ISSN 0001-4346, Mathematical Notes, 2017, Vol. 102, No. 4, pp. 541–546. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © A. A. Sagdeev, 2017, published in Matematicheskie Zametki, 2017, Vol. 102, No. 4, pp. 579–585.

The Chromatic Number of Space with Forbidden Regular Simplex

A. A. Sagdeev[1,2*]

[1] *Lomonosov Moscow State University, Moscow, Russia*
[2] *Moscow Institute of Physics and Technology (State University), Dolgoprudnyi, Moscow Oblast, Russia*

Received October 2, 2016; in final form, October 20, 2016

Abstract—An explicit exponentially growing lower bound for the chromatic number of Euclidean space with forbidden regular simplex is constructed .

DOI: 10.1134/S0001434617090267

Keywords: *chromatic number, distance graph, Euclidean Ramsey theory.*

1. INTRODUCTION

This paper is devoted to a special case of the general problem of finding the chromatic number of space with a forbidden monochromatic configuration. Before stating results, we recall the definitions of basic objects dealt with in this paper and make a few historical comments on this and related problems.

The *chromatic number* $\chi(\mathbb{R}^n)$ of space $\mathbb{R}^n$ is defined as the least number of colors required to color the points of space so that no points of the same color are a unit distance apart.

The problem of finding the chromatic number of the plane was posed by Nelson in 1950 (see [1]); however, at present, it is only known that

$$4 \le \chi(\mathbb{R}^2) \le 7$$

(these bounds can be found, e.g., in [1]). About the behavior of $\chi(\mathbb{R}^n)$ for large n we know that

$$(1.239\ldots + o(1))^n \le \chi(\mathbb{R}^n) \le (3 + o(1))^n \qquad (1)$$

(see [2] and [3]).

During the past decades, many various generalizations of Nelson's original problem have appeared. The branch of mathematics which studies such generalizations is Euclidean Ramsey theory. In this theory, a set $S \subset \mathbb{R}^d$ is said to be *Ramsey* if, given any positive integer m, there exists an $n \in \mathbb{N}$ such that, for any m-coloring of $\mathbb{R}^n$, there exists a monochromatic set congruent to S.

Let us reformulate the definition given above in somewhat different terms. Let $\chi_S(\mathbb{R}^n)$ be the least number of colors required to color $\mathbb{R}^n$ so that none of the sets congruent to S is monochromatic. In this terminology, a set $S \subset \mathbb{R}^d$ is said to be *Ramsey* if

$$\lim_{n\to\infty} \chi_S(\mathbb{R}^n) = \infty.$$

As is known, any Ramsey set lies on a sphere (see [4]). It follows from inequalities (1) that any pair of points is a Ramsey set (and even a so-called exponentially Ramsey set). Many other sets have been proved to be exponentially Ramsey, too. Such sets include the vertex set of any simplex and the Cartesian products of exponentially Ramsey sets (see [4]–[7]). However, the proof is implicit in almost all cases, so that it is impossible to write out an estimate of the form $\chi_S(\mathbb{R}^n) \ge (c + o(1))^n$ with a specific $c > 1$ in the framework of the method applied. The only exception is the paper [8], in which it was proved that, for the vertex set Δ of a regular triangle, we have

$$\chi_\Delta(\mathbb{R}^n) \ge (1.052\ldots + o(1))^n. \qquad (2)$$

*E-mail: xp1@protonmail.com

We are interested in obtaining similar inequalities not only for a regular triangle but also for any regular simplex of any dimension. Let S_k denote the vertex set of a regular k-simplex. Note that, as well as in the case of the chromatic number of space, the quantity $\chi_{S_k}(\mathbb{R}^n)$ does not depend on the edge length of a regular simplex with vertices S_k. For brevity, we denote this quantity by $\chi_k(\mathbb{R}^n)$.

This paper is organized as follows: in Sec. 2, we give exponentially growing lower bounds for $\chi_k(\mathbb{R}^n)$ for each k, and in Sec. 3, we prove these bounds.

In conclusion, we mention that similar problems were considered in [9]–[24]; surveys of related problems of combinatorial geometry can be found in [25]–[31].

