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Trace of Order (-1) for a String with Singular Weight

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Abstract—The Sturm–Liouville problem on a finite closed interval with potential and weight of first order of singularity is studied. Estimates for the s-numbers and eigenvalues of the corresponding integral operator are obtained. The spectral trace of first negative order is evaluated in terms of the integral kernel. The obtained theoretical results are illustrated by examples.

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1. INTRODUCTION

Let us consider the differential equation

$$-y''(x) + q(x)y(x) = \lambda\rho(x)y(x), \qquad x \in [0, \pi], \tag{1.1}$$

and the operator pencil associated with it

$$A(\lambda) = L - \lambda V,$$

where the operator L is generated by the differential expression $-y'' + q(x)y$ and separated boundary conditions, and $V: y \mapsto \rho y$ is the operator of multiplication by the function $\rho \in W_2^{-1}[0, \pi]$.

Equation (1.1), in particular, is a classical mathematical model for describing small transverse oscillations of a loaded string (this phenomenon occurs after the separation of variables in the corresponding wave equation). In such a setting, the values of the function $y(x)$ describe the deviations of the string from its equilibrium position, $q(x)$ is the function describing the density of external forces acting on the string at the point x, $\rho(x)$ is the mass density function of the string, and λ is the spectral parameter. Note that the problem of oscillations of a loaded string has long been studied. So the Stieltjes studies of infinite continued fractions can be regarded as the description of the oscillations of a weightless string loaded with point masses:

$$q(x) \equiv 0, \qquad \rho(x) = \sum m_j\delta(x - x_j).$$

For the case in which $q(x) \equiv 0$, and $\rho(x)$ is a nonnegative Borel measure, the spectral theory for the pencil $A(\lambda)$ was developed in the Krein and Kats papers [1], [2]. Approximately, at the same time, Eq. (1.1) was used to describe one-dimensional Markov processes (see [3] and [4]). For more details on the relationship between Eq. (1.1) with problems in the theory of Gaussian random processes, see [5]. The solution of the inverse problem for Eq. (1.1) on the recovery of the measure ρ from the spectral measure (or, equivalently, from the Weyl–Titchmarsh function) was given by Krein and Kats in [2]. In particular, it was proved that the set of Weyl–Titchmarsh functions of problem (1.1), where ρ is a positive measure, is dense for the class of so-called Stieltjes functions (for more details, see also the surveys [5], [6], and [7]).

The spectral theory of problem (1.1) with an arbitrary (real and alternating) charge ρ has also been intensively developed. We will not mention here numerous papers on this topic, but only two monographs [8] and [9], where the reader can find a detailed survey of the corresponding results and also an extensive bibliography. In addition, note that the studies of the inverse spectral problem for

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this class of functions ρ involve considerable difficulties. For a survey of the state-of-the-art, see, for example, [10]. The study of the direct and inverse problems for Eq. (1.1) with weight of arbitrary sign was significantly motivated by the fact (see, for example, [11]) that such problems are closely connected with the Camassa–Holm equation

$$u_t - u_{xxt} = 2u_x u_{xx} - 3uu_x + uu_{xxx}, \tag{1.2}$$

describing the unidirectional propagation of liquid waves under the shallow water conditions. For a sufficiently complete exposition of the direct and inverse scattering theory for Eq. (1.1) on $\mathbb{R}$, see [12] and [13].

A new stage in the study of Eq. (1.1) was connected with the development of the theory of Sturm–Liouville operators with potentials-distributions (see [14], [15]) and started with the paper [16] in which the function class of coefficients of Eq. (1.1) was significantly enlarged. Namely, the case in which the function ρ is a distribution of first order of singularity, i.e., $\rho \in W_2^{-1}[0,\pi]$, was studied (the function q being identically zero). It is well known (see [2]) that, for positive measures $\rho(x) = dv(x)$,

$$\lim_{n\to\infty} \frac{n}{\sqrt{\lambda_n}} = \int_0^\pi \sqrt{v'(x)}\,dx, \tag{1.3}$$

where the λ_n are the eigenvalues of the problem. In the literature, this case is often referred to as the "Krein string." In the series of papers [16]–[20], asymptotic formulas for λ_n were obtained in the case where the weight ρ is the generalized derivative (distribution) of a self-similar function $v \in \mathrm{L}_2[0,\pi]$ (for the definition of self-similarity, see Sec. 5). Note that the case of self-similar nonnegative measure ρ was also studied in [21]. These studies were initiated by problems dealing with small deviations from zero of Gaussian random processes of certain form (see, for example, [22]); such problems lead, in a natural way, to the study of the asymptotic behavior of the eigenvalues of problem (1.1). In particular, the question whether a particular random process can be expanded in a Karhunen–Loeve series has turned out to be important; a sufficient condition for this is the requirement that the operator of multiplication by the function ρ is an operator of trace class (in the notation of the present paper, this means that the operator $L^{-1}V$ is an operator of trace class). In this connection, the example of a non-trace-class multiplier ρ constructed in [23] is of some interest.

The first goal in this paper is to obtain estimates of the eigenvalues and s-numbers of the operator $L^{-1}V$. Let $v = \int \rho \in \mathrm{L}_2[0,\pi]$. Then

$$\sum_{j=1}^{n} s_j \le C \ln n, \qquad s_n \le C\frac{\ln n}{n}.$$

But if $v \in W_2^\theta[0,\pi]$, $\theta \in [0,1]$ (here W_2^θ are Sobolev spaces with fractional smoothness exponent), then

$$\sum_{j=1}^{\infty} s_j^p < \infty, \qquad s_n \le Cn^{-1/p}$$

for any $p > 1/(1+\theta)$.

The second goal in the paper is to evaluate the sum

$$\operatorname{Tr}_{-1} A(\lambda) := \sum_{n=1}^{\infty} \lambda_n^{-1}, \tag{1.4}$$

i.e., the spectral trace of order (-1) of the pencil $A(\lambda) = L - \lambda V$. The value of the sum (1.4) will be obtained in terms of the function $v(x)$ and the resolvent of the operator L. The calculation of spectral traces of differential operators has a long history. Here we shall cite only the survey [24], which contains a fairly complete bibliography on topics of our interest. In addition, note that the formulas for the first two traces (of negative order) for the quadratic pencil

$$-y'' + \frac{1}{4}y = \lambda\rho y + \lambda^2\sigma y, \qquad x \in \mathbb{R},$$

which is a modification of the spectral problem (1.1) as well as related to the Camassa–Holm equation (1.2), were obtained in [13]. These formulas were used further to model the interactions of two isolated waves (peakons).

2. PRELIMINARIES

Let $\mathcal{H}$ denote the space $L_2[0,\pi]$, let $\|\cdot\|$ be the L_2-norm, and let $(\,\cdot\,,\,\cdot\,)$ denote the inner product in $\mathcal{H}$. For functions $f, g \in \mathcal{H}$, we set

$$\langle f, g\rangle = \int_0^\pi f(x)g(x)\,dx.$$

Further, let $W_2^1[0,\pi]$ be the classical Sobolev space

$$W_2^1 = \{f \in AC[0,\pi] : f' \in \mathcal{H}\} \qquad \text{with norm} \quad \|f\|_1 := (\|f\|^2 + \|f'\|^2)^{1/2}$$

and the corresponding inner product $(\,\cdot\,,\,\cdot\,)_1$. The space of distributions, the completion of the set of functions $f \in \mathcal{H}$ in the norm

$$\|f\|_{-1} := \sup_{\|\varphi\|_1=1} \langle f, \varphi\rangle,$$

will be denoted by $W_2^{-1}[0,\pi]$. Thus, any linear continuous functional F on the space W_2^1 admits the following two representations:

$$F(\varphi) = \langle f, \varphi\rangle = (g, \overline{\varphi})_1, \qquad \text{where} \quad g, \varphi \in W_2^1, \quad f \in W_2^{-1}.$$

A unitary isomorphism $J\colon W_2^1 \to W_2^{-1}$, $J\colon g \mapsto f$, is called a *canonical isomorphism*. The action $\langle f, \varphi\rangle$, $f \in W_2^{-1}$, $\varphi \in W_2^1$, can be written explicitly as

$$\langle f, \varphi\rangle = f_0\varphi(0) + f_\pi\varphi(\pi) - \int_0^\pi \varphi'(x)w(x)\,dx, \qquad w \in \mathcal{H}.$$

Such a representation is not uniquely defined, because the triplets (w, f_0, f_π), $(w + c, f_0 - c, f_\pi + c)$ generate identical functionals. A function w defined in this way, up to a constant, will be called a *generalized antiderivative* of the function $f \in W_2^{-1}$. Further, note that the expression

$$\inf_c(\|w + c\| + |f_0 - c| + |f_\pi + c|)$$

provides an equivalent norm on the space $W_2^{-1}[0,\pi]$. We shall assume that the functions q and ρ are elements of the space $W_2^{-1}[0,\pi]$, and their generalized antiderivatives will be denoted by u and v, respectively.

