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On Balder's Existence Theorem for Infinite-Horizon Optimal Control Problems

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Abstract—Balder's well-known existence theorem (1983) for infinite-horizon optimal control problems is extended to the case in which the integral functional is understood as an improper integral. Simultaneously, the condition of strong uniform integrability (over all admissible controls and trajectories) of the positive part $\max\{f_0, 0\}$ of the utility function (integrand) f_0 is relaxed to the requirement that the integrals of f_0 over intervals $[T, T']$ be uniformly bounded above by a function $\omega(T, T')$ such that $\omega(T, T') \to 0$ as $T, T' \to \infty$. This requirement was proposed by A.V. Dmitruk and N.V. Kuz'kina (2005); however, the proof in the present paper does not follow their scheme, but is instead derived in a rather simple way from the auxiliary results of Balder himself. An illustrative example is also given.

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One of the most general and well-known results on the existence of solutions to infinite-horizon optimal control problems was proved by Balder [1]. Almost all conditions of his theorem are local in time (i.e., they must hold only at each separate instant of time or on each finite time interval) and ensure the existence of solutions to similar problems on finite time intervals. The only condition that regulates the behavior of the system at infinity is the requirement of strong uniform integrability of the positive part of the integrand in the objective functional over all admissible controls and corresponding trajectories. Later several authors achieved some progress in weakening this condition.

The present paper also contributes to this direction. As an alternative to Balder's uniform integrability, we use the condition of "uniform boundedness of pieces of the objective functional" proposed by Dmitruk and Kuz'kina [2]. Note that they considered a significantly narrower class of optimal control problems, while for the general case only a scheme was outlined (without statement of particular results that can be obtained by following this scheme[1]). So the present paper is in a sense a logical completion of the paper [2]. However, we do not follow the scheme proposed in [2] but rather show that the result can be derived from those of Balder himself [3], [1] in a fairly simple way.

Recently, Bogusz [4] also obtained an existence theorem in the case when the integral functional is understood as an improper integral. However, one of the hypotheses in her theorem is the existence of a locally integrable function $\lambda\colon \mathbb{R}_+ \to \mathbb{R}$ that has a finite improper integral $\int_0^\infty \lambda(t)\,dt$ and bounds from above (from below in the case of minimization problem) the integrand in the objective functional for all admissible controls and corresponding trajectories. Such a condition is essentially stronger (although formally this is not so) than strong uniform integrability, because subtracting (adding in the case of minimization problem) the function λ from (to) the integrand reduces the problem to the one with negative (positive) integrand in the objective functional.

Some results on the existence of optimal solutions under conditions of different kind and/or for different problem statements were obtained in [5], [6].

Note that existence theorems are an inherent part of the method for solving optimal control problems based on applying necessary optimality conditions (see, e.g., [7], [8], [9], [10], [11]). Therefore, it

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[1] The absence of exact statements to which one could refer when solving particular optimal control problems was one of the reasons for writing the present note.

is important to have an existence theorem under hypotheses maximally close to those under which necessary optimality conditions are valid. At present, it is the condition of "uniform boundedness of pieces of the objective functional" that is often required for necessary optimality conditions to be valid (see, e.g., [7], [10]).

Let us proceed to the statement of the problem and formulate the conditions under which we will study it.

The main object of our study is the optimal control problem

$$I(x,u) := \int_0^\infty f_0(t,x(t),u(t))\,dt \to \max, \tag{1}$$

$$\dot{x}(t) = f(t,x(t),u(t)) \qquad \text{for a.e.} \quad t \in \mathbb{R}_+ := [0,+\infty), \tag{2}$$

$$x(t) \in A(t), \quad u(t) \in U(t,x(t)) \qquad \text{for a.e.} \quad t \in \mathbb{R}_+, \tag{3}$$

for which the following conditions hold (where $m, n \in \mathbb{N}$ are fixed dimensions of the control and state vectors, respectively):

(i) $A\colon \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ is a set-valued map with $(\mathcal{L} \times \mathcal{B}^n)$-measurable[2] graph $\mathcal{A}$;

(ii) $U\colon \mathcal{A} \rightrightarrows \mathbb{R}^m$ is a set-valued map with $(\mathcal{L} \times \mathcal{B}^{n+m})$-measurable graph $\mathcal{U}$;

(iii) the functions $f\colon \mathcal{U} \to \mathbb{R}^n$ and $f_0\colon \mathcal{U} \to \mathbb{R} \cup \{-\infty\}$ are $(\mathcal{L} \times \mathcal{B}^{n+m})$-measurable[3].

