+5)}{(-z)(-z-1)(-z-2)7\cdot 5\cdot 3}}.$$

Here the constant in $O(1/n)$ is independent of z. On the ray L', for sufficiently large n, the following inequalities hold for some constants $C_1, C_2 > 0$:

$$C_1 n^{-3/2} \le \left|\varphi_n(z)\left(1 + O\left(\frac{1}{n}\right)\right)\right| \le C_2 n^{-3/2}.$$

The change $z = -3.5 + iy$ yields

$$I'_{1,n}(\gamma) = \frac{1}{2\pi i}\int_{L'} \Psi_{1,n}(\zeta,\gamma)\,d\zeta = \frac{1}{2\pi}\int_0^{+\infty} e^{nf_\gamma(z)}(\varphi_n(z)n)\,dy.$$

Thus, let $q_\gamma(y) = \operatorname{Re} f_\gamma(z)$, where $z = -3.5 + iy$. Then

$$\begin{aligned} q'_\gamma(y) = -\operatorname{Im}\left[\frac{\partial f_\gamma}{\partial x}\right] = -\operatorname{Im}\left[\frac{\partial f_\gamma}{\partial z}\right] = {} & -(\arg(z+7) + \arg(z+6) + \arg(z+5) \\ & - \arg(-z) - \arg(-z-1) - \arg(-z-2) - \arg\gamma - \pi - 2\pi) = -(g_\gamma(y) - 2\pi). \end{aligned}$$

The equality $q'_\gamma(y) = 0$ can hold only if $g_\gamma(y) = 2\pi$, i.e., $y = y_{1,\gamma}$. In this case, $z_{1,\gamma} = -3.5 + iy_{1,\gamma}$ is a root of Eq. (4.2), i.e.,

$$e^{f'_\gamma(z_{1,\gamma})} = \frac{(z_{1,\gamma}+7)(z_{1,\gamma}+6)(z_{1,\gamma}+5)}{(-z_{1,\gamma})(-z_{1,\gamma}-1)(-z_{1,\gamma}-2)(-\gamma)} = 1;$$

therefore, $f'_\gamma(z_{1,\gamma}) = 0$. It is also easy to verify that $y_{1,\gamma}$ is a point of maximum of $q_\gamma(y)$ on $[0; +\infty)$. For $z = -3.5 + iy$, we have

$$f_\gamma(-3.5 + iy) = f_\gamma(z_{1,\gamma}) - \frac{f''_\gamma(z_{1,\gamma})}{2}(y - y_{1,\gamma})^2 + O((y - y_{1,\gamma})^3),$$

$$f''_\gamma(z) = -\frac{1}{z} - \frac{1}{z+1} - \frac{1}{z+2} + \frac{1}{z+7} + \frac{1}{z+6} + \frac{1}{z+5}$$
$$= \frac{3.5}{(3.5)^2+y^2} + \frac{2.5}{(2.5)^2+y^2} + \frac{1.5}{(1.5)^2+y^2} > 0.$$

Using the saddle-point method (see Theorem 3 in [11, p. 59], which is convenient to apply in the case under consideration), we obtain

$$\lim_{n\to+\infty} \frac{1}{n} \ln |I'_{1,n}(\gamma)| = \operatorname{Re} f_\gamma(z_{1,\gamma}) = \operatorname{Re} h(z_{1,\gamma}).$$

The last equality follows from $f'_\gamma(z_{1,\gamma}) = 0$.

In a similar way, for $\operatorname{Im}\zeta \le 0$, we obtain

$$\lim_{n\to+\infty} \frac{1}{n} \ln |I''_{1,n}(\gamma)| = \operatorname{Re} h(z_{3,\gamma}).$$

In view of the inequality $y_{1,\gamma} > |y_{3,\gamma}| > 0$, it is easy to verify that

$$\lim_{n\to+\infty} \frac{1}{n} \ln |I_{1,n}(\gamma)| = \operatorname{Re} h(z_{1,\gamma}).$$

This completes the proof of Proposition 2. □

5. END OF THE PROOF

It is easy to check that

$$M(\lambda_k) := \lim_{n\to\infty} \frac{1}{n} \ln(D_{2,n,k}) = -2.5 \ln m + 7 - \lim_{n\to\infty} \frac{1}{n} \ln \Delta \qquad \text{for} \quad k = 2m;$$

the last summand can be calculated by using the following lemma.