Our nearest goal is to reveal the exact meaning of the spectral problem (1.1). Consider the linear operator pencil

$$A(\lambda) = L - \lambda V, \qquad \text{where} \quad Ly = -y'' + q(x)y$$

and V is the operator of multiplication by the function ρ. In this paper, we shall consider the case of separated boundary conditions. Strictly speaking (see [14], [15]),

$$Ly = l(y) = -(y^{[1]}(x))' - u(x)y^{[1]}(x) - u^2(x)y(x), \qquad y^{[1]}(x) := y'(x) - u(x)y(x),$$

$$\mathfrak{D}(L) = \{y, y^{[1]} \in AC[0,\pi] : l(y) \in \mathcal{H},\ y^{[1]}(0) + h_0y(0) = y^{[1]}(\pi) + h_\pi y(\pi) = 0\}. \tag{2.1}$$

Let us agree that the equality $h_0 = \infty$ (or $h_\pi = \infty$) means that the first (or the second) boundary condition takes the form $y(0) = 0$ (respectively, $y(\pi) = 0$). The potential q (i.e., the function u and the numbers q_0, q_π) will be assumed real, while and function ρ, complex.[1]

Let us define the operator T acting in the space $\mathcal{H}$ as follows. Set

$$Ty = -y'' + y, \qquad \mathfrak{D}(T) = \{y \in W_2^2[0,\pi] : y'(0) = y'(\pi) = 0\} \quad \text{if } h_0, h_\pi \neq \infty, \tag{2.2}$$

$$Ty = -y'', \qquad \mathfrak{D}(T) = \{y \in W_2^2[0,\pi] : y(0) = y'(\pi) = 0\} \quad \text{if } h_0 = \infty,\ h_\pi \neq \infty, \tag{2.3}$$

$$Ty = -y'', \qquad \mathfrak{D}(T) = \{y \in W_2^2[0,\pi] : y'(0) = y(\pi) = 0\} \quad \text{if } h_0 \neq \infty,\ h_\pi = \infty, \tag{2.4}$$

$$Ty = -y'', \qquad \mathfrak{D}(T) = \{y \in W_2^2[0,\pi] : y(0) = y(\pi) = 0\} \quad \text{if } h_0 = h_\pi = \infty. \tag{2.5}$$

[1]Problem (1.1) can also be posed in the case of a complex potential $q(x)$, but this requires a special technique.

In any case, the operator T is self-adjoint and uniformly positive, and hence its root $T^{1/2}$ is well defined. The Hilbert space with the norm of the graph $(\|T^{1/2}x\|^2 + \|x\|^2)^{1/2}$ will be denoted by $\mathcal{H}_1$. It is easy to see that the space $\mathcal{H}_1$ coincides with $W_2^1[0,\pi]$ in case (2.2), while, otherwise, it is a subspace in $W_2^1[0,\pi]$ defined by the conditions $y(0) = 0$ in case (2.3), $y(\pi) = 0$ in case (2.4), and $y(0) = y(\pi) = 0$ in case (2.5). In all the cases, the norm of the graph is equivalent to the norm of the space W_2^1. The space dual to $\mathcal{H}_1$ with respect to the action $(\,\cdot\,,\,\cdot\,)$ will be denoted by $\mathcal{H}_{-1}$ (obviously $\mathcal{H}_{-1} \subseteq W_2^{-1}[0,\pi]$). By definition, the operator $T^{1/2} = (T^*)^{1/2}$ is an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}$, and hence[2] also between $\mathcal{H}$ and $\mathcal{H}_{-1}$. The operator $T = T^*$ is an isomorphism from $\mathcal{H}_1$ onto $\mathcal{H}_{-1}$.

Integrating by parts, we obtain the following representation for the quadratic form of the operator L on the domain $\mathfrak{D}(L) \subset \mathcal{H}$:

$$(Ly, y) = \mathfrak{l}[y, y] = \|y'\|^2 + \mathfrak{b}[y, y], \tag{2.6}$$

where

$$\mathfrak{b}[y, y] = -h_0|y(0)|^2 + h_\pi|y(\pi)|^2 - \int_0^\pi u(x)(y(x)\overline{y}(x))'\,dx.$$

We have written the answer in the case $h_0 \neq \infty$, $h_\pi \neq \infty$. In the other cases, the corresponding terms after integration vanish. As proved in [15, Lemma 1.10], the form $\mathfrak{b}$ admits the estimate

$$|\mathfrak{b}[y, y]| \le \varepsilon\|y\|_1^2 + M(\varepsilon)\|y\|^2$$

for an arbitrary $\varepsilon > 0$. Then, by the second representation theorem (see, for example, [25, Chap. VI, Theorem 2.23]), the form $\mathfrak{l} + c$ for any sufficiently large $c > 0$ generates a positive self-adjoint operator in $\mathcal{H}$ (coinciding naturally with the operator $L + c$); further,

$$\mathfrak{D}((L + c)^{1/2}) = \mathfrak{D}(\mathfrak{l}) = \mathcal{H}_1,$$

and $(L + c)^{1/2}$ is an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}$. Passing to the dual spaces, we see that the operator $(L + c)^{1/2}$ is also an isomorphism between $\mathcal{H}$ and $\mathcal{H}_{-1}$, while $L + c$ is an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_{-1}$.

Since, for $y \in \mathcal{H}_1$,

$$\langle qy, \overline{y}\rangle = q_0|y(0)|^2 + q_\pi|y(\pi)|^2 - \int_0^\pi u(x)(y(x)\overline{y}(x))'dx,$$

we see that the choice of the numbers h_0 and h_π is equivalent to the specification of the parameters q_0 and q_π of the functional q. In other words, the definition of the operator L depends only on the choice of the potential $q \in W_2^{-1}$ and of one of the four spaces $\mathcal{H}_1$ described above.

It is well known (see [15]) that the spectrum of the operator L acting in the space $\mathcal{H}$ is discrete with only one accumulation point $+\infty$. *We shall further assume that* $0 \notin \sigma(L)$. This will mean that the operator L^{-1} is bounded in $\mathcal{H}$. Then it follows from the equality

$$L^{-1} = (L + c)^{-1} + c(L + c)^{-1}L^{-1}$$

that L^{-1} boundedly acts from $\mathcal{H}$ to $\mathcal{H}_1$, and since

$$L^{-1} = (L + c)^{-1} + cL^{-1}(L + c)^{-1},$$

it follows that L^{-1} boundedly acts from $\mathcal{H}_{-1}$ to $\mathcal{H}_1$. Thus, the operator L is an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_{-1}$.

Since the operator L defined in the space $\mathcal{H}$ on the domain (2.1), is self-adjoint, it follows that all of its eigenvalues μ_k are real. Let us number them in increasing order. The existence and uniqueness theorem for the system of differential equations readily implies that the initial condition

$$\varphi^{[1]}(0, \mu) + h_0\varphi(0, \mu) = 0$$

[2] Here and further, changing the spaces in which the operator acts, we preserve its notation, indicating between what spaces the operator acts.

(in the case $h_0 = \infty$, we set $\varphi(0,\mu) = 0$) defines the solution of the equation $l(\varphi(x,\mu)) = \mu\varphi(x,\mu)$ up to its coefficient. This implies that the geometric multiplicity of each eigenvalue of the operator L is 1. In view of self-adjointness, it follows that the algebraic multiplicity of each number μ_k is also 1. Let $\{\varphi_k\}_1^\infty$ denote the system of corresponding eigenfunctions normalized by the condition $\|\varphi_k\| = 1$. It follows from [15, Theorem 2.9] that the system $\{\varphi_k\}_1^\infty$ is an orthonormal basis in the space $\mathcal{H}$.