2. STATEMENT OF RESULTS

The central result of this paper is the following constructive lower bound for $\chi_k(\mathbb{R}^n)$ which exponentially grows with respect to n.

Theorem 1. *Let k be a positive integer. Then*

$$\chi_k(\mathbb{R}^n) \geq \left(1 + \frac{1}{3^{5^k}} + o(1)\right)^n.$$

It is seen from the statement of the theorem that the exponential bases in the bounds are very close to 1. However, for $k > 2$, even such bounds were not known previously. Only the existence of a $c_k > 1$ for which

$$\chi_k(\mathbb{R}^n) \geq (c_k + o(1))^n$$

had been proved. No explicit expressions for c_k can be derived from the preceding works.

It should also be mentioned that our bound is far from optimal. For example, as mentioned above, the best known estimates for k equal to 1 and 2, which were obtained in [2] and [8], are

$$\chi_1(\mathbb{R}^n) \geq (1.239\ldots + o(1))^n, \tag{3}$$

$$\chi_2(\mathbb{R}^n) \geq (1.052\ldots + o(1))^n. \tag{4}$$

The bounds provided by Theorem 1 for the same k are much worse. However, for $k \geq 3$, Theorem 1 gives the only known constructive lower bounds for $\chi_k(\mathbb{R}^n)$.

3. PROOF OF THEOREM 1

The proof of Theorem 1 is based on the same idea as the proof of bound (2) in [8]. Before describing this idea, we give a definition needed in what follows.

A graph $G = (V, E)$ is said to be *distance* if V is a subset in $\mathbb{R}^n$ and

$$E \subseteq \{\{x, y\} | x, y \in V,\ |x - y| = a\}$$

for some fixed a. An important special example of a distance graph is the following construction. Let $s \leq r \leq n$ be positive integers. We define the distance graph $G(n, r, s)$ as follows: the vertices of this graph are all points in $\mathbb{R}^n$ which have r coordinates equal to 1 and $n - r$ zero coordinates; two vertices are joined by an edge if and only if the inner product of their radius vectors equals s.

We set $v(n) = [n/2]$. Let $v'(n)$ be the greatest positive integer smaller than $n/4 - 1$ for which $v(n) - v'(n)$ is a prime. It is known from number theory that

$$\lim_{n\to\infty} \frac{v'(n)}{n} = \frac{1}{4}$$

(see [32] and [33]). Consider the distance graph $G_n = G(n, v(n), v'(n))$.

Stirling's formula implies

$$C_{n(a+o(1))}^{n(b+o(1))} = (c_a^b + o(1))^n, \qquad \text{where} \quad c_a^b = \frac{a^a}{b^b \cdot (a-b)^{(a-b)}}.$$

Now it is easy to see that the graph G_n has

$$C_n^{v(n)} = (c_1^{1/2} + o(1))^n = (2 + o(1))^n$$

vertices, and the number of its edges equals

$$\frac{1}{2} C_n^{v(n)} \cdot C_{v(n)}^{v'(n)} \cdot C_{n-v(n)}^{v(n)-v'(n)} = (c_1^{1/2} \cdot c_{1/2}^{1/4} \cdot c_{1/2}^{1/4} + o(1))^n = (4 + o(1))^n.$$

Using a theorem proved by Frankl and Wilson in [34], we can estimate the independence number of many graphs $G(n, r, s)$ from above. In particular, this theorem applies to our graphs G_n; we state the result of its application as a separate lemma.

Lemma 1. *If a subset W of the vertex set of G_n contains no vertices joined by edges, then*

$$|W| \le \sum_{i=0}^{v(n)-v'(n)-1} C_n^i < n C_n^{v(n)-v'(n)} = (c_1^{1/4} + o(1))^n < (1.755 + o(1))^n.$$

Let us show that, in fact, a stronger assertion holds: any sufficiently large subset of the vertex set of G_n contains a vertex of sufficiently large degree. To this end, we employ a theorem essentially proved in [10].

Theorem 2. *If*

$$z \le 0.0288, \qquad \tau > 2 - \frac{z^4}{2450},$$

then any set W of vertices of G_n which consists of at least $(\tau + o(1))^n$ vertices contains at least

$$(2\tau \cdot (c_{1/4}^{z/4})^{-4} + o(1))^n$$

edges.

