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On the Compactness of Convolution-Type Operators in Morrey Spaces

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Abstract—In a Morrey space, the product of the convolution operator with summable kernel and the operator of multiplication by an essentially bounded function is considered. Sufficient conditions for such a product to be compact are obtained. In addition, it is shown that the commutator of the convolution operator and the operator of multiplication by a function of weakly oscillating type is compact in a Morrey space.

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1. INTRODUCTION

At present there are many papers dealing with Morrey spaces and their generalizations (see, for example, the surveys of Burenkov [1], [2] and the bibliography given there). The studies of these spaces go back to Morrey's paper [3] and have been pursued intensively during the last 20–30 years. The development of the theory of Morrey-type spaces stimulates, in a natural way, the study of operators in these spaces. In particular, for classical operators in the analysis, such as the Hardy operator, the maximal operator, the fractional maximal operator, the Riesz potential, and the singular integral operator, boundedness conditions in Morrey-type spaces were obtained (see [4]–[7] as well as the survey [2]). Convolution operators in generalized global Morrey-type spaces were studied in [8] and there analogs of Young's inequality for convolutions in these spaces were established. The compactness of commutators of classical operators in Morrey spaces was investigated in the papers [9]–[11].

In the present paper, we study the compactness of certain operators in a Morrey space, namely, the product operator of the convolution operator with summable kernel and the operator of multiplication by a function from $L_\infty(\mathbb{R}^n)$. We show that if this function tends to zero at infinity, then the product is a compact operator. In the concluding part of the paper, we consider the commutator of the convolution operator and the operator of multiplication by a function. It is proved that if the function defining the multiplication operator satisfies a condition of weakly oscillating type at infinity, then the commutator is compact in a Morrey space.

In what follows, we shall use the following notation: $\mathbb{R}^n$ is n-dimensional Euclidean space,

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n, \qquad |x| = \sqrt{x_1^2 + \cdots + x_n^2}\,;$$

and $\mathbb{B}(x, r)$ is the open ball in $\mathbb{R}^n$ of radius r centered at the point x.

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2. PRELIMINARIES

Let $1 \le p \le \infty$, and let $X \subseteq \mathbb{R}^n$ be a measurable set. Then $L_p(X)$ is the space (of classes) of measurable complex-valued functions with norm

$$\|f\|_{L_p(X)} = \left(\int_X |f(x)|^p \, dx\right)^{1/p}, \quad 1 \le p < \infty, \qquad \|f\|_{L_\infty(X)} = \operatorname*{ess\,sup}_{x \in X} |f(x)|.$$

In the case $X = \mathbb{R}^n$, we shall use the notation $\|\cdot\|_p$ instead of $\|\cdot\|_{L_p(X)}$. Further, we shall say that f belongs to $L_p^{\mathrm{loc}}(\mathbb{R}^n)$ if $f \in L_p(K)$ for any compact set $K \subset \mathbb{R}^n$.

Definition 1. Let $1 \le p \le \infty$, and let $\lambda \in \mathbb{R}$. A function f is said to belong to $L_{p,\lambda}(\mathbb{R}^n)$ if $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} \equiv \|f\|_{p,\lambda} = \sup_{x \in \mathbb{R}^n,\, r > 0} \frac{\|f\|_{L_p(\mathbb{B}(x,r))}}{r^\lambda} < \infty. \tag{2.1}$$

With respect to the usual linear operations and the norm (2.1), the set $L_{p,\lambda}(\mathbb{R}^n)$ constitutes a Banach space, which is called a *Morrey space*.

The spaces $L_{p,\lambda}(\mathbb{R}^n)$ are nontrivial, i.e., they consist not only of functions equivalent to zero on $\mathbb{R}^n$ if and only if $0 \le \lambda \le n/p$. For $\lambda = 0$ and $\lambda = n/p$, Morrey spaces coincide with L_p-spaces; namely,

$$L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n), \qquad L_{p,n/p}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n). \tag{2.2}$$

Let us present precompactness conditions for a set in a Morrey space.