Denote by $\kappa \geq 0$ the number of negative eigenvalues of the operator L. In other words,

$$\mu_1 < \mu_2 < \dots < \mu_\kappa < 0 < \mu_{\kappa+1} < \cdots .$$

We have already noted that the operator $(L+c)^{-1/2}$, $c > -\mu_1$, is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}_1$. This, in particular, implies that the system

$$\{(L+c)^{-1/2}\varphi_k = (\mu_k + c)^{-1/2}\varphi_k\}_1^\infty$$

is a Riesz basis in $\mathcal{H}_1$. Then also the systems

$$\pm\{i|\mu_1|^{-1/2}\varphi_1, \dots, i|\mu_\kappa|^{-1/2}\varphi_\kappa\} \cup \{\mu_k^{-1/2}\varphi_k\}_{k=\kappa+1}^\infty$$

are Riesz bases in $\mathcal{H}_1$. This means that the operators $L_\pm^{-1/2}$ given by the equalities

$$L_\pm^{-1/2} x = \pm i \sum_{k=1}^{\kappa} |\mu_k|^{-1/2}(x,\varphi_k)\varphi_k + \sum_{k=\kappa+1}^{\infty} \mu_k^{-1/2}(x,\varphi_k)\varphi_k$$

are isomorphisms between $\mathcal{H}$ and $\mathcal{H}_1$. Passing to the dual spaces (obviously, $(L_\pm^{-1/2})^* = L_\mp^{-1/2}$), we see that both the operators are isomorphisms between $\mathcal{H}_{-1}$ and $\mathcal{H}$. In addition, it is obvious that $L_+^{-1/2} \cdot L_+^{-1/2} = L^{-1}$ (in this equality, we assume that the left operator $L_+^{-1/2}$ acts from $\mathcal{H}$ to $\mathcal{H}_1$, while the right one, from $\mathcal{H}_{-1}$ to $\mathcal{H}$). The same holds for the operator $L_-^{-1/2}$.

Let us turn to the operator V. We shall define it by the following rule:

$$\langle Vy, \varphi\rangle = \langle \rho, y\varphi\rangle = \rho_0 y(0)\varphi(0) + \rho_\pi y(\pi)\varphi(\pi) - \int_0^\pi v(x)(y(x)\varphi(x))'\,dx.$$

Thus, the operator $V\colon W_2^1 \to W_2^{-1}$ is well defined. Using the inequality $\|\cdot\|_{\mathrm{C}} \leq C_{\mathrm{abs}}\|\cdot\|_1$, we obtain the estimate

$$\begin{aligned} |\langle Vy,\varphi\rangle| &\leq (|\rho_0| + |\rho_\pi|)\|y\|_{\mathrm{C}}\|\varphi\|_{\mathrm{C}} + \|v\|_{\mathrm{L}_2}(\|y\|_{\mathrm{C}}\|\varphi'\|_{\mathrm{L}_2} + \|y'\|_{\mathrm{L}_2}\|\varphi\|_{\mathrm{C}}) \\ &\leq C_{\mathrm{abs}}(|\rho_0| + |\rho_\pi| + \|v\|)\|y\|_1\|\varphi\|_1 \leq C_{\mathrm{abs}}\|\rho\|_{-1}\|y\|_1\|\varphi\|_1, \end{aligned} \tag{2.7}$$

from which we see that the operator $V\colon W_2^1 \to W_2^{-1}$ is bounded. Restricting the functional $Vy = \rho \cdot y$ to the subspace $\mathcal{H}_1 \subseteq W_2^1$ (which is equivalent to projecting the vector Vy onto $\mathcal{H}_{-1}$), we obtain a bounded operator from W_2^1 to $\mathcal{H}_{-1}$. In turn, this operator can restricted to the subspace $\mathcal{H}_1 \ni y$, yielding a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. For all such operators, we shall preserve the common notation $V : y \mapsto Vy$, because each operator is identified by its image and argument spaces.

Thus, for any $\lambda \in \mathbb{C}$, we have defined the bounded operator $A(\lambda) = L - \lambda V$, acting from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. Note that, for each fixed λ, the quadratic form of the operator $A(\lambda)$, defined on the space $\mathcal{H}_1$ is of the form

$$\begin{aligned} \langle Ay, \overline{y}\rangle = \|y'\|^2 - (h_0 + \lambda\rho_0)|y(0)|^2 + (h_\pi - \lambda\rho_\pi)|y(\pi)|^2 \\ + \int_0^\pi (\lambda v(x) - u(x))(y(x)\overline{y}(x))'\,dx \end{aligned}$$

(with specifications described above in the cases $h_0 = \infty$ and/or $h_\pi = \infty$). In view of [15], this form defines the closed operator

$$Ay = -(y' + (\lambda v - u)y)' + (\lambda v - u)y'$$

in the space $\mathcal{H}$ with the domain

$$\mathfrak{D}(A) = \{y, y' + (\lambda v - u)y \in AC : Ay \in \mathcal{H},$$

$$(y' + (\lambda v - u)y + (h_0 + \lambda\rho_0)y)(0) = (y' + (\lambda v - u)y + (h_\pi - \lambda\rho_\pi)y)(\pi) = 0\big\}.$$

Thus, the operator pencil $A(\lambda) = L - \lambda V$ can be regarded as a family of closed densely defined unbounded operators in $\mathcal{H}$. However, the domain $\mathfrak{D}(A)$ may vary, depending on the parameter λ.

The spectrum of our operator pencil are all the points $\lambda \in \mathbb{C}$ for which the operator $A(\lambda)\colon \mathcal{H}_1 \to \mathcal{H}_{-1}$ is noninvertible. Note that the operator $L^{-1}V\colon \mathcal{H}_1 \to \mathcal{H}_1$ is compact. Indeed, in the case $\rho \in \mathrm{L}_2$, the operator V acts boundedly from $\mathcal{H}_1$ to $\mathcal{H}$ and it remains to use the compactness of the embedding $\mathcal{H} \subset \mathcal{H}_{-1}$ and the boundedness of the operator $L^{-1}\colon \mathcal{H}_{-1} \to \mathcal{H}_1$. Now, let us approximate the arbitrary functional $\rho \in W_2^{-1}$ by the sequence $\rho_n \to \rho$ in the norm $\|\cdot\|_{-1}$, $\rho_n \in \mathrm{L}_2$, and let us use estimate (2.7), obtaining

$$\|V_n - V\|_{\mathcal{H}_1 \to \mathcal{H}_{-1}} \le C_{\mathrm{abs}} \|\rho_n - \rho\|_{-1}.$$

Thus, the compactness of the operator $L^{-1}V$ follows from the operator convergence $L^{-1}V_n \to L^{-1}V$ and the compactness of the operators $L^{-1}V_n$. The spectrum of the compact operator $L^{-1}V$ consists (with the exception of the point zero) of the eigenvalues. Let μ be one of them, i.e., $L^{-1}Vy = \mu y$ for some $y \in \mathcal{H}_1$. Then $Ly - \mu^{-1}Vy = 0$, i.e., μ^{-1} lies in the spectrum of the pencil $A(\lambda)$. Conversely, if the operator $L^{-1}V - \mu$ is invertible, then the operator

$$A(\mu^{-1}) = L - \mu^{-1}V = -\mu^{-1}L(L^{-1}V - \mu)$$

is also invertible. Thus, $\sigma(A(\lambda))$ coincides with the image of the set $\sigma(L^{-1}V)$ under the mapping $z \mapsto z^{-1}$ in $\mathbb{C}$. We shall number the eigenvalues λ_n of the operator pencil $A(\lambda)$ in increasing order of their moduli, counting multiplicities (by the *multiplicity* of an eigenvalue λ_n we mean $\dim\mathrm{Ker}(L^{-1}V - \lambda_n^{-1})$). The number of points of the spectrum $\sigma(A(\lambda))$ can be both finite (see the examples in Sec. 5) and infinite. Nevertheless, for brevity, we shall abide by the numbering $\{\lambda_n\}_1^\infty$, assuming, where necessary, that $\lambda_n = \infty$ for $n > N$.