Lemma 4 ([5, Lemma 6]). *Let u and v be real numbers satisfying the inequalities $0 < u < v < 1$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{u \le \{n/p\} < v} \ln p = \psi(v) - \psi(u),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function and the summation is over all primes p such that the fractional part $\{n/p\}$ satisfies the inequality under the summation sign.

Theorem 1. *If $M_2(\lambda_k) + M(\lambda_k) < 0$, then*

$$\mu(\alpha_k) \le 1 - \frac{M_1(\lambda_k) + M(\lambda_k)}{M_2(\lambda_k) + M(\lambda_k)}.$$

Proof. In what follows, we use Hata's lemma stated below (see also [1, Statement 2.1]). □

Lemma 5. *If $n \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $l_n = q_n\alpha + p_n$ for $q_n, p_n \in \mathbb{Z}$, and*

$$\lim_{n\to\infty} \frac{1}{n} \ln |q_n| = \sigma, \quad \limsup_{n\to\infty} \frac{1}{n} \ln |l_n| \le -\tau, \qquad \sigma, \tau > 0,$$

then $\mu(\alpha) \le 1 + \sigma/\tau$.

Theorem 1 follows from Lemma 5 applied to the sequence

$$L_n = D_{2,n,k}(i\sqrt{2k-1}\, I_2(\lambda_k)) = D_{2,n,k} U(\lambda_k)\alpha_k - D_{2,n,k}(i\sqrt{2k-1}\, V(\lambda_k)) = P_n\alpha_k + Q_n.$$

Lemma 3 implies that P_n and Q_n are integers, and Proposition 2 and Lemma 4 give the asymptotics of L_n and P_n; thus, Lemma 5 implies Theorem 1. □

Table 2.

k	$\mu_2(\alpha_k) \le$	a	b
2	—		
4	$44.87472\ldots$	1	7
6	$19.19130\ldots$	2	23
8	$14.37384\ldots$	1	13
10	$12.28656\ldots$	1	13
12	$11.11119\ldots$	1	13

Remark. Using similar methods, we can obtain upper bounds for the *nonquadraticity measures* of the numbers α_k (for more details, see [11]).

The *nonquadraticity measure* of a number α which is not a root of a quadratic equation with integer coefficients is defined as the least upper bound for the set of numbers $\varkappa$ such that the inequality

$$|\alpha-\beta| < H^{-\varkappa}(\beta)$$

has an infinite number of solutions in quadratic irrationalities β. The number $H(\beta)$ is the greatest (in absolute value) integer coefficient of a primitive quadratic trinomial with root β. The nonquadraticity measure is denoted by $\mu_2(\alpha)$.

To obtain upper bounds for the nonquadraticity measures, we consider the following integrals for $t=\lambda_k$:

$$J_{i,n}(t) := \frac{t^{-(bn+1)/2}}{2\pi i}\int_L \Psi_{i,n}(\zeta,t)\,d\zeta, \qquad i=1,2,3,$$

where

$$\Psi_{1,n}(\zeta,t) := A_n(\zeta)\biggl(\frac{\pi}{\sin(\pi\zeta)}\biggr)(-t)^{-\zeta}, \qquad \Psi_{2,n}(\zeta,t) = A_n(\zeta)\biggl(\frac{\pi}{\sin(\pi\zeta)}\biggr)^2 t^{-\zeta},$$

$$\Psi_{3,n}(\zeta,t) = A_n(\zeta)\biggl(\frac{\pi}{\sin(\pi\zeta)}\biggr)^3(-t)^{-\zeta},$$

$$\begin{aligned} A_n(x) &:= \binom{x+(b-2a)n}{(b-4a)n}\binom{x+(b-a)n}{(b-2a)n}\binom{x+bn}{bn} \\ &= \frac{(x+2an+1)\cdots(x+(b-2a)n)}{((b-4a)n)!}\,\frac{(x+an+1)\cdots(x+(b-a)n)}{((b-2a)n)!}\,\frac{(x+1)\cdots(x+bn)}{(bn)!}. \end{aligned}$$

Here a, b, and n are positive integers ($b>4a$ and n is odd) and the vertical line L is given by $\operatorname{Re}\zeta = -bn/2$ and is passed from below upward. To obtain bounds for the nonquadraticity measures of the numbers α_k, the author used a lemma similar to Lemma 5 and stated in [12, Lemma 2.3]. Table 2 contains bounds for the nonquadraticity measures and the parameters a and b for which these bounds were obtained.

ACKNOWLEDGMENTS

The author wishes to express deep gratitude to his scientific adviser Yu. V. Nesterenko for help during the work on this paper.

This work was supported by the Russian Foundation for Basic Research under grant 09-01-00743.

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