By definition, the set Ω of *admissible pairs* (x,u) consists of pairs of vector functions x and u such that $x \in \mathrm{AC}^n_{\mathrm{loc}}(\mathbb{R}_+)$ and $u\colon \mathbb{R}_+ \to \mathbb{R}^m$ is a Lebesgue measurable function for which conditions (2), (3) are satisfied. Here $\mathrm{AC}^n_{\mathrm{loc}}(\mathbb{R}_+)$ is the space of locally absolutely continuous (i.e., absolutely continuous on any finite interval) functions $x\colon \mathbb{R}_+ \to \mathbb{R}^n$ with the topology described in [1].

The integral in (1) is understood in [1] in the following sense:

$$\int_0^\infty g(t)\,dt := \int_0^\infty g^+(t)\,dt - \int_0^\infty g^-(t)\,dt, \qquad \text{where} \quad g^\pm := \max\{\pm g, 0\}, \tag{4}$$

with the convention[4] that $(+\infty) - (+\infty) = -\infty$. Thus, the value of the functional (4) (which is equal to a finite number or to $\pm\infty$) is defined on any admissible pair.

We fix an $\alpha \in \mathbb{R}$ and define $\Omega_\alpha := \{(x,u) \in \Omega \mid I(x,u) \geq \alpha\}$. The existence of a solution of problem (1)–(3) was proved in [1] under the following assumptions:

(iv) the function $f(t,\,\cdot\,,\,\cdot\,)$ is continuous on $\mathcal{U}(t) := \{(\chi,\upsilon) \in \mathbb{R}^n \times \mathbb{R}^m \mid (t,\chi,\upsilon) \in \mathcal{U}\}$ for every $t \in \mathbb{R}_+$;

(v) the function $f_0(t,\,\cdot\,,\,\cdot\,)$ is upper semicontinuous on $\mathcal{U}(t)$ for every $t \in \mathbb{R}_+$;

(vi) the sets $A(t)$ and $\mathcal{U}(t)$ are closed for every $t \in \mathbb{R}_+$;

(vii) the set $\{x(0) \mid (x,u) \in \Omega_\alpha\}$ is bounded;

(viii) for any $T > 0$, the set of functions $F^T_\alpha := \{f(\,\cdot\,,x(\cdot),u(\cdot))|_{[0,T]} \mid (x,u) \in \Omega_\alpha\}$ is uniformly integrable on the interval $[0,T]$, i.e.,

$$\inf_{c>0} \sup_{g \in F^T_\alpha} \int_{C^T_{g,c}} |g(t)|\,dt = 0, \qquad \text{where} \quad C^T_{g,c} = \{t \in [0,T] \mid |g(t)| > c\};$$

[2] This means that the graph belongs to the σ-algebra in $\mathbb{R}_+ \times \mathbb{R}^n$ generated by Cartesian products of Lebesgue measurable subsets in $\mathbb{R}_+$ and Borel subsets in $\mathbb{R}^n$.

[3] This means that the preimages of Borel subsets are $(\mathcal{L} \times \mathcal{B}^{n+m})$-measurable.

[4] Here and below, without further comment, we reformulate all results obtained for minimization problems in [1], [2] in terms of maximization problems.

(ix) the set

$$Q(t,\chi) := \{(z^0, z) \in \mathbb{R}\times\mathbb{R}^n \mid z^0 \le f_0(t,\chi,\upsilon),\ z = f(t,\chi,\upsilon),\ \upsilon \in U(t,\chi)\}$$

is convex for all $(t,\chi) \in \mathcal{A}$;

(x) we have

$$Q(t,\chi) = \bigcap_{\delta>0} \mathrm{cl}\biggl(\bigcup_{\chi' \in A(t)\cap B_\delta(\chi)} Q(t,\chi')\biggr),$$

where $B_\delta(\chi)$ is the ball of radius δ centered at a point χ;