3. ESTIMATES FOR THE s-NUMBERS AND EIGENVALUES

Let us recall the definition of symmetrically normed ideals of compact operators (see [26]). A function $\Phi(\xi)$ defined on the linear space of finite sequences is called a *symmetric norming function* if, in addition to the norm axioms, it satisfies the conditions

$$\Phi(1, 0, 0, \dots) = 1, \qquad \Phi(\xi) = \Phi(\xi^*), \tag{3.1}$$

where the sequence $\xi^* = (\xi_n^*)$ is a nondecreasing rearrangement of the sequence $(|\xi_n|)$. Obviously, the function

$$\Phi(\xi) := \sup_{n\in\mathbb{N}} \frac{1}{\psi(n)} \sum_{j=1}^{n} \xi_j^*, \qquad \text{where} \quad \psi(t) = \frac{\ln(t+1)}{\ln 2},$$

satisfies conditions (3.1). For it, the norm axioms also hold. The space of all sequences (ξ_n) for which the expression for $\Phi(\xi)$ is finite endowed with the norm $\Phi(\xi)$, is the *Marcinkiewicz space* M_ψ constructed from the function $\psi(t)$ (see [27]). The function Φ, just as any other symmetric norming function, defines a two-sided ideal $\mathfrak{S}_\psi(\mathfrak{H})$ in the space of bounded operators $B(\mathfrak{H})$ (here $\mathfrak{H}$ is an arbitrary Hilbert space). This ideal consists of all compact operators K whose s-numbers satisfy the condition $\Phi((s_j(K))_1^\infty) < \infty$. The last expression specifies the norm on $\mathfrak{S}_\psi$, and the ideal $\mathfrak{S}_\psi$ is closed with respect to this norm.

Remark 1. Any sequence ξ whose elements satisfy the estimate $|\xi_n| \le Cn^{-1}$ belongs to the space M_ψ (here $\|\xi\|_{M_\psi} \le C$). The converse statement is false. However, if $\xi \in M_\psi$, then

$$\|\xi\|_{M_\psi} \ln(n+1) \ge \ln 2 \sum_{j=1}^{n} \xi_j^* = \ln 2 \biggl(n\xi_n^* - \sum_{j=1}^{n-1} j(\xi_{j+1}^* - \xi_j^*) \biggr) \ge \ln 2 \cdot n\xi_n^*, \tag{3.2}$$

i.e., $\xi_n^* \le C\ln(n+1)/n$, where $C = \|\xi\|_{M_\psi} / \ln 2$.

Theorem 1. *For any function $v \in \mathrm{L}_2[0,\pi]$ and any operator L of the form (2.1),*

$$L^{-1}V \in \mathfrak{S}_\psi(\mathcal{H}_1) \qquad \textit{and} \qquad L_+^{-1/2}VL_+^{-1/2}, L_-^{-1/2}VL_-^{-1/2} \in \mathfrak{S}_\psi(\mathcal{H}).$$

In other words, the s-numbers of each of these operators satisfy the estimates

$$\sum_{j=1}^n s_j \le C\ln n, \qquad s_n \le C\frac{\ln n}{n}. \tag{3.3}$$

The numbers $|\lambda_n|^{-1}$, where the λ_n are the eigenvalues of the pencil $A(\lambda)$, satisfy the same estimates.

Proof. Let us begin with the first assertion of the theorem. Since $\mathfrak{S}_\psi$ is a two-sided ideal, it suffices to carry out the proof for any one of the operators

$$L^{-1}V,\ L_+^{-1/2}VL_+^{-1/2},\ L_-^{-1/2}VL_-^{-1/2},\ T^{-1}V \in \mathcal{B}(\mathcal{H}_1) \qquad \text{is or} \qquad J^{-1}V \in \mathcal{B}(\mathcal{H}_1),$$

where J is a canonical isomorphism from $W_2^1[0,\pi]$ onto $W_2^{-1}[0,\pi]$. The space $\mathcal{H}_1$ depends on the following boundary conditions:

- if $h_0 \ne \infty$ and $h_\pi \ne \infty$, then $\mathcal{H}_1 = W_2^1[0,\pi]$;
- if $h_0 = \infty$, $h_\pi \ne \infty$, then $\mathcal{H}_1$ is a subspace in $W_2^1[0,\pi]$ and is defined by the constraint $y(0) = 0$;
- if $h_0 \ne \infty$, $h_\pi = \infty$, then $\mathcal{H}_1$ is defined by the constraint $y(\pi) = 0$;
- if $h_0 = h_\pi = \infty$, then $\mathcal{H}_1$ is defined by the constraints $y(0) = y(\pi) = 0$.

It suffices to carry out the proof for one (any) of the four cases, because the operators $J^{-1}V$ for the different spaces $\mathcal{H}_1$ differ from one another by the extension (restriction) to the subspace of dimension (codimension) ≤ 2.

It will be convenient to work with the operator $T^{-1}V$ for the case $h_0 = \infty$, $h_\pi \ne \infty$. Here, first, considering the operator $T\colon \mathcal{H} \to \mathcal{H}$, we can explicitly present an operator S for which $T^{-1} = SS^*$:

$$(Sf)(x) = \int_0^x f(t)\,dt, \qquad (S^*f)(x) = \int_x^\pi f(t)\,dt.$$

It is easy to see that the operator $S\colon \mathcal{H} \to \mathcal{H}_1$ is bounded and bijective and S^* can be extended by continuity to the operator $S^*\colon \mathcal{H}_{-1} \to \mathcal{H}$. Since $T^{-1}V = S(S^*VS)S^{-1}$, it suffices to verify that $S^*VS \in \mathfrak{S}_\psi(\mathcal{H})$. If $\rho \in \mathrm{L}_2$, then

$$(S^*VSf, g) = (VSf, Sg) = \langle VSf, \overline{Sg}\,\rangle. \tag{3.4}$$

In the general case, we construct the sequence $\rho_n \in \mathrm{L}_2$, $\|\rho - \rho_n\|_{-1} \to 0$. In view of (2.7) we see that $\|V - V_n\|_{\mathcal{H}_1 \to \mathcal{H}_{-1}} \to 0$, and hence equality (3.4) remains valid. Then

$$\begin{aligned}(S^*VSf, g) &= \rho_0 \cdot (Sf)(0)\overline{(Sg)(0)} + \rho_\pi \cdot (Sf)(\pi)\overline{(Sg)(\pi)} - \int_0^\pi v(x)((Sf)(x)\overline{(Sg)(x)}\,)'\,dx \\ &= \rho_\pi \int_0^\pi f(t)\,dt \int_0^\pi \overline{g}(x)\,dx - \int_0^\pi v(x)f(x)\int_0^x \overline{g}(t)\,dt\,dx \\ &\quad - \int_0^\pi v(x)\overline{g}(x)\int_0^x f(t)\,dt\,dx.\end{aligned}$$

Thus,

$$(S^*VSf)(x) =: (Kf)(x) = \int_0^\pi K(x,t)f(t)\,dt, \qquad K(x,t) = \begin{cases} v(x) + \rho_\pi & \text{for } t \le x, \\ v(t) + \rho_\pi & \text{for } t \ge x.\end{cases}$$

Let us express the operator K as the sum $K = K_+ + K_-$, where

$$(K_- f)(x) = (v(x) + \rho_\pi) \int_0^x f(t)\, dt, \qquad (K_+ f)(x) = \int_x^\pi (v(t) + \rho_\pi) f(t)\, dt.$$

Since

$$(K_+^* f)(x) = (\overline{v}(x) + \overline{\rho}_\pi) \int_0^x f(t)\, dt,$$

i.e., the operator K_+^* is of the same form as the operator K_- (with the function $v(x) + \rho_\pi$ replaced by $\overline{v}(x) + \overline{\rho}_\pi$), it remains to carry out the proof for the operator K_-. To do this, we write out the operator $M := K_-^* K_-$:

$$(Mf)(x) = \int_x^\pi |v(t) + \rho_\pi|^2 \int_0^t f(s)\, ds\, dt$$

and find its eigenvalues s_n^2. After the replacement $y(x) = \int_0^x f(t)\, dt$, the equation $Mf = s^2 f$ takes the form

$$\int_x^\pi |v(t) + \rho_\pi|^2 y(t)\, dt = s^2 y'(x) \qquad \Longleftrightarrow \qquad \begin{cases} -y'' = s^{-2} |v(x) + \rho_\pi|^2 y(x), \\ y(0) = y'(\pi) = 0. \end{cases} \tag{3.5}$$

Thus, we have returned to a spectral problem of the form (1.1), but now with the classical nonnegative summable weight $|v(x) + \rho_\pi|^2$. The eigenvalues of this problem have the asymptotics $s_n^{-2} \sim Cn^2$, i.e., $s_n \sim Cn^{-1}$ (see [2]). Then $K_- \in \mathfrak{S}_\psi(\mathcal{H})$, which proves the first of estimates (3.3).