Proposition 1 (see [11]). *Suppose that $1 \le p < \infty$, $0 < \lambda < n/p$, and $\Psi = \{\psi\}$ is a set of functions from $L_{p,\lambda}(\mathbb{R}^n)$. Let the following conditions hold:*

i) *the set Ψ is bounded, i.e., there exists a $C > 0$ such that $\|\psi\|_{p,\lambda} \le C$ for any $\psi \in \Psi$;*

ii) $\lim_{\delta \to 0} \|\psi(\cdot + \delta) - \psi(\cdot)\|_{p,\lambda} = 0$ *uniformly with respect to $\psi \in \Psi$;*

iii) $\lim_{\rho \to \infty} \|\psi \chi_\rho\|_{p,\lambda} = 0$ *uniformly with respect to $\psi \in \Psi$, where χ_ρ is the characteristic function of the set $\mathbb{R}^n \setminus \mathbb{B}(0,\rho)$.*

Then the set Ψ is precompact in the space $L_{p,\lambda}(\mathbb{R}^n)$.

Let us point out that the precompactness conditions given in Proposition 1 are only sufficient.

In the space $L_{p,\lambda}(\mathbb{R}^n)$, we consider the convolution operator

$$(H\varphi)(x) = \int_{\mathbb{R}^n} h(x-y)\varphi(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{2.3}$$

where $h \in L_1(\mathbb{R}^n)$. It was shown in [8] that the operator H is bounded in $L_{p,\lambda}(\mathbb{R}^n)$, where $1 \le p \le \infty$, and, for any function $\varphi \in L_{p,\lambda}(\mathbb{R}^n)$, the following inequality holds:

$$\|H\varphi\|_{p,\lambda} \le \|h\|_1 \|\varphi\|_{p,\lambda}. \tag{2.4}$$

3. THE PRODUCT OF THE CONVOLUTION OPERATOR AND THE OPERATOR OF MULTIPLICATION

Since the convolution operator H commutes with shift operators, it is not compact (see, for example, [12]). The following question naturally arises: Is the product of the convolution operator and the operator of multiplication by a bounded function compact? Since the convolution operators in L_p-spaces are well studied, taking into account equalities (2.2), we will not consider the cases $\lambda = 0$ and $\lambda = n/p$.

Denote by M_a the operator of multiplication by a function $a \in L_\infty(\mathbb{R}^n)$. It is easy to see that this operator is bounded in the space $L_{p,\lambda}(\mathbb{R}^n)$ and, for any function $\varphi \in L_{p,\lambda}(\mathbb{R}^n)$, the following inequality holds:

$$\|M_a\varphi\|_{p,\lambda} \le \|a\|_\infty \|\varphi\|_{p,\lambda}.$$

The main object of study in this section will be the operator

$$H_a = M_a H. \tag{3.1}$$

Denote by $C_0(\mathbb{R}^n)$ the set of all continuous functions $a(x)$ on $\mathbb{R}^n$ such that $\lim_{x\to\infty} a(x) = 0$.

Lemma 1. *Suppose that $1 \le p < \infty$, $0 < \lambda < n/p$, $a \in C_0(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$. Then the operator H_a of the form (3.1) is compact in the space $L_{p,\lambda}(\mathbb{R}^n)$.*

Proof. Let $\Phi = \{\varphi\}$ be an arbitrary bounded set in $L_{p,\lambda}(\mathbb{R}^n)$, i.e., $\|\varphi\|_{p,\lambda} \le C$ for any $\varphi \in \Phi$. Using Proposition 1, we shall show that the set $\{H_a\varphi\}$, where $\varphi \in \Phi$, is precompact in the space $L_{p,\lambda}(\mathbb{R}^n)$. We shall verify all three conditions.

The validity of condition i) follows from the boundedness of the operator H_a. Let us verify condition ii). For any function $\varphi \in \Phi$, we have

$$\begin{aligned}
&\|(H_a\varphi)(\,\cdot+\delta) - (H_a\varphi)(\,\cdot\,)\|_{p,\lambda}\\
&\qquad \le \|(a(\,\cdot+\delta) - a(\,\cdot\,))(H\varphi)(\,\cdot+\delta)\|_{p,\lambda} + \|a(\,\cdot\,)((H\varphi)(\,\cdot+\delta) - (H\varphi)(\,\cdot\,))\|_{p,\lambda}\\
&\qquad \le \|a(\,\cdot+\delta) - a(\,\cdot\,)\|_\infty \|H\varphi\|_{p,\lambda} + \|a\|_\infty \|(H\varphi)(\,\cdot+\delta) - (H\varphi)(\,\cdot\,)\|_{p,\lambda}.
\end{aligned}$$