(xi) the set of functions $F^+_{0,\alpha} := \{f^+_0(\,\cdot\,, x(\cdot), u(\cdot)) \mid (x,u) \in \Omega_\alpha\}$ is strongly uniformly integrable on $\mathbb{R}_+$, i.e.,

$$\inf_{h\in L_1(\mathbb{R}_+)} \sup_{g\in F^+_{0,\alpha}} \int_{C_{g,h}} |g(t)|\,dt = 0, \qquad \text{where} \quad C_{g,h} := \{t\in\mathbb{R}_+ \mid |g(t)| > h(t)\}.$$

Theorem A ([1, Theorem 3.6]). *If there is an $\alpha\in\mathbb{R}$ such that $\Omega_\alpha \ne \varnothing$ and conditions* (i)–(xi) *are satisfied, then, in problem* (1)–(3)*, there exists an admissible pair $(x_*, u_*) \in \Omega$ such that $I(x_*,u_*) = \sup_{(x,u)\in\Omega} I(x,u)$.*

As was already mentioned, the only condition in Theorem A which regulates the behavior of system (1)–(3) at infinity is condition (xi). At the same time, in many optimal economic growth problems, it seems more natural to define the value of the objective functional not in the sense of (4), but rather in the sense of the limit

$$J(x,u) := \lim_{T\to+\infty} \int_0^T f_0(t, x(t), u(t))\,dt, \tag{5}$$

provided that this limit exists (see, e.g., [7], [4]). We also follow this definition; then problem (1)–(3) is replaced by the problem

$$J(x,u) \to \max \tag{6}$$

under the conditions (2), (3).

Remark 1. It is clear that if the value of the functional $I(x,u)$ is finite for an admissible pair (x,u), then $J(x,u) = I(x,u)$.

As noticed in [2], instead of condition (xi), one can consider the following condition:

(xii) the inequality

$$\varlimsup_{T\to+\infty} \sup_{T'>T} \sup_{(x,u)\in\Omega} \int_T^{T'} f_0(t,x(t),u(t))\,dt \le 0$$

is satisfied.

It is easy to see that, for admissible pairs in Ω_α, condition (xii) is weaker[5] than condition (xi). Indeed,

$$\varlimsup_{T\to+\infty} \sup_{T'>T} \sup_{(x,u)\in\Omega_\alpha} \int_T^{T'} f_0(t,x(t),u(t))\,dt \le \varlimsup_{T\to+\infty} \sup_{T'>T} \sup_{g\in F^+_{0,\alpha}} \int_T^{T'} g(t)\,dt$$

[5]On the whole, it would be incorrect to say that condition (xii) is weaker than condition (xi), because condition (xii) is considered for the set Ω, while condition (xi), only for the subset $\Omega_\alpha \subset \Omega$. Therefore, formally, neither of these conditions follows from the other.

$$\le \inf_{h\in L_1(\mathbb{R}_+)} \varlimsup_{T\to+\infty} \sup_{T'>T} \sup_{g\in F^+_{0,\alpha}} \biggl(\int_{[T,T']\cap C_{g,h}} g(t)\,dt + \int_{[T,T']\setminus C_{g,h}} h(t)\,dt\biggr)$$

$$\le \inf_{h\in L_1(\mathbb{R}_+)} \varlimsup_{T\to+\infty} \sup_{T'>T} \sup_{g\in F^+_{0,\alpha}} \int_{C_{g,h}} g(t)\,dt + 0 = \inf_{h\in L_1(\mathbb{R}_+)} \sup_{g\in F^+_{0,\alpha}} \int_{C_{g,h}} g(t)\,dt.$$

However, below we will still need a local version of condition (xi), namely,

(xi$'$) for every $T>0$, the set of functions

$$F_0^{T,+} := \{f_0^+(\,\cdot\,,x(\cdot),u(\cdot))|_{[0,T]} \mid (x,u)\in\Omega\}$$

is uniformly integrable on $[0,T]$.

(In [2], since the mappings considered there are continuous and compact-valued, condition (xi$'$) is satisfied automatically.)

Let us make the following important observation.