The second estimate in (3.3) follows from Remark 1, whereby, for any operator $B \in \mathfrak{S}_\psi$, the inequality $s_n(B) \le C \ln n / n$ holds. The last assertion of the theorem follows from Weyl's inequality

$$\sum_{j=1}^n |\lambda_j(L^{-1}V)|^p \le \sum_{j=1}^n s_j^p(L^{-1}V), \qquad p > 0, \tag{3.6}$$

(see [26, Chap. II, Sec. 3]), where the eigenvalues $\lambda_j(L^{-1}V)$ are numbered in decreasing order of their moduli. Indeed, choosing $p = 1$, we see that the sequence $\{|\lambda_n|(L^{-1}V)\}$ belongs to the space M_ψ. It remains to recall that the eigenvalues $\lambda_n(A)$ of the pencil $A(\lambda) = L - \lambda V$ coincide with $\lambda_n^{-1}(L^{-1}V)$. The theorem is proved. □

It is not clear whether the estimate $|\lambda_n| \ge Cn/\ln n$ is sharp; however, in Vladimirov's paper [23], an example of the function $\rho(x)$ was constructed for which the eigenvalues $|\lambda_n(A)|$ were estimated from above: $|\lambda_n| \le Cn \ln n$.

The first assertion of Theorem 1 can be reformulated thus: the mapping $\rho \mapsto L^{-1}V$ acts from the space W_2^{-1} to the space $\mathfrak{S}_\psi(\mathcal{H}_1)$ (respectively, the mappings $\rho \mapsto L_+^{-1/2} V L_+^{-1/2}$ and $\rho \mapsto L_-^{-1/2} V L_-^{-1/2}$ act from W_2^{-1} to $\mathfrak{S}_\psi(\mathcal{H})$). Obviously, these mappings are linear operators.

Proposition 1. *The operators*

$$\rho \mapsto L^{-1}V, \qquad \rho \mapsto L_+^{-1/2} V L_+^{-1/2}, \qquad \rho \mapsto L_-^{-1/2} V L_-^{-1/2}$$

boundedly act from W_2^{-1} to $\mathfrak{S}_\psi$. The constants in the estimates of the characteristic numbers $s_n \le C \ln n / n$ of the operators $L^{-1}V$ and $L_\pm^{-1/2} V L_\pm^{-1/2}$ and in the estimates of the eigenvalues $|\lambda_n| \ge Cn/\ln n$ of the pencil $A(\lambda)$ can be chosen the same on the ball $\|\rho\|_{-1} \le 1$.

Proof. Since $\mathfrak{S}_\psi$ is a Banach ideal, it suffices to carry out the proof for any one of the three operators. Moreover, repeating arguments from the proof of Theorem 1, we see that it suffices carry out the proof for any one of the operators $\rho \mapsto T^{-1}V$, $\rho \mapsto K$, and $\rho \mapsto K_-$. Certainly, the form of the integral kernel $K(x,t) \in \mathrm{L}_2[0,\pi]^2$ immediately implies the continuity of the mapping $\rho \mapsto K_-$ acting from W_2^{-1} to the ideal of the Hilbert–Schmidt operators. This, in particular, implies the fact that the numbers $s_1(K_-)$ on the ball $\|\rho\|_{-1} \le 1$ are bounded. To prove the boundedness of the mapping $\rho \mapsto K_-$ in the metric of the ideal $\mathfrak{S}_\psi$, we shall need slightly finer arguments. It suffices to obtain the estimate $s_n \le Cn^{-1}$ for the characteristic numbers of the spectral problem (3.5) uniformly on the ball $\|\rho\|_{-1} \le 1$ (see the remark preceding Theorem 1). The last estimate is equivalent to the estimate $n(\lambda) \le C\lambda^{1/2}$ for the counting function of the spectral problem

$$-y''(x) = \lambda r(x)y(x), \quad y(0) = y'(\pi) = 0, \qquad r(x) = |v(x) + \rho_\pi|^2.$$

It is well known that the eigenvalues of the Sturm–Liouville operator $-y'' - try$ with boundary conditions $y(0) = y'(\pi) = 0$ are continuous in the parameter t, and since $r(x) \ge 0$, it follows that they are monotone decreasing for $t \in [0, \lambda]$. This means that $n(\lambda)$ coincides with the number of negative eigenvalues of the operator $-y'' + \lambda ry$; we will denote this number by $\nu_-(\lambda)$. By the classical oscillation theory (see, for example, [28, Chap. XI, Sec. 5]), for any solution of the equation $-y'' - \lambda r(x)y$ where r is a nonnegative continuous function, the number of zeros on the half-interval $(0, \pi]$ satisfies the estimate

$$N \le \lambda^{1/2}\frac{\sqrt{\pi}}{2}\left(\int_0^\pi r(x)\,dx\right)^{1/2} + 1.$$

Now, approximating the arbitrary nonnegative function $r \in L_1[0, \pi]$ by continuous functions and passing to the limit in the inequality, we see that the estimate remains valid for $r(x) = |v(x) + \rho_\pi|^2$. It follows from Sturm's theory that

$$n(\lambda) = \nu_-(\lambda) \le N + 1 \le \lambda^{1/2}\frac{\sqrt{\pi}}{2}\left(\int_0^\pi r(x)\,dx\right)^{1/2} + 2$$

(for Sturm's theory for operators with potentials-distributions, see [29]). Substituting $\lambda = s_n^{-2}$ into this inequality, we obtain

$$s_n \le \frac{\frac{\sqrt{\pi}}{2}\left(\int_0^\pi |v(x) + \rho_\pi|^2\,dx\right)^{1/2}}{n-2}, \qquad n \ge 3,$$

We have already referred to the uniform boundedness of the numbers s_1 and $s_2 \le s_1$ on the ball $\|\rho\|_{-1} \le 1$. The first assertion is proved. The second assertion follows from estimates (3.2) and (3.6). The proposition is proved. □

Now consider the case $v \in W_2^\theta[0, \pi]$, $\theta \in [0, 1]$. Here by W_2^θ we denote Sobolev spaces with fractional smoothness exponent θ. These spaces can be defined in a variety of ways. The following definition is convenient for us:

$$W_2^\theta = [\mathrm{L}_2, W_2^1]_\theta = (\mathrm{L}_2, W_2^1)_{\theta,2},$$

where by $[\,\cdot\,,\,\cdot\,]_\theta$ we denote the complex and by $(\,\cdot\,,\,\cdot\,)_{\theta,p}$ the real method of interpolation of Banach spaces. In our proof, we shall also need the quasinormed Besov spaces

$$B^s_{2,\,q} = (\mathrm{L}_2, W_2^1)_{s,q}, \qquad s \in (0, 1), \quad q > 0.$$

Estimates of the operator $L^{-1}V$ will be searched for on the scale of quasinormed two-sided Neumann–Schatten ideals $\mathfrak{S}_p$, $p \in (0, \infty)$. Recall that a compact operator K belongs to an ideal $\mathfrak{S}_p$ if $\sum_{n=1}^\infty s_n^p(K) < \infty$.