Applying inequality (2.4) to each summand and taking into account the boundedness of the set Φ, we obtain

$$\begin{aligned}
\|(H_a\varphi)(\,\cdot+\delta) - (H_a\varphi)(\,\cdot\,)\|_{p,\lambda} &\le \|a(\,\cdot+\delta) - a(\,\cdot\,)\|_\infty \|h\|_1 \|\varphi\|_{p,\lambda} + \|a\|_\infty \|h(\,\cdot+\delta) - h(\,\cdot\,)\|_1 \|\varphi\|_{p,\lambda}\\
&\le C(\|a(\,\cdot+\delta) - a(\,\cdot\,)\|_\infty \|h\|_1 + \|a\|_\infty \|h(\,\cdot+\delta) - h(\,\cdot\,)\|_1).
\end{aligned}$$

On the right-hand side of this inequality, the first summand tends to zero as $\delta \to 0$, because $a \in C_0(\mathbb{R}^n)$, and so does the second, because of the continuity of the function $h \in L_1(\mathbb{R}^n)$ in the L_1-norm. Therefore,

$$\|(H_a\varphi)(\,\cdot+\delta) - (H_a\varphi)(\,\cdot\,)\|_{p,\lambda} \to 0$$

uniformly with respect to $\varphi \in \Phi$.

Let us verify condition iii). Again using inequality (2.4), we obtain

$$\|\chi_\rho H_a\varphi\|_{p,\lambda} \le \|\chi_\rho a\|_\infty \|H\varphi\|_{p,\lambda} \le C\|h\|_1 \sup_{|x|\ge\rho} |a(x)|.$$

Since $\lim_{x\to\infty} a(x) = 0$, we have $\|\chi_\rho H_a\varphi\|_{p,\lambda} \to 0$ as $\rho \to \infty$ uniformly with respect to $\varphi \in \Phi$. The lemma is proved. □

Let us pass to a more general case. Following [13, p. 37], we say that a function $a \in L_\infty(\mathbb{R}^n)$ belongs to the class $B_0^{\sup}(\mathbb{R}^n)$ if

$$\lim_{N\to\infty} \operatorname*{ess\,sup}_{|x|>N} |a(x)| = 0.$$

Note that the class $B_0^{\sup}(\mathbb{R}^n)$ is the closure in the L_∞-norm of the set of all compactly supported functions in $L_\infty(\mathbb{R}^n)$.

Theorem 1. *Suppose that $1 \le p < \infty$, $0 < \lambda < n/p$, $a \in B_0^{\sup}(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$. Then the operator H_a of the form* (3.1) *is compact in the space $L_{p,\lambda}(\mathbb{R}^n)$.*

Proof. Set

$$a_N(x) = \begin{cases} a(x) & \text{if } |x| \le N, \\ 0 & \text{if } |x| > N, \end{cases}$$

and let us show that the operator H_{a_N} is compact. Indeed, let the function $b \in C_0(\mathbb{R}^n)$ satisfy $b(x) \equiv 1$ for $|x| \le N$. Then $H_{a_N} = M_{a_N} H_b$. By Lemma 1, the operator H_b is compact, and hence H_{a_N} is also a compact operator. Since

$$\|H_a - H_{a_N}\|_{L_{p,\lambda} \to L_{p,\lambda}} \le \operatorname*{ess\,sup}_{|x|>N} |a(x)| \|H\|_{L_{p,\lambda} \to L_{p,\lambda}},$$

and $a \in B_0^{\sup}(\mathbb{R}^n)$, it follows that $\|H_a - H_{a_N}\|_{L_{p,\lambda} \to L_{p,\lambda}} \to 0$ as $N \to \infty$. Therefore, the operator H_a is compact in the space $L_{p,\lambda}(\mathbb{R}^n)$. □

This theorem immediately implies the following statement.

Corollary 1. *Let X be a bounded measurable set in $\mathbb{R}^n$, and let P_X be the operator of multiplication by the characteristic function of the set X. Then the operator $P_X H$ is compact in the space $L_{p,\lambda}(\mathbb{R}^n)$.*

4. THE COMMUTATOR OF THE CONVOLUTION OPERATOR AND THE OPERATOR OF MULTIPLICATION

We recall that the commutator $[M_a, H]$ of the operators M_a and H is defined by the formula $[M_a, H] = M_a H - H M_a$. In view of (2.3), this commutator is of the form

$$\begin{aligned}([M_a, H]\varphi)(x) &= \int_{\mathbb{R}^n} (a(x) - a(y)) h(x-y) \varphi(y)\, dy \\ &= \int_{\mathbb{R}^n} (a(x) - a(x-t)) h(t) \varphi(x-t)\, dt, \qquad x \in \mathbb{R}^n.\end{aligned}$$

Denote by $\Omega_\infty(\mathbb{R}^n)$ the set of all functions $a \in L_\infty(\mathbb{R}^n)$ such that, for any compact set $K \subset \mathbb{R}^n$, the function

$$\mathcal{A}(x) := \operatorname*{ess\,sup}_{t \in K} |a(x) - a(x-t)|$$

belongs to the class $B_0^{\sup}(\mathbb{R}^n)$.