Proposition 1. *Under condition* (xi$'$)*, condition* (xii) *is equivalent to each of the following conditions*:

(xii$'$) *there is a continuous function* $\omega\colon \mathbb{R}_+^2\to\mathbb{R}_+$ *such that* $\omega(T,T')\to 0$ *as* $T,T'\to\infty$ *and*

$$\sup_{(x,u)\in\Omega} \int_T^{T'} f_0(t,x(t),u(t))\,dt \le \omega(T,T') \qquad \forall\, T,T'\colon\ T'>T\ge 0;$$

(xii$''$) *there is a continuous function* $\widetilde\omega\colon \mathbb{R}_+\to\mathbb{R}_+$ *such that* $\widetilde\omega(T)\to 0$ *as* $T\to\infty$ *and*

$$\sup_{T'>T} \sup_{(x,u)\in\Omega} \int_T^{T'} f_0(t,x(t),u(t))\,dt \le \widetilde\omega(T) \qquad \forall\, T\ge 0.$$

Proof. Obviously, condition (xii$''$) implies (xii$'$) (it suffices to set $\omega(T,T') := \widetilde\omega(T)$), and condition (xii$'$) implies (xii) (because $\varlimsup_{T\to\infty}\sup_{T'>T}$ is the same as $\varlimsup_{T,T'\to\infty,\,T'>T}$, and the latter does not exceed $\varlimsup_{T,T'\to\infty}$). Let us show that condition (xii) implies (xii$''$). We set

$$\widehat\omega(T) := \biggl(\sup_{T'>T} \sup_{(x,u)\in\Omega} \int_T^{T'} f_0(t,x(t),u(t))\,dt\biggr)^+, \qquad T\ge 0.$$

By condition (xii), we have $\lim_{T\to\infty}\widehat\omega(T)=0$. Therefore, there is a T_1 such that $\widehat\omega(T)\le 1$ for $T\ge T_1$. We show that this function is bounded for all $T\ge 0$. For $T<T_1$, we have

$$\widehat\omega(T) \le \sup_{(x,u)\in\Omega} \int_0^{T_1} f_0^+(t,x(t),u(t))\,dt + \widehat\omega(T_1) \le \inf_{c>0} \sup_{g\in F_0^{T_1,+}} \biggl(\int_{C^{T_1}_{g,c}} g(t)\,dt + cT_1\biggr) + 1.$$

By condition (xi$'$), there is a constant $c_1>0$ such that

$$\sup_{g\in F_0^{T_1,+}} \int_{C^{T_1}_{g,c_1}} g(t)\,dt \le 1.$$

Then $\widehat\omega(T)\le c_1T_1+2$ for all $T\ge 0$.

We set $\widehat\omega_1(T) := \sup_{T'\ge T}\widehat\omega(T')$ for $T\ge 0$. Then $\widehat\omega_1$ is a bounded monotonically nonincreasing function on $\mathbb{R}_+$ which tends to zero as $T\to\infty$.

Finally, we set $\widetilde\omega(T) := \int_{T-1}^T \widehat\omega_1(t^+)\,dt$ (recall that $t^+=\max\{t,0\}$). It is clear that $\widetilde\omega$ is a continuous function on $\mathbb{R}_+$ for which all requirements in condition (xii$''$) are satisfied. ◻

An important consequence of condition (xii) is the fact that the value of the functional $J(\,\cdot\,,\,\cdot\,)$ is defined on any admissible pair. For completeness, we present here a proof of this fact which is slightly shorter than that in [2].

Proposition 2. *Under conditions* (xi$'$) *and* (xii)*, the value of the functional* $J(x,u)$ *is defined for any admissible pair* $(x,u)\in\Omega$ *and equal to either a finite number or* $-\infty$.