Introducing the new parameter $w = z^4/2450$ and using Dirichlet's principle, we obtain the following lemma.

Lemma 2. *If $w \le 2.8 \cdot 10^{-10}$, then any set W of vertices of G_n which consists of at least $(2 - w + o(1))^n$ vertices contains a vertex of degree at least*

$$(2 \cdot (c_{1/4}^{\sqrt[4]{2450w}/4})^{-4} + o(1))^n.$$

Note that, for $w = 2.8 \cdot 10^{-10}$, Lemma 2 guarantees that one of the vertices has $(1.755\ldots + o(1))^n$ neighbors; therefore, by Lemma 1, at least two neighbors of this vertex are joined by an edge. Thus, we have proved the following statement.

Lemma 3. *Any set W of vertices of G_n which consists of at least $(2 - 2.8 \cdot 10^{-10} + o(1))^n$ vertices contains three pairwise adjacent vertices.*

It is easy to derive from this lemma that

$$\chi_2(\mathbb{R}^n) \ge \left(\frac{2}{2 - 2.8 \cdot 10^{-10}} + o(1)\right)^n. \tag{5}$$

Indeed, suppose that, on the contrary, given arbitrarily large n, $\mathbb{R}^n$ can be colored with less than

$$\left(\frac{2}{2 - 2.8 \cdot 10^{-10}} + o(1)\right)^n$$

colors so that the side length of any monochromatic regular triangle is different from the distance between any two vertices of the graph G_n. Then the vertices of G_n can be colored in a similar way. Since the graph contains $(2+o(1))^n$ vertices, it follows by Dirichlet's principle that at least $(2-2.8\cdot 10^{-10}+o(1))^n$ vertices of G_n are of the same color. Applying Lemma 3, we see that G_n has three pairwise adjacent vertices of the same color. This contradiction completes the proof of inequality (5).

Now, we obtain a lower bound for $\chi_3(\mathbb{R}^n)$ in a similar way. It can be shown that, for $w = 5\cdot 10^{-53}$, Lemma 2 ensures that one of the vertices has more than $(2-2.8\cdot 10^{-10}+o(1))^n$ neighbors. By Lemma 3, among the neighbors of this vertex there are three pairwise adjacent vertices. Thus, we have proved the following statement.

Lemma 4. *Any set W of vertices of G_n which consists of at least $(2-5\cdot 10^{-53}+o(1))^n$ vertices contains four pairwise adjacent vertices.*

Precisely the same argument as that used to prove (5) also proves the inequality

$$\chi_3(\mathbb{R}^n) \geq \left(\frac{2}{2-5\cdot 10^{-53}}+o(1)\right)^n. \tag{6}$$

Now, fix a positive integer $k \geq 4$. Let us obtain a lower bound for $\chi_k(\mathbb{R}^n)$. Computer calculations show that, for $0 < w \leq 5\cdot 10^{-53}$, the number of neighbors for one of the vertices ensured by Lemma 2 is larger than $(2-\sqrt[5]{w}+o(1))^n$.

It follows that, considering any set W of vertices of G_n which consists of at least

$$(2-(5\cdot 10^{-53})^{5^{k-3}}+o(1))^n$$

vertices and applying Lemma 2 to this set $k-3$ times, we find $k-3$ pairwise adjacent vertices with at least $(2-5\cdot 10^{-53}+o(1))^n$ common neighbors in W. Applying Lemma 4 to these common neighbors, we obtain the following lemma.

Lemma 5. *Let $k \geq 4$. Then any set W of vertices of G_n which consists of at least*

$$(2-(5\cdot 10^{-53})^{5^{k-3}}+o(1))^n$$

vertices contains $k+1$ pairwise adjacent vertices.

The same argument as that used to prove (5) proves the following theorem.