The class $\Omega_\infty(\mathbb{R}^n)$ is a generalization of the class $\Omega(\mathbb{R}^n) = \Omega_\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, which was introduced in [14]. Functions from $\Omega(\mathbb{R}^n)$ are said to be *weakly oscillating* at infinity. Therefore, functions from $\Omega_\infty(\mathbb{R}^n)$ will be called *functions of weakly oscillating type* at infinity.

Lemma 2. *If $a \in \Omega_\infty(\mathbb{R}^n)$ and $h \in L_1(\mathbb{R}^n)$, then the function*

$$J(x) = \int_{\mathbb{R}^n} |a(x) - a(x-t)| |h(t)|\, dt$$

belongs to the class $B_0^{\sup}(\mathbb{R}^n)$.

Proof. We take an arbitrary $\varepsilon > 0$ and choose a $\rho > 0$ such that

$$\int_{|t|>\rho} |h(t)|\, dt < \frac{\varepsilon}{4\|a\|_\infty}.$$

Then, for almost all $x \in \mathbb{R}^n$, we have

$$J(x) \le \operatorname*{ess\,sup}_{|t|\le\rho} |a(x)-a(x-t)| \int_{|t|\le\rho} |h(t)|\,dt + 2\|a\|_\infty \int_{|t|>\rho} |h(t)|\,dt$$
$$< \operatorname*{ess\,sup}_{|t|\le\rho} |a(x)-a(x-t)|\|h\|_1 + \frac{\varepsilon}{2} = \mathcal{A}_\rho(x)\|h\|_1 + \frac{\varepsilon}{2},$$

where $\mathcal{A}_\rho(x) = \operatorname{ess\,sup}_{|t|\le\rho} |a(x)-a(x-t)|$.

Let us fix the number ρ. Since $\mathcal{A}_\rho \in B_0^{\sup}(\mathbb{R}^n)$, there exists an $N_0 > 0$ such that, for all $N > N_0$, the following inequality holds:

$$\operatorname*{ess\,sup}_{|x|>N} \mathcal{A}_\rho(x) < \frac{\varepsilon}{2\|h\|_1}.$$

Therefore, $\operatorname{ess\,sup}_{|x|>N} J(x) < \varepsilon$ for all $N > N_0$, i.e., $J \in B_0^{\sup}(\mathbb{R}^n)$. The lemma is proved. □

The main result of this section is the following statement.

Theorem 2. *Suppose that $1 < p < \infty$, $0 < \lambda < n/p$, $a \in \Omega_\infty(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$. Then the commutator $[M_a, H]$ is compact in the space $L_{p,\lambda}(\mathbb{R}^n)$.*

Proof. Let us show that the operator $[M_a, H]$ can be approximated in the operator norm by compact operators with any degree of accuracy. We take an arbitrary $\varepsilon > 0$. By Lemma 2, there exists an $N > 0$ such that

$$\operatorname*{ess\,sup}_{|x|\ge N} \int_{\mathbb{R}^n} |a(x)-a(x-t)||h(t)|\,dt < \frac{\varepsilon^{p'}}{(2\|a\|_\infty\|h\|_1)^{p'/p}}.$$