Theorem 2. *For any function $v \in W_2^\theta[0, \pi]$, $\theta \in [0, 1]$, and any operator L of the form* (2.1)*, the inclusion $L^{-1}V \in \mathfrak{S}_p(\mathcal{H}_1)$ holds, and*

$$L_+^{-1/2}VL_+^{-1/2},\ L_-^{-1/2}VL_-^{-1/2} \in \mathfrak{S}_p(\mathcal{H}) \qquad \textit{for any} \quad p > \frac{1}{1+\theta}.$$

The s-numbers of each one of the operators

$$L^{-1}V, \qquad L_+^{-1/2}VL_+^{-1/2}, \qquad L_-^{-1/2}VL_-^{-1/2}$$

satisfy the estimate $s_n \le Cn^{-1/p}$, *and the eigenvalues of the pencil* $A(\lambda)$ *satisfy the estimate*

$$|\lambda_n| \ge Cn^{1/p}, \qquad \textit{where} \quad p > \frac{1}{1+\theta} \quad \textit{is arbitrary.}$$

In these estimates, common constants on the ball $\|\rho\|_{-1} \le 1$ *can be chosen.*

Proof. The case $\theta = 0$ follows from the assertion of Theorem 1. Consider the case $\theta = 1$ in which

$$\rho(x) = \rho_0\delta_0(x) + \rho_\pi\delta_\pi(x) + v'(x), \qquad \text{and} \qquad v'(x) \in \mathrm{L}_2[0,\pi].$$

Here the function $\rho(x)$ is a complex-valued σ-additive charge on $[0,\pi]$, and hence it admits the Hahn decomposition

$$\rho(x) = \rho_1(x) - \rho_2(x) + i\rho_3(x) - i\rho_4(x),$$

where the $\rho_j(x)$, $1 \le j \le 4$, are nonnegative measures. Denote by V_j the operator of multiplication by the measure ρ_j. In the same way, just as in the proof of Theorem 1, we note that it suffices to present our arguments for the operator $T^{-1/2}V_jT^{-1/2}$ of one of the types (2.2)–(2.5). Here it is convenient to choose $h_0 = h_\pi = \infty$, so that the numbers ρ_0 and ρ_π can be assumed zero, i.e., $\rho_j \in \mathrm{L}_2[0,\pi]$. Since $\rho_j(x) \ge 0$, it follows that the operators $T^{-1/2}V_jT^{-1/2}$ are nonnegative. Searching for their s-numbers leads to the equation

$$T^{-1/2}V_jT^{-1/2}f = sf,$$

or, equivalently to,

$$\rho_j(x)y(x) = s(Ty)(x), \qquad \text{where} \quad y = T^{-1/2}f.$$

The last equation is of the form

$$-y'' = s^{-1}\rho_j y, \qquad y(0) = y(\pi) = 0.$$

Now it follows from (1.3) that $s_n \le Cn^{-2}$, i.e., $T^{-1/2}V_jT^{-1/2} \in \mathfrak{S}_p$ for any $p > 1/2$. Moreover, the proof of Proposition 1 implies the estimate $s_n \le Cn^{-2}$ uniformly in the ball $\|v_j\|_{W_2^1} \le 1$. Thus, the mapping $v_j \mapsto T^{-1/2}V_jT^{-1/2}$ is continuous as a linear operator from W_2^1 to $\mathfrak{S}_p$, and hence the sum of four mappings, i.e., the mapping $v \mapsto T^{-1/2}VT^{-1/2}$, is also continuous.

We have proved the first assertion of the theorem in the cases $\theta = 0$ and $\theta = 1$. The intermediate cases $\theta \in (0,1)$ follow from the interpolation theorem. It is well known (see [30, Sec. 1.19.7] or the original paper [31]) that

$$(\mathfrak{S}_{p_0}, \mathfrak{S}_{p_1})_{\theta,q} = \mathfrak{S}_q, \qquad \text{if} \quad \frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \theta \in (0,1).$$

The scale of the spaces W_2^θ is closed with respect to the interpolation $(\,\cdot\,,\,\cdot\,)_{\theta,q}$ only for $q = 2$, so that we must use the embeddings $W_2^\theta \subset B_{2,q}^{\theta-\varepsilon}$, valid for any $q \in (0,\infty)$; see, for example, [30, Sec. 4.6.1] (here and further, we choose ε to be a suitable small positive number). We now choose $p_0 = 1/(1-\varepsilon)$, $p_1 = 1/(2-\varepsilon)$, $q = 1/(1+\theta-2\varepsilon)$ and note that

$$(\mathrm{L}_2, W_2^1)_{\theta-\varepsilon,q} = B_{2,q}^{\theta-\varepsilon}.$$

We have proved the boundedness of the operator $\rho \mapsto T^{-1/2}VT^{-1/2}$, acting from L_2 to $\mathfrak{S}_{p_0}$ and from W_2^1 to $\mathfrak{S}_{p_1}$. It follows from the main interpolation theorem that it is bounded from $B_{2,q}^{\theta-\varepsilon}$ to $\mathfrak{S}_q$. Taking into account the continuity of the embedding $W_2^\theta \subset B_{2,q}^{\theta-\varepsilon}$, we obtain that the operator from W_2^θ to $\mathfrak{S}_{1/(1+\theta-2\varepsilon)}$ is bounded, which implies the first assertion of the theorem.

The estimate for the numbers s_n is a consequence of the following simple remark: if the sequence $\{s_n\}_1^\infty$ is positive, monotone nonincreasing, and pth power summable, $p > 0$, then $s_n = O(n^{-1/p})$. Indeed,

$$\|s\|_{l_p}^p \geq \sum_{j=1}^{n} s_j^p = ns_n^p - \sum_{j=1}^{n-1} j(s_{j+1}^p - s_j^p) \geq ns_n^p,$$

whence $s_n \leq \|s\|_{l_p} n^{-1/p}$. Estimates for the numbers $|\lambda_n|$ follow from the same remark and Weyl's inequality (3.6). The theorem is proved. □

We have obtained estimates of the numbers s_n and λ_n with "ε-losses" in the exponent. We consider that our result can be strengthened, but this question remains open.

4. CALCULATION OF THE TRACE OF ORDER (−1)

We have already noticed that the spectrum of the pencil $A(\lambda)$, up to a mapping $z \mapsto z^{-1}$, coincides with that of the operator $L^{-1}V$. An example from [23] shows that $L^{-1}V$ is, possibly, not a trace-class operator, and hence the series (1.4) can fail to be absolutely convergent. Certainly, we can pose the question about its conditional convergence in the sense of $\lim_{r\to+\infty} \sum_{\lambda_n:\,|\lambda_n|\leq r} {\lambda_n}^{-1}$, but, in this paper, we shall evaluate the sum (1.4) only under the condition that $L^{-1}V$ is a trace-class operator.

We already noted in Sec. 2 that the system of eigenfunctions $\{\varphi_k\}_1^\infty$ of the operator L^{-1} constitutes an orthonormal basis in $\mathcal{H}$. In this basis, an operator L^{-1} can be expressed as the series

$$L^{-1} = \sum_{k=1}^{\infty} \mu_k^{-1} \varphi_k(\,\cdot\,, \varphi_k) \tag{4.1}$$

convergent in the uniform operator norm of the space $\mathcal{B}(\mathcal{H})$. Since $\|\varphi_k\|_{\mathcal{H}} = 1$, and $\mu_k \sim Ck^2$ (see [15, Theorem 2.6]), it follows that the function series

$$\sum_{k=1}^{\infty} \mu_k^{-1} \varphi_k(t)\varphi_k(s) =: K(t,s) \tag{4.2}$$

converges in the norm of the space $\mathrm{L}_2[0,\pi]^2$, and hence the series (4.1) converges in the norm of the Hilbert–Schmidt ideal $\mathfrak{S}_2(\mathcal{H})$. Thus, $L^{-1} \in \mathfrak{S}_2(\mathcal{H})$, and its integral kernel $K(t,s)$ belongs to $\mathrm{L}_2[0,\pi]^2$.

Lemma 1. *The series* (4.2) *converges uniformly on the square* $[0,\pi]^2$*, so that the function* $K(t,s)$ *is continuous. Moreover,* $K(t,t) \in W_2^1[0,\pi]$*, and the series* $\sum_{k=1}^{\infty} \mu_k^{-1}\varphi_k^2(t)$ *converges to* $K(t,t)$ *in the norm of the space* W_2^1.

Proof. Here we shall need to know about the asymptotic behavior of the functions $\varphi_k(x)$ as $x \to \infty$. Recall that $\varphi_k^{[1]}(x) := \varphi_k'(x) - u(x)\varphi_k(x)$, where u is the generalized antiderivative of q. Denote by $\widetilde{\varphi}_k^0$ the normalized eigenfunctions of the operator L_0 with potential $q = 0$ and the boundary conditions in which the quasiderivatives are replaced by the ordinary derivatives. By Theorem 2.7 from [15], the following expressions are valid:

$$\varphi_k(x) = \widetilde{\varphi}_k^0(x) + \psi_k(x), \qquad \varphi_k^{[1]}(x) = (\widetilde{\varphi}_k^0(x))' + k\psi_k^1(x), \tag{4.3}$$

where

$$\sup_{x\in[0,\pi]} \sum_{k=1}^{\infty} (|\psi_k(x)|^2 + |\psi_k^1(x)|^2) < \infty.$$

The functions $\widetilde{\varphi}_k^0$ can be written out explicitly. By a direct calculation (omitted here), we can show that, in equalities (4.3), the functions $\widetilde{\varphi}_k^0$ can be replaced by the normalized eigenfunction of the operator $-d^2/dx^2$ with the following simplified boundary conditions:

- $y'(0) = y'(\pi) = 0$ for $h_0 \neq \infty$, $h_\pi \neq \infty$;
- $y(0) = y'(\pi) = 0$ for $h_0 = \infty$, $h_\pi \neq \infty$;
- $y'(0) = y(\pi) = 0$ for $h_0 \neq \infty$, $h_\pi = \infty$;
- $y(0) = y(\pi) = 0$ for $h_0 = h_\pi = \infty$.