Proof. The existence of the limit in (5) follows from the estimate

$$\begin{aligned}\overline{\lim_{T\to+\infty}}\int_0^T f_0(t,x(t),u(t))\,dt &= \underline{\lim}_{T_1\to+\infty}\;\overline{\lim_{T\to+\infty}}\biggl(\int_0^{T_1}+\int_{T_1}^{T}\biggr)f_0(t,x(t),u(t))\,dt\\ &\le \underline{\lim}_{T_1\to+\infty}\int_0^{T_1} f_0(t,x(t),u(t))\,dt+\overline{\lim_{T_1\to+\infty}}\;\sup_{T>T_1}\int_{T_1}^{T} f_0(t,x(t),u(t))\,dt\\ &\le \underline{\lim}_{T\to+\infty}\int_0^{T} f_0(t,x(t),u(t))\,dt,\end{aligned}$$

where condition (xii) was used at the last step. At the same time, the limit under study does not exceed $\widetilde{\omega}(0)$ for some continuous function $\widetilde{\omega}\colon \mathbb{R}_+\to\mathbb{R}_+$. ☐

Now we formulate our main result. To this end, we introduce the following set similar to Ω_α:

$$\widetilde{\Omega}_\alpha := \{(x,u)\in\Omega \mid J(x,u)\ge\alpha\}\qquad\text{for}\quad \alpha\in\mathbb{R}.$$

Theorem 1. *If there is an* $\alpha\in\mathbb{R}$ *such that* $\widetilde{\Omega}_\alpha\ne\varnothing$ *and conditions* (i)–(x)*,* (xi$'$)*, and* (xii) (*or* (xii$'$)*, or* (xii$''$)) *hold with* Ω_α *replaced by* $\widetilde{\Omega}_\alpha$*, then, in problem* (6)*,* (2)*,* (3)*, there exists an admissible pair* $(x_*,u_*)\in\Omega$ *such that* $J(x_*,u_*)=\sup_{(x,u)\in\Omega}J(x,u)$.

The main role in the proof is played by another result of Balder.

Theorem B ([1, Theorem 3.2])**.** *Suppose that conditions* (i)–(vi)*,* (ix)*, and* (x) *hold. Suppose also that* $\{(x_k,u_k)\}_{k=1}^\infty$ *is a sequence in* Ω *such that the sequence* $\{x_k\}_{k=1}^\infty$ *converges weakly*[6] *to a function* $x_0\in \mathrm{AC}^n_{\mathrm{loc}}(\mathbb{R}_+)$ *and the set of functions* $\{f_0^+(\,\cdot\,,x_k(\cdot),u_k(\cdot))\}_{k=1}^\infty$ *is strongly uniformly integrable on* $\mathbb{R}_+$*. Then there exists a Lebesgue measurable function* $u_*\colon\mathbb{R}_+\to\mathbb{R}^m$ *such that* $(x_0,u_*)\in\Omega$ *and*

$$I(x_0,u_*)\ge\overline{\lim_{k\to\infty}}\,I(x_k,u_k).$$

Proof of Theorem 1. Let $\{(x_k,u_k)\}_{k=1}^\infty$ be a maximizing sequence for $J(\,\cdot\,,\,\cdot\,)$ in $\widetilde{\Omega}_\alpha$. It follows from conditions (vii), (viii) (with Ω_α replaced by $\widetilde{\Omega}_\alpha$) and Theorem 2.1 in [1] that the sequence $\{x_k\}_{k=1}^\infty$ contains a subsequence converging weakly to a function $x_0\in\mathrm{AC}^n_{\mathrm{loc}}(\mathbb{R}_+)$. We pass to this subsequence and denote it again by $\{(x_k,u_k)\}_{k=1}^\infty$.

For $N\in\mathbb{N}$, we introduce the function

$$f_0^N(t,\chi,\upsilon):=\begin{cases} f_0(t,\chi,\upsilon), & t\in[N-1,N),\ (t,\chi,\upsilon)\in\mathcal{U},\\ 0, & t\in\mathbb{R}_+\setminus[N-1,N),\ (t,\chi,\upsilon)\in\mathcal{U},\end{cases}$$

and consider problem (1)–(3) with f_0^N instead of f_0. Let us denote the corresponding functional (in which the integral is actually taken over the interval $[N-1,N)$) by I_N. We first assume that, for each $N\in\mathbb{N}$, all assumptions of Theorem B are satisfied for this truncated problem (with the objective

[6] For the definition of weak convergence in $\mathrm{AC}^n_{\mathrm{loc}}(\mathbb{R}_+)$, see [1].