Let us fix N, and let P_N and Q_N denote the operators of multiplication by the characteristic functions of the sets $\mathbb{B}(0,N)$ and $\mathbb{R}^n \setminus \mathbb{B}(0,N)$, respectively. Let us estimate the norm of the operator $Q_N[M_a,H]$. Applying Hölder's inequality, for almost all $x \in \mathbb{R}^n \setminus \mathbb{B}(0,N)$, we obtain

$$|(Q_N[M_a,H]\varphi)(x)| \le \int_{\mathbb{R}^n} |a(x)-a(x-t)||h(t)||\varphi(x-t)|\,dt$$
$$\le \biggl(\int_{\mathbb{R}^n} |a(x)-a(x-t)||h(t)|\,dt\biggr)^{1/p'} \biggl(\int_{\mathbb{R}^n} |a(x)-a(x-t)||h(t)||\varphi(x-t)|^p\,dt\biggr)^{1/p}$$
$$\le \biggl(\operatorname*{ess\,sup}_{|x|\ge N}\int_{\mathbb{R}^n} |a(x)-a(x-t)||h(t)|\,dt\biggr)^{1/p'} \biggl(2\|a\|_\infty \int_{\mathbb{R}^n} |h(t)||\varphi(x-t)|^p\,dt\biggr)^{1/p}$$
$$< \varepsilon\|h\|_1^{-1/p} \biggl(\int_{\mathbb{R}^n} |h(t)||\varphi(x-t)|^p\,dt\biggr)^{1/p}.$$

Then, for arbitrary $x \in \mathbb{R}^n$ and $r > 0$, we can write

$$\|Q_N[M_a,H]\varphi\|_{L_p(\mathbb{B}(x,r))} < \varepsilon\|h\|_1^{-1/p} \biggl(\int_{\mathbb{B}(x,r)} dy \int_{\mathbb{R}^n} |h(t)||\varphi(y-t)|^p\,dt\biggr)^{1/p}$$
$$= \varepsilon\|h\|_1^{-1/p} \biggl(\int_{\mathbb{R}^n} |h(t)|\,dt \int_{\mathbb{B}(x,r)} |\varphi(y-t)|^p\,dy\biggr)^{1/p}.$$

Replacing $y - t = z$ in the inner integral, we obtain the inequality

$$\|Q_N[M_a,H]\varphi\|_{L_p(\mathbb{B}(x,r))} < \varepsilon\|h\|_1^{-1/p} \biggl(\int_{\mathbb{R}^n} |h(t)|\|\varphi\|^p_{L_p(\mathbb{B}(x-t,r))}\,dt\biggr)^{1/p}.$$

Using this inequality, we estimate the norm of the function $Q_N[M_a,H]\varphi$ in a Morrey space, obtaining

$$\|Q_N[M_a,H]\varphi\|_{p,\lambda} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\lambda}\|Q_N[M_a,H]\varphi\|_{L_p(\mathbb{B}(x,r))}$$

$$< \varepsilon\|h\|_1^{-1/p} \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda}\biggl(\int_{\mathbb{R}^n} |h(t)|\|\varphi\|^p_{L_p(\mathbb{B}(x-t,r))}\,dt\biggr)^{1/p}$$
$$\le \varepsilon\|h\|_1^{-1/p}\|\varphi\|_{p,\lambda}\biggl(\int_{\mathbb{R}^n}|h(t)|\,dt\biggr)^{1/p} = \varepsilon\|\varphi\|_{p,\lambda}.$$

This implies that

$$\|Q_N[M_a,H]\|_{L_{p,\lambda}\to L_{p,\lambda}} < \varepsilon.$$

Taking into account the fact that $P_N + Q_N = I$, where I is the identity operator, we rewrite this inequality as

$$\|[M_a,H] - P_N[M_a,H]\|_{L_{p,\lambda}\to L_{p,\lambda}} < \varepsilon.$$

By Corollary 1, the operator $P_N[M_a,H]$ is compact. Then, since the number ε is arbitrary, it follows that the operator $[M_a,H]$ is also compact. The theorem is proved. □

Combined with results from the previous section, this theorem enables us to easily solve the question of whether operators of the form HM_a are compact.

Lemma 3. *Suppose that $1 < p < \infty$, $0 < \lambda < n/p$, $a \in C_0(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$. Then the operator HM_a is compact in the space $L_{p,\lambda}(\mathbb{R}^n)$.*

Proof. Since $a \in C_0(\mathbb{R}^n)$, we have $a \in \Omega_\infty(\mathbb{R}^n)$. Then the equality $HM_a = H_a - [M_a,H]$, Lemma 1, and Theorem 2 immediately imply the compactness of the operator HM_a. □

Theorem 3. *Suppose that $1 < p < \infty$, $0 < \lambda < n/p$, $a \in B_0^{\sup}(\mathbb{R}^n)$, and $h \in L_1(\mathbb{R}^n)$. Then the operator HM_a is compact in the space $L_{p,\lambda}(\mathbb{R}^n)$.*

Proof. The proof is quite similar to that of of Theorem 1. □

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