We shall denote these functions by φ_k^0 and note that, up to normalization bounded away both from zero and from infinity by a sequence of multipliers, they are of the form

$$\begin{aligned}
&\varphi_k^0(x) = \cos(kx), && k = 0, 1, \dots, && \text{for} \quad h_0 \neq \infty, \quad h_\pi \neq \infty;\\
&\varphi_k^0(x) = \sin\left(k - \frac{1}{2}\right)x, && k = 1, 2, \dots, && \text{for} \quad h_0 = \infty, \quad h_\pi \neq \infty;\\
&\varphi_k^0(x) = \cos\left(k - \frac{1}{2}\right)x, && k = 1, 2, \dots, && \text{for} \quad h_0 \neq \infty, \quad h_\pi = \infty;\\
&\varphi_k^0(x) = \sin(kx), && k = 1, 2, \dots, && \text{for} \quad h_0 = h_\pi = \infty,
\end{aligned} \tag{4.4}$$

whence we have the estimate $\|\varphi_k^0\|_{\mathrm{C}} \le 1$. Then it follows from (4.3) that $\|\varphi_k\|_{\mathrm{C}} \le M$. The uniform convergence of the series (4.2) is proved.

Restricting the series (4.2) to the diagonal $t = s$, we see that the series $\sum_{k=1}^\infty \mu_k^{-1}\varphi_k^2(t)$ converges to $K(t,t)$ uniformly on $[0,\pi]$. It remains to verify that the series of the derivatives $\sum_{k=1}^\infty \mu_k^{-1}\varphi_k(t)\varphi_k'(t)$ converges in the norm of the space $\mathrm{L}_2[0,\pi]$. In view of (4.3), we can write

$$\begin{aligned}
\sum_{k=1}^\infty \mu_k^{-1}\varphi_k(t)\varphi_k'(t) &= \sum_{k=1}^\infty \mu_k^{-1}\varphi_k(t)\varphi_k^{[1]}(t) + \sum_{k=1}^\infty u(t)\mu_k^{-1}\varphi_k^2(t)\\
&= \sum_{k=1}^\infty \mu_k^{-1}\varphi_k^0(t)(\varphi_k^0(t))' + \sum_{k=1}^\infty \mu_k^{-1}\psi_k(t)(\varphi_k^0(t))'\\
&\qquad + \sum_{k=1}^\infty k\mu_k^{-1}\varphi_k(t)\psi_k^1(t) + \sum_{k=1}^\infty u(t)\mu_k^{-1}\varphi_k^2(t).
\end{aligned} \tag{4.5}$$

The convergence of the fourth series on the right-hand side of the last equality follows from the uniform convergence of the series $\sum_{k=1}^\infty \mu_k^{-1}\varphi_k^2(t)$. For the third series, we have

$$\begin{aligned}
\left\|\sum_{k=1}^\infty k\mu_k^{-1}\varphi_k(t)\psi_k^1(t)\right\|_{\mathrm{L}_2} &\le \left(\sum_{k=1}^\infty |k\mu_k^{-1}|^2\right)^{1/2}\left(\sum_{k=1}^\infty \|\varphi_k\|_{\mathrm{C}}^2\|\psi_k^1\|_{\mathrm{L}_2}^2\right)^{1/2}\\
&\le M\left(\sum_{k=1}^\infty \|\psi_k^1\|_{\mathrm{L}_2}^2\right)^{1/2},
\end{aligned} \tag{4.6}$$

because $k\mu_k^{-1} \sim k^{-1}$. Applying Levi's theorem to the sequence of partial sums of the series $\sum_{k=1}^\infty |\psi_k^1(x)|^2$ and taking into account (4.3), we obtain the convergence of the series $\sum_{k=1}^\infty \|\psi_k^1\|_{\mathrm{L}_2}^2$. Thus, the third series on the right-hand side of (4.5) converges in the norm of $\mathrm{L}_2[0,\pi]$. The arguments for the second series are quite similar, because $\|k^{-1}(\varphi_k^0)'\|_{\mathrm{C}} \le M$.

To estimate the first summand, we shall use the explicit form (4.4) of the functions φ_k^0. We have

$$\sum_{k=1}^\infty \mu_k^{-1}\varphi_k^0(t)(\varphi_k^0(t))' = \mp\frac{1}{2}\sum_{k=1}^\infty k\mu_k^{-1}\sin(2kx)$$

for $h_0 \neq \infty$, $h_\pi \neq \infty$ and for $h_0 = h_\pi = \infty$, respectively. In the two remaining cases,

$$\sum_{k=1}^{\infty} \mu_k^{-1} \varphi_k^0(t)(\varphi_k^0(t))' = \pm\frac{1}{2}\sum_{k=1}^{\infty} k\mu_k^{-1}\sin(2k-1)x.$$

Taking into account the square summability of the numbers $k\mu_k^{-1}$, we see that both series obtained above converge in $L_2[0,\pi]$. We have proved the L_2-convergence of all four series in representation (4.5). The lemma is proved. □

Theorem 3. *Let $\rho, q \in W_2^{-1}[0,\pi]$, and let the operator L be of the form* (2.1)*, where $L^{-1}V$ is a trace-class operator. Then the series* (1.4)*, where the λ_n are the eigenvalues of the pencil $A(\lambda)$, converges absolutely to the value of $\langle \rho(x), K(x,x)\rangle$, where $K(t,s)$ is the integral kernel of the operator L^{-1}.*

Proof. The fact that $L^{-1}V$ is a trace-class operator implies that any one of the operators $L_+^{-1/2}VL_+^{-1/2}$ or $L_-^{-1/2}VL_-^{-1/2}$ are trace-class operators (to be definite, we shall further consider only the first of these two). As already noted, the numbers λ_n^{-1} (with the point 0 added) constitute its spectrum, so that the series (1.4) is the spectral trace of this operator. By the Lidskii theorem, the spectral trace of a trace-class operator coincides with its matrix trace calculated in an arbitrary orthonormal basis. For such a basis, we shall choose $\{\varphi_k\}_1^\infty$. Then

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = \sum_{k=1}^{\infty} (L_+^{-1/2}VL_+^{-1/2}\varphi_k, \varphi_k) = \sum_{k=1}^{\infty} \langle VL_+^{-1/2}\varphi_k, \overline{L_-^{-1/2}\varphi_k}\,\rangle$$

(to prove the second equality, we argue just as in the proof of (3.4)). Recall that $L_\pm^{-1/2}\varphi_k = \pm i|\mu_k|^{-1/2}\varphi_k$ for $k \le \kappa$ (when $\mu_k < 0$) and $L_\pm^{-1/2}\varphi_k = \mu_k^{-1/2}\varphi_k$ for $k > \kappa$ (when $\mu_k > 0$). Taking into account the fact that the functions φ_k are real, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} \langle VL_+^{-1/2}\varphi_k, \overline{L_-^{-1/2}\varphi_k}\,\rangle &= -\sum_{k=1}^{\kappa} |\mu_k|^{-1}\langle V\varphi_k, \varphi_k\rangle + \sum_{k>\kappa} \mu_k^{-1}\langle V\varphi_k, \varphi_k\rangle \\ &= \sum_{k=1}^{\infty} \mu_k^{-1}\langle \rho, \varphi_k^2\rangle.\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} \mu_k^{-1}\varphi_k^2(x)$ converges to $K(x,x)$ in the norm of the space $W_2^1[0,\pi]$, we have

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = \left\langle \rho, \sum_{k=1}^{\infty} \mu_k^{-1}\varphi_k^2(x)\right\rangle = \langle \rho(x), K(x,x)\rangle.$$

□

Corollary. *For any function $v \in W_2^\theta[0,\pi]$, $\theta > 0$, and any operator L of the form* (2.1)*, the series* (1.4) *converges absolutely to the value of $\langle \rho(x), K(x,x)\rangle$.*