functional I_N) and for our sequence $\{(x_k, u_k)\}_{k=1}^{\infty}$. Then there exists a Lebesgue measurable function $u_{N*}\colon \mathbb{R}_+ \to \mathbb{R}^m$ such that $(x_0, u_{N*}) \in \Omega$ and

$$I_N(x_0, u_{N*}) \ge \varlimsup_{k\to\infty} I_N(x_k, u_k).$$

We set $u_*(t) := u_{N*}(t)$ for $t \in [N-1, N)$, $N \in \mathbb{N}$. It is clear that $(x_0, u_*) \in \Omega$ and

$$\begin{aligned} J(x_0, u_*) &= \lim_{K\to\infty} \sum_{N=1}^{K} I_N(x_0, u_*) = \lim_{K\to\infty} \sum_{N=1}^{K} I_N(x_0, u_{N*}) \\ &\ge \lim_{K\to\infty} \sum_{N=1}^{K} \varlimsup_{k\to\infty} I_N(x_k, u_k) \ge \lim_{K\to\infty} \varlimsup_{k\to\infty} \sum_{N=1}^{K} I_N(x_k, u_k) \\ &\ge \lim_{K\to\infty} \varlimsup_{k\to\infty} (J(x_k, u_k) - \widetilde{\omega}(K)) = \varlimsup_{k\to\infty} J(x_k, u_k), \end{aligned}$$

where $\widetilde{\omega}$ is a function in condition (xii$''$).

So (x_0, u_*) is the desired admissible pair. It remains to explain why the conclusion of Theorem B holds for the truncated problem with the objective functional I_N. Among all assumptions of Theorem B, only conditions (ix), (x) need to be checked for $t \notin [N-1, N)$. Condition (ix) holds, because the projection of a convex set is a convex set. But condition (x) (if it is at all satisfied) cannot be verified so simply.

To overcome this difficulty, we proceed as follows. We note that, in the above reasoning, the values $u_{N*}(t)$ are used only for $t \in [N-1, N)$. Therefore, we can arbitrarily change the sequence $\{(x_k, u_k)\}_{k=1}^{\infty}$ and the parameters of problem (1)–(3) outside the interval $[N-1, N)$. In particular, we can set $f(t, \cdot, \cdot) = 0$, $A(t) = \mathbb{R}^n$, $U(t, \cdot) = \{0\}$, and $u_k(t) = 0$ for $t \notin [N-1, N)$, and also $x_k(t) = x_k(N-1)$ for $0 \le t < N-1$ and $x_k(t) = x_k(N)$ for $t \ge N$. For the problem thus modified (with the functional I_N as before), all assumptions of Theorem B undoubtedly hold, and we obtain the required function u_{N*} on the interval $[N-1, N)$. □

Remark 2. From the formal point of view, Theorem 1 cannot be said to strengthen Theorem A not only for reasons explained in footnote 5, but also in view of the following important remark. Theorems 1 and A deal with problems in which the objective functionals are defined differently. In particular, it may happen that for the same parameters of the problem, an optimal solution exists in one problem and does not exist in the other, or that optimal solutions exist in both problems but are different. Nevertheless, the hypothesis in Theorem 1 concerning the behavior of the control system at infinity seems to be essentially weaker than that in Theorem A. As an illustration, we give the following example.

Example 1. We consider the problem

$$\int_0^{\infty} \frac{u(t)}{t+1}\,dt \to \max, \tag{7}$$

$$\dot{x}(t) = u(t) \qquad \text{for a.e.} \quad t \in \mathbb{R}, \tag{8}$$

$$x(t) \in [-t, t] \cap [-1, 1], \quad u(t) \in [-1, 1] \qquad \text{for a.e.} \quad t \in \mathbb{R}_+. \tag{9}$$

It is clear that $x(0) = 0$ and the integrand in (7) is bounded in absolute value by $1/(t+1)$ for any admissible pair (x, u). All local conditions (i)–(x) and (xi$'$) are satisfied. Let us show that condition (xii$''$) also holds:

$$\begin{aligned} \int_T^{T'} \frac{u(t)}{t+1}\,dt &= \int_T^{T'} \frac{\dot{x}(t)}{t+1}\,dt = \frac{x(T')}{T'+1} - \frac{x(T)}{T+1} + \int_T^{T'} \frac{x(t)}{(t+1)^2}\,dt \\ &\le \frac{1}{T'+1} + \frac{1}{T+1} + \int_T^{T'} \frac{dt}{(t+1)^2} = \frac{2}{T+1} \qquad \forall\, T > 0. \end{aligned} \tag{10}$$

Thus, if we consider the functional (7) as an improper integral, i.e., in the sense of (5), then Theorem 1 can be applied, which guarantees the existence of an optimal solution.