Theorem 3. *Let $k \geq 4$. Then*

$$\chi_k(\mathbb{R}^n) \geq \left(\frac{2}{2-(5\cdot 10^{-53})^{5^{k-3}}}+o(1)\right)^n.$$

Proof of Theorem 1. To reduce the above bound to the form announced in Theorem 1, we use the following chain of inequalities, which hold for $k \geq 4$:

$$\begin{aligned}\frac{2}{2-(5\cdot 10^{-53})^{5^{k-3}}} &= \frac{2}{2-x} = 1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\cdots > 1+\frac{x}{2} = 1+\frac{(5\cdot 10^{-53})^{5^{k-3}}}{2}\\ &> 1+10^{-53\cdot 5^{k-3}} = 1+(10^{-53/125})^{5^k} > 1+\left(\frac{1}{3}\right)^{5^k} = 1+\frac{1}{3^{5^k}}.\end{aligned}$$

Thus, we have proved Theorem 1 for $k \geq 4$. To prove it in the case $k = 3$, it suffices to note that inequality (6) proved above is stronger than that appearing in Theorem 1. As already mentioned, the known estimates (3) and (4) for k equal to 1 and 2 are substantially better than those required in Theorem 1.

This completes the proof of Theorem 1. □

REFERENCES

1. A. Soifer, "The Chromatic Number of the Plane: Its Past, Present, and Future," in *Mat. Pros.*, Ser. 3 (MTsNMO, Moscow, 2004), Vol. 8, pp. 186–221 [in Russian].
2. A. M. Raigorodskii, "On the chromatic number of a space," Uspekhi Mat. Nauk **55** (2 (332)), 147–148 (2000) [Russian Math. Surveys **55** (2), 351–352 (2000)].
3. D. G. Larman and C. A. Rogers, "The realization of distances within sets in Euclidean space," Mathematika **19**, 1–24 (1972).
4. R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey Theory* (John Wiley and Sons, New York, 1990).
5. P. Frankl and V. Rödl, "All triangles are Ramsey," Trans. Amer. Math. Soc. **297** (2), 777–779 (1986).
6. P. Frankl and V. Rödl, "Forbidden intersections," Trans. of Amer. Math. Soc. **300** (1), 259–286 (1987).
7. P. Frankl and V. Rödl, "A partition property of simplices in Euclidean space," J. Amer. Math. Soc. **3** (1), 1–7 (1990).
8. A. E. Zvonarev, A. M. Raigorodskii, D. V. Samirov, and A. A. Kharlamova, "On the chromatic number of a space with forbidden equilateral triangle," Mat. Sb. **205** (9), 97–120 (2014) [Sb. Math. **205** (9), 1310–1333 (2014)].
9. A. E. Zvonarev, A. M. Raigorodskii, D. V. Samirov, and A. A. Kharlamova, "Improvement of the Frankl–Rödl theorem on the number of edges of a hypergraph with forbidden intersection," Dokl. Akad. Nauk **457** (2), 144–146 (2014) [Dokl. Math. **90**, 432–434 (2014)].
10. A. A. Sagdeev, "Lower Bounds for the Chromatic Numbers of Distance Graphs with Large Girth," Mat. Zametki **101** (3), 430–445 (2017) [Math. Notes **101** (3–4), 515–528 (2017)].
11. A. E. Zvonarev and A. M. Raigorodskii, "Improvements of the Frankl–Rödl theorem on the number of edges of a hypergraph with forbidden intersections, and their consequences in the problem of finding the chromatic number of a space with forbidden equilateral triangle," in *Trudy Mat. Inst. Steklov.*, Vol. 288: *Geometry, Topology, and Applications* (2015), pp. 109–119 [Proc. Steklov Inst. Math. **288** (1), 94–104 (2015)].
12. A. B. Kupavskii, "Distance graphs with large chromatic number and arbitrary girth," Mosc. J. Comb. Number Theory **2** (2), 52–62 (2012).
13. A. M. Raigorodskii, "On distance graphs with large chromatic number but without large simplices," Uspekhi Mat. Nauk **62** (6 (378)), 187–188 (2007) [Russian Math. Surveys **62** (6), 1224–1225 (2007)].
14. A. M. Raigorodskii and O. I. Rubanov, "Small clique and large chromatic number," in *European Conference on Combinatorics, Graph Theory and Applications*, *Electron. Notes Discrete Math.* (Elsevier Sci. B. V., Amsterdam, 2009), Vol. 34, pp. 441–445.