5. EXAMPLES

Let us present several examples dealing with the calculation of traces of the form (1.4). Let us choose the trivial potential $q(x) \equiv 0$, because, in this case, the integral kernel $K(t,s)$ can be written out explicitly. We shall consider three cases: the Dirichlet boundary conditions $y(0) = y(\pi) = 0$; the Dirichlet–Neumann boundary conditions $y(0) = y'(\pi) = 0$; and the Neumann–Dirichlet boundary conditions $y'(0) = y(\pi) = 0$. We do not consider the Neumann boundary conditions, because, in this case, the spectrum of the operator L will contain the point 0. Certainly, it is also easy to write out the answer for any third boundary-value problem (by considering the case of nonzero finite h_0 and h_π). By a direct calculation, we obtain

$$K_D(t,s)=\min(t,s)-\frac{ts}{\pi},\qquad K_{DN}=\min(t,s),\qquad K_{ND}=\pi-\max(t,s),$$

where K_D, K_{DN}, and K_{ND} denote the corresponding integral kernels. These representations imply that the trace of the pencil $A(\lambda)$ can be expressed by the power moments of orders 0, 1, and 2 of the generalized function (distribution) ρ. Namely, we have

$$\begin{gathered}\operatorname{Tr}_{-1}(L_D-\lambda V)=\left\langle\rho,x-\frac{x^2}{\pi}\right\rangle,\\ \operatorname{Tr}_{-1}(L_{DN}-\lambda V)=\langle\rho,x\rangle,\qquad \operatorname{Tr}_{-1}(L_{ND}-\lambda V)=\langle\rho,\pi-x\rangle,\end{gathered}\tag{5.1}$$

where L_D, L_{DN}, and L_{ND} denote the corresponding second differentiation operators. The further simplification of expressions (5.1) involves the calculation of the moments of the function ρ. We shall consider several cases.

1) Let $\rho(x)=\sum_{k=1}^n m_k\delta(x-x_k)$, where m_k and $x_k\in[0,\pi]$, $1\le k\le n$, are known constants. Then

$$\operatorname{Tr}_{-1}(L_D-\lambda V)=\left\langle\sum_{k=1}^n m_k\delta(x-x_k),x-\frac{x^2}{\pi}\right\rangle=\sum_{k=1}^n m_kx_k-\frac{1}{\pi}\sum_{k=1}^n m_kx_k^2,$$

$$\operatorname{Tr}_{-1}(L_{DN}-\lambda V)=\left\langle\sum_{k=1}^n m_k\delta(x-x_k),x\right\rangle=\sum_{k=1}^n m_kx_k,$$

$$\operatorname{Tr}_{-1}(L_{ND}-\lambda V)=\left\langle\sum_{k=1}^n m_k\delta(x-x_k),\pi-x\right\rangle=\sum_{k=1}^n m_k(\pi-x_k).$$

2) The weight ρ is the derivative of a self-similar real function $v\in \mathrm{L}_2[0,\pi]$. Recall (see [18]) that the function v is a self-similar function if it satisfies the following functional equation:

$$v(x)=\sum_{k=1}^n\chi_{(\alpha_k,\alpha_{k+1})}(x)\left\{\beta_k+d_kv\left(\frac{x-\alpha_k}{a_k}\right)\right\}.\tag{5.2}$$

Here $n\ge 2$, while the points α_k, $k=1,\dots,n+1$, specify the subdivision of the closed interval $[0,\pi]$ of the form

$$0=\alpha_1<\alpha_2<\dots<\alpha_{n+1}=\pi.$$

Then the points a_k, $k=1,\dots,n$, are uniquely determined by the equalities $a_k=(\alpha_{k+1}-\alpha_k)/\pi$. The numbers d_k, $k=1,\dots,n$, satisfy the condition $\sum_{k=1}^n a_k|d_k|^2<1$, while the numbers β_k are arbitrary. It is well known (see [16, Lemma 3.1]) that, under these conditions on the self-similarity parameters, Eq. (5.2) uniquely determines the self-similar function $v\in \mathrm{L}_2[0,\pi]$. Let us consider the function ρ, the generalized derivative of a self-similar function f; namely, we set

$$\langle\rho,\varphi\rangle=\rho_0\varphi(0)+\rho_\pi\varphi(\pi)-\int_0^\pi v(x)\varphi'(x)\,dx.$$

It is necessary to find the expressions for the first and second moments of the function ρ in terms of the self-similarity parameters. The formulas that we obtain below are well known (see, for example, [29] in the case of a Dirichlet conditions). The calculations will be given for the benefit of the reader. It is easy to see that $\langle\rho,1\rangle=\rho_0+\rho_\pi$. Further,

$$\langle\rho,x\rangle=\rho_\pi-\int_0^\pi v(x)\,dx,$$

and the last integral can be evaluated using Eq. (5.2):

$$\int_0^\pi v(x)\,dx=\sum_{k=1}^n\beta_k(\alpha_{k+1}-\alpha_k)+\sum_{k=1}^n d_ka_k\int_0^\pi v(y)\,dy$$

(in the integral, we make the replacement $x = a_k y + \alpha_k$). Thus, for the first moment of the function ρ, we have

$$\langle \rho, x\rangle = \rho_\pi - \frac{\pi \sum_{k=1}^n a_k\beta_k}{1-\sum_{k=1}^n d_k a_k}\,. \tag{5.3}$$

The denominator of the fraction is nonzero, because

$$\biggl|\sum_{k=1}^n d_k a_k\biggr| \le \biggl(\sum_{k=1}^n a_k|d_k|^2\biggr)^{1/2}\biggl(\sum_{k=1}^n a_k\biggr)^{1/2} < 1.$$

Similarly, for the second moment, we have

$$\langle \rho, x^2\rangle = \rho_\pi - 2\int_0^\pi xv(x)\,dx,$$

and the last integral is again evaluated using the self-similarity equation

$$\int_0^\pi xv(x)\,dx = \frac{1}{2}\sum_{k=1}^n \beta_k(\alpha_{k+1}^2-\alpha_k^2) + \sum_{k=1}^n d_k a_k^2\int_0^\pi yv(y)\,dy + \sum_{k=1}^n d_k a_k\alpha_k\int_0^\pi v(y)\,dy,$$

whence

$$\langle \rho, x^2\rangle = \rho_\pi - \frac{\displaystyle\sum_{k=1}^n \beta_k(\alpha_{k+1}^2-\alpha_k^2)}{\displaystyle 1-\sum_{k=1}^n a_k^2 d_k} - \frac{\displaystyle 2\pi\sum_{k=1}^n a_k\beta_k\cdot\sum_{k=1}^n a_k d_k\alpha_k}{\displaystyle\biggl(1-\sum_{k=1}^n a_k d_k\biggr)\biggl(1-\sum_{k=1}^n a_k^2 d_k\biggr)}\,. \tag{5.4}$$

Using (5.3) and (5.4), we can easily obtain formulas for the traces of order (-1) for all boundary conditions considered in the present paper. Let us illustrate this for the Cantor weight in detail, i.e., for the case in which ρ is the generalized derivative of a function v, which is a Cantor function on the closed interval $[0,\pi]$. For the self-similarity parameters, we have

$$n=3,\qquad \alpha_1=0,\quad \alpha_2=\frac{\pi}{3},\quad \alpha_3=\frac{2\pi}{3},\quad \alpha_4=\pi,$$
$$a_1=a_2=a_3=\frac{1}{3},\qquad \beta_1=0,\quad \beta_2=\beta_3=\frac{1}{2},\qquad d_1=d_3=\frac{1}{2},\quad d_2=0;$$

also, $\rho_0=0$, $\rho_\pi=\pi$. Then

$$\langle\rho,1\rangle=\pi,\qquad \langle\rho,x\rangle=\frac{\pi}{2},\qquad \langle\rho,x^2\rangle=\pi-\frac{5\pi^2}{8}\,.$$

Hence

$$\mathrm{Tr}_{-1}(L_D-\lambda V)=\frac{9\pi}{8}-1,$$
$$\mathrm{Tr}_{-1}(L_{DN}-\lambda V)=\frac{\pi}{2},\qquad \mathrm{Tr}_{-1}(L_{ND}-\lambda V)=\pi^2-\frac{\pi}{2}\,.$$

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