This optimal solution can easily be found explicitly. Indeed, since

$$\lim_{T\to\infty}\int_0^T \frac{u(t)}{t+1}\,dt = \lim_{T\to\infty}\left(\frac{x(T)}{T+1} + \int_0^T \frac{x(t)}{(t+1)^2}\,dt\right) = \lim_{T\to\infty}\int_0^T \frac{x(t)}{(t+1)^2}\,dt,$$

it suffices to maximize $x(t)$ for every t (which is possible here), i.e., to set $u_*(t) = 1$ for $t < 1$ and $u_*(t) = 0$ for $t \geq 1$. The corresponding optimal trajectory is $x_*(t) = \min\{t, 1\}$.

Since the integrand is positive, by Remark 1 the same solution is also optimal in the case where the objective functional is understood in the sense of (4). Let us show that, nevertheless, Theorem A is inapplicable in this case for any α (except for α equal to the exact value α_* ($= \ln 2$) of the functional on the optimal solution, but, in that case, the theorem is almost worthless, because the set Ω_{α_*} consists of a single admissible pair). The reason is that condition (xi) of strong uniform integrability does not hold for $\alpha < \alpha_*$. Let us show this.

First, we consider an admissible pair with $u(t) = \cos t$, i.e., the pair

$$u_0(t) = \cos t, \quad x_0(t) = \sin t, \qquad t \geq 0.$$

In this case,

$$\int_0^\infty \left(\frac{u_0(t)}{t+1}\right)^+ dt = \int_0^\infty \frac{\max\{\cos t, 0\}}{t+1}\,dt = +\infty, \tag{11}$$

i.e., no family of functions containing the integrand from (11) can be strongly uniformly integrable.

To show that condition (xi) is also violated for admissible pairs for which the value of the functional is (in any sense) close to the optimal one, it suffices to construct such an admissible pair by gluing parts together as follows:

- first, on a sufficiently large interval $[0, T_1]$, where $T_1 = \pi/2 + 2\pi k,\ k \in \mathbb{N}$, we use the optimal control u_* and follow the optimal trajectory x_*;
- further, on a sufficiently large interval $[T_1, T_2]$, we use the control u_0 and follow the trajectory x_0 (since $x_0(T_1) = 1 = x_*(T_1)$, we can switch from one trajectory to the other);
- for $t > T_2$, we use the control $u = 0$.

Because of the zero control on the last interval, the value of the functional (in any sense) is finite on such an admissible pair. In view of the estimate (10) (we note that the replacement of u by $-u$ results in the change of the trajectory x to $-x$, so the estimate (10) is also valid for the absolute value of the integral on its left-hand side), the value of the functional (in any sense) on such an admissible pair differs from the optimal value by at most $4/(T_1 + 1)$. Choosing a sufficiently large T_2 (depending on T_1), we can make the integral analogous to (11) arbitrarily large. This implies that condition (xi) of strong uniform integrability is not satisfied for Ω_α for any $\alpha < \alpha_*$.

Remark 3. One can also construct a similar example without any state constraint. For example, it suffices to replace $u(t)$ by $u(t)(1 - x(t)^2)$ in (7) and (8) and to introduce the initial condition $x(0) = 0$ in (9) instead of the state constraint.

Remark 4. In the problem considered in Example 1, the existence result from [4, Theorem 7.9] cannot be applied either, because it requires that there should exist a locally integrable function $\lambda\colon \mathbb{R}_+ \to \mathbb{R}$ with finite improper integral $\int_0^\infty \lambda(t)\,dt$ which would majorize the integrand in the objective functional for all admissible pairs from $\widetilde{\Omega}_\alpha$. It is clear that, in our case, such a function does not exist for $\alpha < \alpha_*$ (while, for $\alpha = \alpha_*$, as mentioned above, the set $\widetilde{\Omega}_\alpha$ consists of a single pair and the theorem becomes almost worthless).

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