15. A. M. Raigorodskii and O. I. Rubanov, "On the clique and the chromatic numbers of high-dimensional distance graphs," in *Number Theory and Applications* (Hindustan Book Agency, New Delhi, 2009), pp. 149–157.
16. A. M. Raigorodskii and O. I. Rubanov, "Distance Graphs with Large Chromatic Number and without Large Cliques," Mat. Zametki **87** (3), 417–428 (2010) [Math. Notes **87** (3–4), 392–402 (2010)].
17. A. B. Kupavskii and A. M. Raigorodskii, "On distance graphs with large chromatic numbers and small clique numbers," Dokl. Akad. Nauk **444** (5), 483–487 (2012) [Dokl. Math. **85** (3), 394–398 (2012)].
18. A. E. Zvonarev and A. M. Raigorodskii, "On distance graphs with large chromatic and small clique numbers," Trudy Mosk. Fiz.-Tekh. Inst. **4** (1), 122–126 (2012).
19. A. B. Kupavskii and A. M. Raigorodskii, "Obstructions to the realization of distance graphs with large chromatic numbers on spheres of small radii," Mat. Sb. **204** (10), 47–90 (2013) [Sb. Math. **204** (10), 1435–1479 (2013)].
20. E. E. Demekhin, A. M. Raigorodskii, and O. I. Rubanov, "Distance graphs having large chromatic numbers and not containing no cliques or cycles of a given size," Mat. Sb. **204** (4), 49–78 (2013) [Sb. Math. **204** (4), 508–538 (2013)].
21. A. B. Kupavskii, "Explicit and probabilistic constructions of distance graphs with small clique numbers and large chromatic numbers," Izv. Ross. Akad. Nauk Ser. Mat. **78** (1), 65–98 (2014) [Izv.: Math. **78** (1), 59–89 (2014)].
22. N. Alon and A. Kupavskii, "Two notions of unit distance graphs," J. Combin. Theory Ser. A **125**, 1–17 (2014).
23. D. D. Cherkashin and A. M. Raigorodskii, "On the chromatic number of low-dimensional spaces," Dokl. Akad. Nauk **472** (1), 11–12 (2017) [Dokl. Math. **95** (1), 5–6 (2017)].
24. A. M. Raigorodskii and D. V. Samirov, "New bounds for the chromatic number of a space with forbidden isosceles triangles," Dokl. Akad. Nauk **456** (3), 280–283 (2014) [Dokl. Math. **89** (3), 313–316 (2014)].
25. P. K. Agarwal and J. Pach, *Combinatorial Geometry* (John Wiley and Sons, New York, 1995).
26. P. Brass, W. Moser, and J. Pach, *Research Problems in Discrete Geometry* (Springer, New York, 2005).
27. A. Soifer, *The Mathematical Coloring Book* (Springer, New York, 2009).
28. A. M. Raigorodskii, "Borsuk's problem and the chromatic numbers of some metric spaces," Uspekhi Mat. Nauk **56** (1 (337)), 107–146 (2001) [Russian Math. Surveys **56** (1), 103–139 (2001)].

29. A. M. Raigorodskii, "Around Borsuk's hypothesis," in *Contemporary Mathematics. Fundamental Directions*, Vol. 23: *Geometry and Mechanics* (2007), pp. 147–164 [J. Math. Sci. **154** (4), 604–623 (2008)].
30. A. M. Raigorodskii, "Coloring distance graphs and graphs of diameters," in *Thirty Essays on Geometric Graph Theory* (Springer, New York, 2013), pp. 429–460.
31. A. M. Raigorodskii, "Cliques and cycles in distance graphs and graphs of diameters," in *Discrete Geometry and Algebraic Combinatorics*, *Contemp. Math.* (Amer. Math. Soc., Providence, RI, 2014), Vol. 625, pp. 93–109.
32. K. Prachar, *Primzahlverteilung* (Springer-Verlag, Berlin, 1957; Mir, Moscow, 1967).
33. R. C. Baker, G. Harman, and J. Pintz, "The difference between consecutive primes. II," Proc. London Math. Soc. (3) **83** (3), 532–562 (2001).
34. P. Frankl and R. M. Wilson, "Intersection theorems with geometric consequences," Combinatorica **1** (4), 357–368 (1981).